SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 10: Rigid body dynamics II & MAV dynamics

King Fahd University of Petroleum & Minerals

جامعة الملك فهد للبترول والمعادن

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Outline

- Recap last lectures
- Covector nature of wrenches and momenta
- Rigid body dynamics
- Case study: Multi-rotor aerial vehicles (MAVs)

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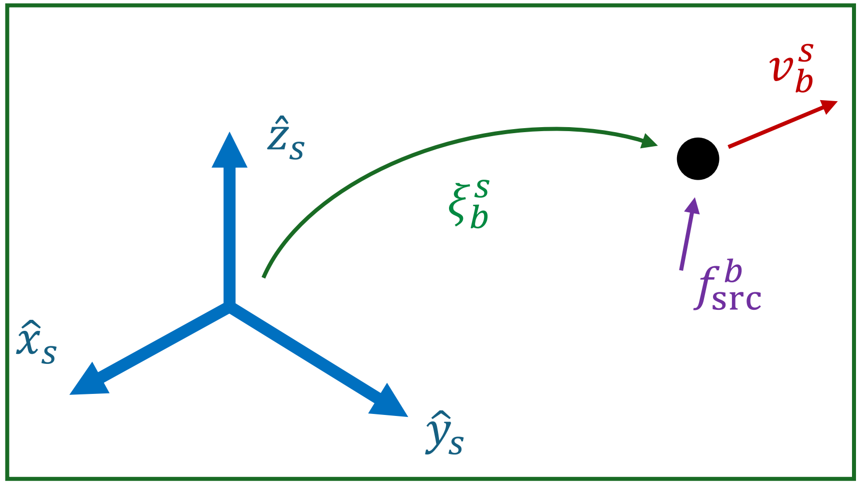
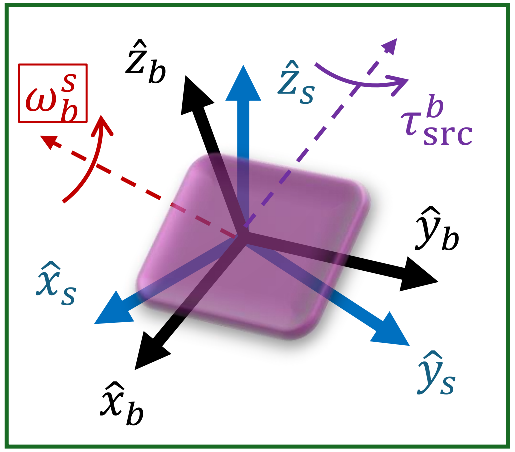
Recap: Dynamic modeling

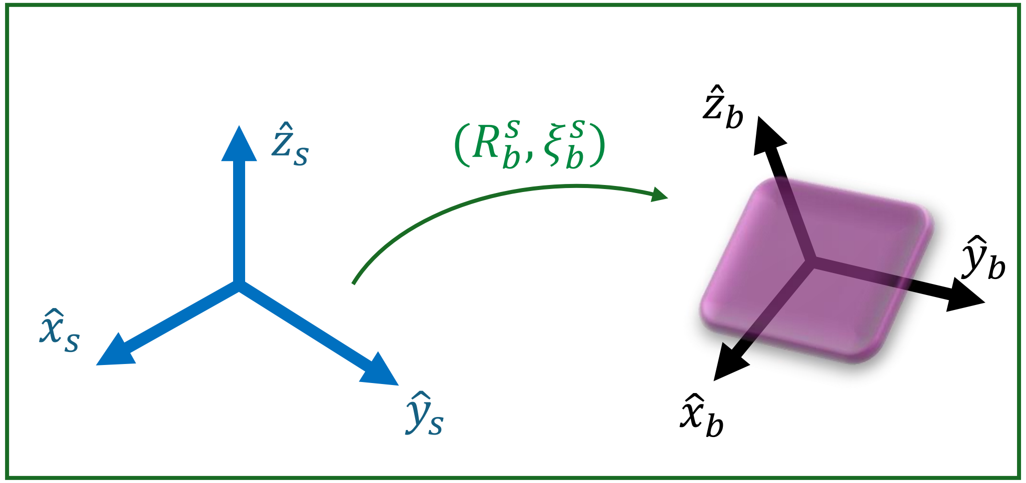
- A dynamic model describes the motion of a system while considering the forces and torques that cause the motion.
- It includes both kinematics and conservation laws



<u>Forces</u> on translating point mass

<u>Torques</u> on rotating rigid body



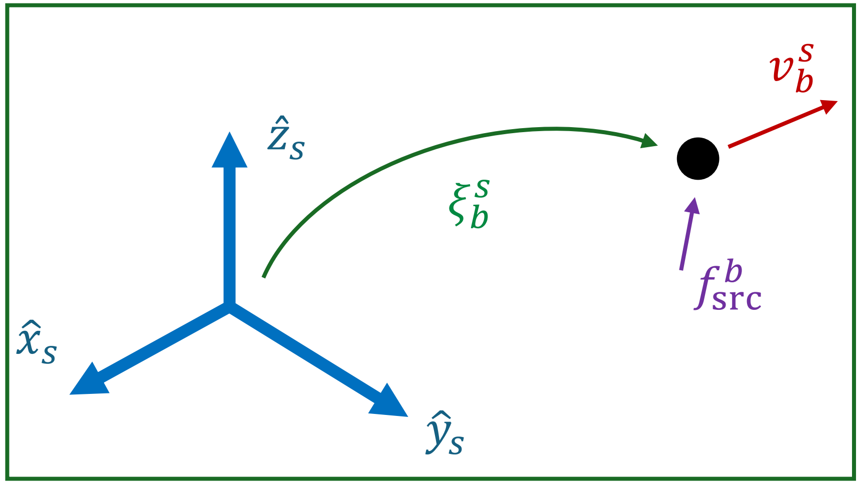


<u>Wrenches</u> on moving rigid body

Recap: Point mass dynamics

- We will denote by:
  - $f_{\text{src}}^{b} \in \mathbb{R}^3$: the abstract force from source src acting on point mass b
  - $f_{\text{src}}^{s,b} \in \mathbb{R}^3$: the force from source src acting on point mass b, expressed in $\{s\}$.
- The kinetic energy of point b, $E_k: \mathbb{R}^3 \rightarrow \mathbb{R}$ is given by:
  - $E_k\left(v_b^{s,s}\right) = \frac{1}{2}\,\mathrm{m}\left(v_b^{s,s}\right)^{\top} v_b^{s,s}$
- The linear momentum of point b expressed in $\{s\}$, $P_v^{s,b} \in \mathbb{R}^3$ is given by:
  - $P_v^{s,b} \coloneqq \frac{\partial E_k}{\partial v_b^{s,s}}\left(v_b^{s,s}\right) = \mathrm{m} v_b^{s,s}$
- Newton's law:
  - $\dot{P}_v^{s,b} = f_{\text{tot}}^{s,b}$

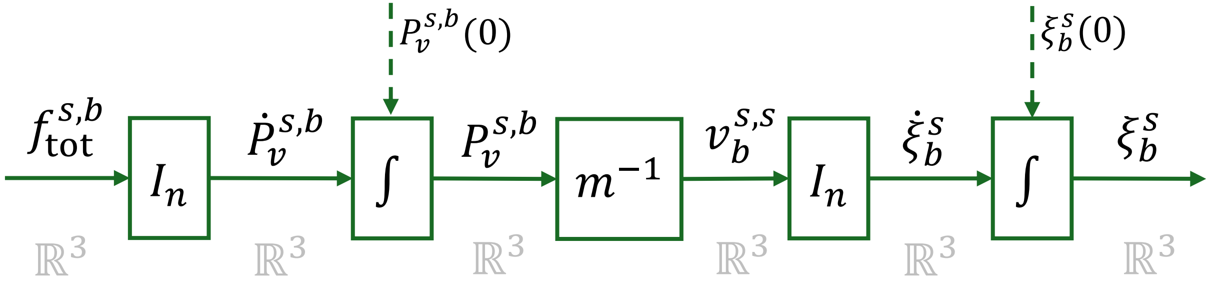
  or
  - $\dot{v}_b^{s,s} = \frac{1}{m} f_{\text{tot}}^{s,b}$



Forces on translating point mass

Recap: Point mass dynamics

- In summary,



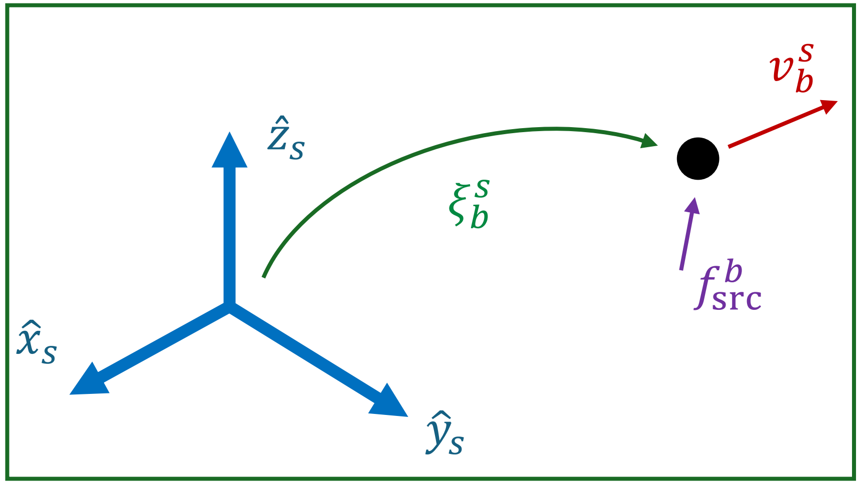
- $\dot{\xi}_b^s = v_b^{s,s}$ — Kinematic relation
- $\dot{P}_v^{s,b} = f_{\text{tot}}^{s,b}$ — Momentum balance
- $v_b^{s,s} = m^{-1} P_v^{s,b}$ — Constitutive relation

Rewritten as

- $\dot{\xi}_b^s = v_b^{s,s}$
- $\dot{v}_b^{s,s} = m^{-1}\, f_{\text{tot}}^{s,b}$

Rewritten as

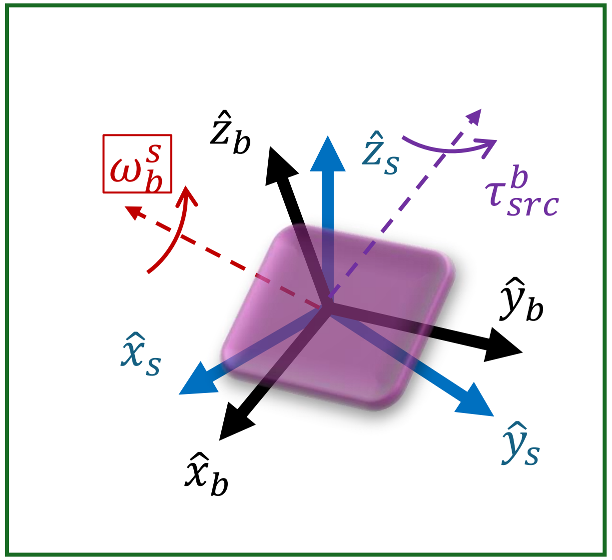
- $\ddot{\xi}_b^s = m^{-1}\, f_{\text{tot}}^{s,b}$



Forces on translating point mass

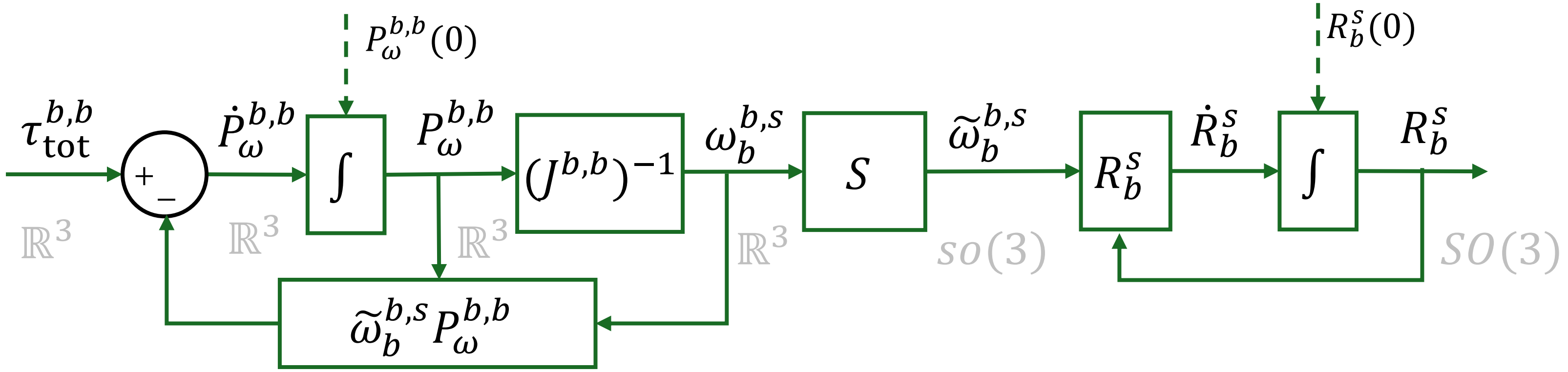
Recap: Rigid body rotation dynamics

- We will denote by:
  - $\tau_{\mathrm{src}}^{b} \in \mathbb{R}^3$: the abstract torque from source src acting on body attached to $\{b\}$
  - $\tau_{\mathrm{src}}^{*,b} \in \mathbb{R}^3$: the torque from source src acting on body attached to $\{b\}$, expressed in $\{*\}$.
- The kinetic energy of the rigid body, $E_k: SO(3) \times \mathbb{R}^3 \to \mathbb{R}$ is given by:
  - $E_k\left(R_b^s, \omega_b^{*,s}\right) = \frac{1}{2}\left(\omega_b^{*,s}\right)^{\top} J^{*,b}(R_b^s)\, \omega_b^{*,s}$
- The angular momentum of the rigid body expressed in $\{*\}$, $P_\omega^{*,b} \in \mathbb{R}^3$ is given by:
  - $P_\omega^{*,b} \coloneqq \frac{\partial E_k}{\partial \omega_b^{*,s}}\left(R_b^s, \omega_b^{*,s}\right) = J^{*,b}(R_b^s)\, \omega_b^{*,s}$
- Euler's law in $\{s\}$:
  - $\dot{P}_\omega^{s,b} = \tau_{\mathrm{tot}}^{s,b}$



[Torques](#) on rotating rigid body

Recap: Rigid body rotation dynamics



- $\dot{R}_b^s = R_b^s \tilde{\omega}_b^{b,s}$ — Kinematic relation
- $\dot{P}_\omega^{s,b} = \tau_{\text{tot}}^{s,b}$ — Momentum balance
- $\omega_b^{b,s} = (J^{b,b})^{-1} P_\omega^{b,b}$ — Constitutive relation

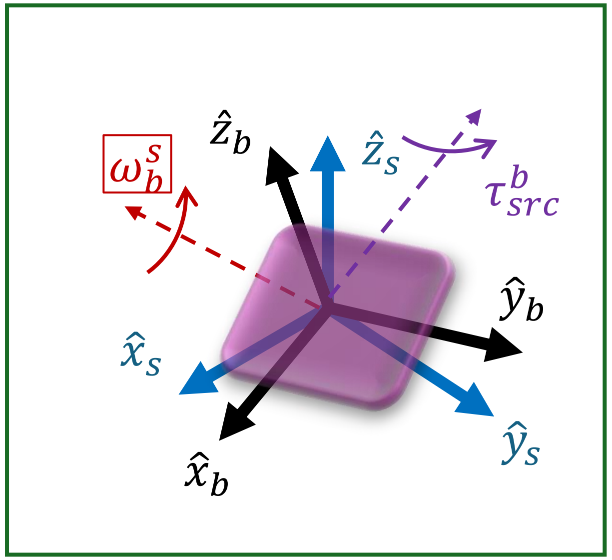
Rewritten as

1

- $\dot{R}_b^s = R_b^s \tilde{\omega}_b^{b,s}$
- $\dot{P}_\omega^{b,b} = \tau_{\text{tot}}^{b,b} - \tilde{\omega}_b^{b,s} P_\omega^{b,b}$
- $\omega_b^{b,s} = (J^{b,b})^{-1} P_\omega^{b,b}$

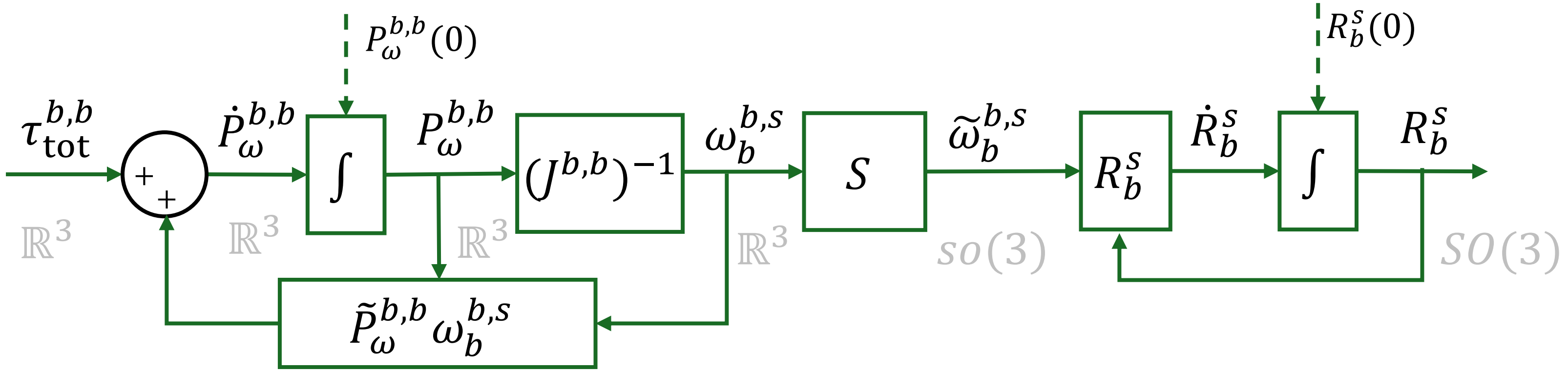
$P_\omega^{*,b} \in \mathbb{R}^3$ is the angular momentum of body attached to $\{b\}$ expressed in $\{*\}$

$J^{*,b} \in \mathbb{R}^{3\times 3}$ is the moment of inertia of body attached to $\{b\}$ expressed in $\{*\}$



Torques on rotating rigid body

Recap: Rigid body rotation dynamics



- $\dot{R}_b^s = R_b^s \tilde{\omega}_b^{b,s}$ — Kinematic relation
- $\dot{P}_\omega^{s,b} = \tau_{\text{tot}}^{s,b}$ — Momentum balance
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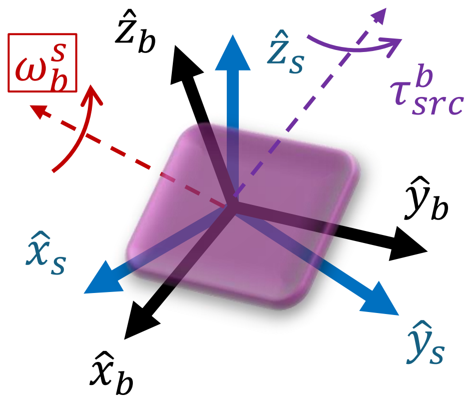
Or rewritten as

2

- $\dot{R}_b^s = R_b^s \tilde{\omega}_b^{b,s}$
- $\dot{P}_\omega^{b,b} = \tau_{\text{tot}}^{b,b} + \tilde{P}_\omega^{b,b}\,\omega_b^{b,s}$
- $\omega_b^{b,s} = (J^{b,b})^{-1} P_\omega^{b,b}$



Torques on rotating rigid body

$P_\omega^{*,b} \in \mathbb{R}^3$ is the angular momentum of body attached to $\{b\}$ expressed in $\{*\}$

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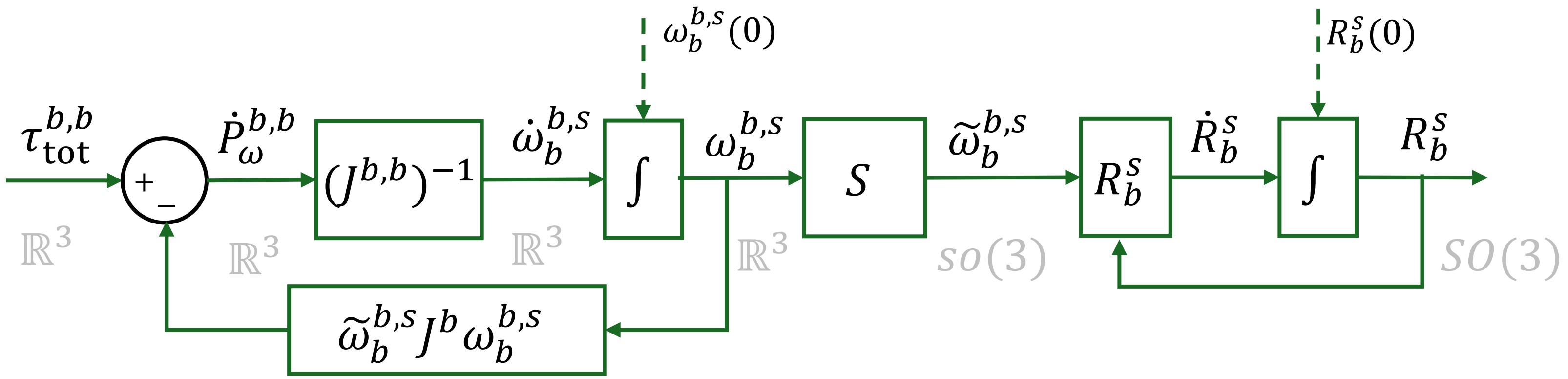
Recap: Rigid body rotation dynamics

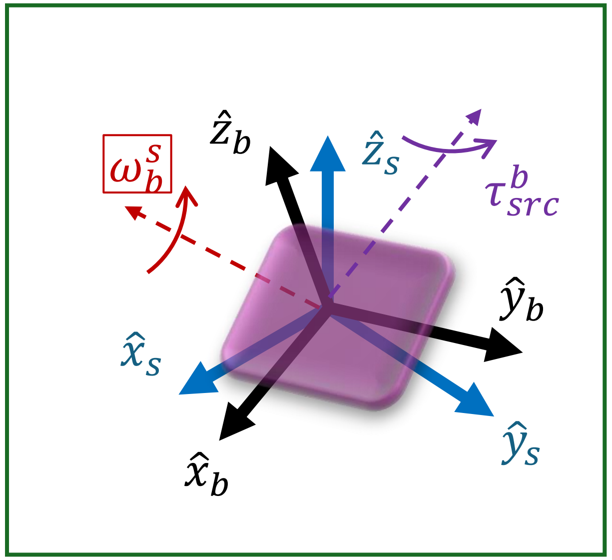


- $\dot{R}_b^s = R_b^s \widetilde{\omega}_b^{b,s}$ Kinematic relation
- $\dot{P}_\omega^{s,b} = \tau_{\text{tot}}^{s,b}$ Momentum balance
- $\omega_b^{b,s} = (J^{b,b})^{-1} P_\omega^{b,b}$ Constitutive relation

Which can also be written as

3

- $\dot{R}_b^s = R_b^s \widetilde{\omega}_b^{b,s}$
- $\dot{\omega}_b^{b,s} = \left(J^{b,b}\right)^{-1}\left(\tau_{\text{tot}}^{b,b} - \widetilde{\omega}_b^{b,s} J^b \omega_b^{b,s}\right)$



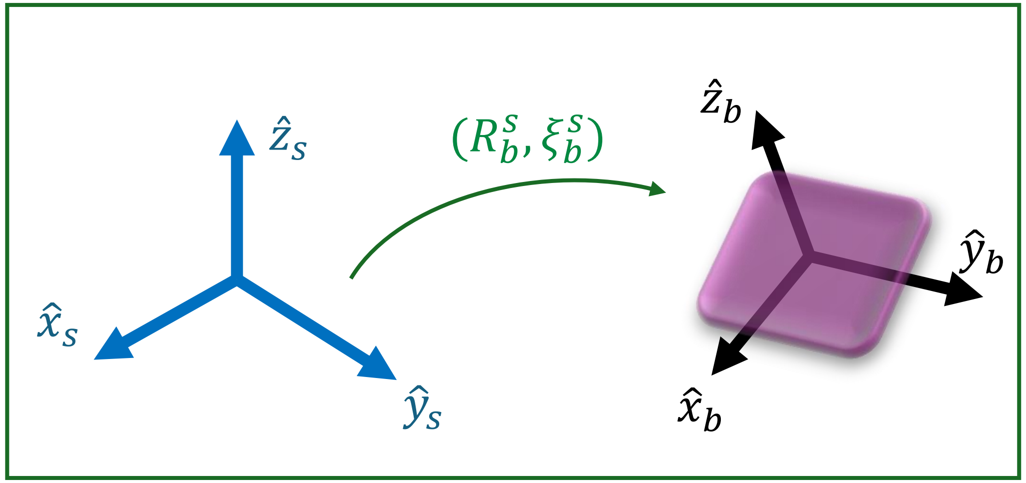
<u>Torques</u> **on rotating rigid body**

$P_\omega^{*,b} \in \mathbb{R}^3$ is the angular momentum of body attached to $\{b\}$ expressed in $\{*\}$

$J^{*,b} \in \mathbb{R}^{3\times3}$ is the moment of inertia of body attached to $\{b\}$ expressed in $\{*\}$

Recap: Rigid body motion dynamics

- We will denote by:
  - $\mathcal{W}^{b}_{\text{src}} \in (\mathbb{R}^6)^*$: the abstract wrench from source src acting on body attached to $\{b\}$
  - $\mathcal{W}^{*,b}_{\text{src}} \in (\mathbb{R}^6)^*$: the wrench from source src acting on body attached to $\{b\}$, expressed in $\{*\}$.
- The kinetic energy of the rigid body, $E_k : SE(3) \times \mathbb{R}^6 \to \mathbb{R}$ is given by:
  - $E_k\left(H^s_b, \mathcal{V}^{*,s}_b\right) = \frac{1}{2}\left(\mathcal{V}^{*,s}_b\right)^\top \mathfrak{I}^{*,b}(H^s_b)\, \mathcal{V}^{*,s}_b$
- The generalized momentum of the rigid body expressed in $\{*\}$, $P^{*,b} \in (\mathbb{R}^6)^*$ is given by:
  - $P^{*,b} := \frac{\partial E_k}{\partial \mathcal{V}^{*,s}_b}\left(H^s_b, \mathcal{V}^{*,s}_b\right) = \mathfrak{I}^{*,b}(H^s_b)\mathcal{V}^{*,s}_b$
- Netwon-Euler's law in $\{s\}$:
  - $\dot{P}^{s,b} = \mathcal{W}^{s,b}_{\text{tot}}$



Wrenches on moving rigid body

Note:
Wrenches and generalized momenta are covectors !!

$\boxed{\mathcal{W}^{*,b}, P^{*,b} \in (\mathbb{R}^6)^*}$

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Kinetic energy invariance

- Kinetic energy is a scalar, so it is a coordinate-free concept.

$$E_k\left(H_b^s, \mathcal{V}_b^{b,s}\right) = E_k\left(H_b^s, \mathcal{V}_b^{s,s}\right) = E_k\left(H_b^s, \mathcal{V}_b^{*,s}\right)$$

- Let's examine the kinetic energy of the twist in $\{b\}$:

$$E_k\left(H_b^s, \mathcal{V}_b^{b,s}\right) = E_k\left(\mathcal{V}_b^{b,s}\right) = \frac{1}{2}\left(\mathcal{V}_b^{b,s}\right)^{\top} \mathfrak{I}^{b,b}\, \mathcal{V}_b^{b,s}$$

$$\boxed{\mathfrak{I}^{b,b} = \begin{pmatrix} J^{b,b} & m\,\tilde{\xi}_{\mathrm{cm}}^{b} \\ -m\tilde{\xi}_{\mathrm{cm}}^{b} & m\,I_3 \end{pmatrix}}$$

Generalized inertia

$\mathfrak{I}^{b,b} \in \mathbb{R}^{6\times 6}$ is the generalized inertia of body attached to $\{b\}$ expressed in $\{b\}$ and is independent of H_b^s

Kinetic energy invariance

- Kinetic energy is a scalar, so it is a coordinate-free concept.

$$E_k\left(H_b^s, \mathcal{V}_b^{b,s}\right) = E_k\left(H_b^s, \mathcal{V}_b^{s,s}\right) = E_k\left(H_b^s, \mathcal{V}_b^{*,s}\right)$$

- Let's examine the kinetic energy of the twist in $\{b\}$:

$$E_k\left(H_b^s, \mathcal{V}_b^{b,s}\right) = E_k\left(\mathcal{V}_b^{b,s}\right) = \frac{1}{2}\left(\mathcal{V}_b^{b,s}\right)^\top \mathfrak{I}^{b,b}\, \mathcal{V}_b^{b,s}$$

- Using the definition of the generalized momentum, we have that

$$2E_k = \left(\mathcal{V}_b^{b,s}\right)^\top P^{b,b} = \left(P^{b,b}\right)^\top \mathcal{V}_b^{b,s}$$

$$\boxed{\mathfrak{I}^{b,b} = \begin{pmatrix} J^{b,b} & m\,\tilde{\xi}_{\mathrm{cm}}^{b} \\ -m\tilde{\xi}_{\mathrm{cm}}^{b} & m\, I_3 \end{pmatrix}}$$

Generalized inertia

The duality pairing between twist $\mathcal{V}_b^{*,s}$ and generalized momentum $P^{*,b}$ is twice the kinetic energy $2E_k$

Generalized momentum

- Due to the invariance of kinetic energy, if we express the momentum and twist in the $\{s\}$ frame we have that:

$$2E_k = \left(\mathcal{P}^{b,b}\right)^{\top}\mathcal{V}_b^{b,s} = \left(\mathcal{P}^{b,b}\right)^{\top}Ad_{H_s^b}\mathcal{V}_b^{s,s} = \left(Ad_{H_s^b}^{\top}\mathcal{P}^{b,b}\right)^{\top}\mathcal{V}_b^{s,s} = \left(\mathcal{P}^{s,b}\right)^{\top}\mathcal{V}_b^{s,s}$$

- Therefore,

Vector $\boxed{\mathcal{V}_b^{s,s} = Ad_{H_b^s}\mathcal{V}_b^{b,s} \in \mathbb{R}^6}$

$\boxed{\mathcal{P}^{s,b} = Ad_{H_b^s}^{-\top}\mathcal{P}^{b,b} \in \left(\mathbb{R}^6\right)^*}$ Covector

$$\begin{tikzcd}
\mathcal{V}_b^{b,s} \arrow[r] & \boxed{Ad_{H_b^s}} \arrow[r] & \mathcal{V}_b^{s,s} \\
\mathcal{P}^{b,b} & \boxed{Ad_{H_b^s}^{\top}} \arrow[l] & \mathcal{P}^{s,b} \arrow[l]
\end{tikzcd}$$

A vector and a covector change coordinates differently !!

Wrench

- Just as with twists, we can merge torques and forces into a six-dimensional object we shall call a wrench.

$$\mathcal{W}^{*,b} = \begin{pmatrix} \tau^{*,b} \\ f^{*,b} \end{pmatrix} \in \left(\mathbb{R}^6\right)^*$$

- The duality pairing of a wrench and a twists gives mechanical power:

$$\text{Power} = \left(\mathcal{W}^{*,b}\right)^\top \mathcal{V}_b^{*,s}$$

- Since power is also a coordinate-free concept, we have that

Vector $\boxed{\mathcal{V}_b^{s,s} = Ad_{H_b^s} \mathcal{V}_b^{b,s} \in \mathbb{R}^6}$

$\boxed{\mathcal{W}^{s,b} = Ad_{H_b^s}^{-\top} \mathcal{W}^{b,b} \in \left(\mathbb{R}^6\right)^*}$ **Covector**

$$\begin{tikzcd} \mathcal{V}_b^{b,s} \arrow[r] & \boxed{Ad_{H_b^s}} \arrow[r] & \mathcal{V}_b^{s,s} \\ \mathcal{W}^{b,b} & \boxed{Ad_{H_b^s}^\top} \arrow[l] & \mathcal{W}^{s,b} \arrow[l] \end{tikzcd}$$

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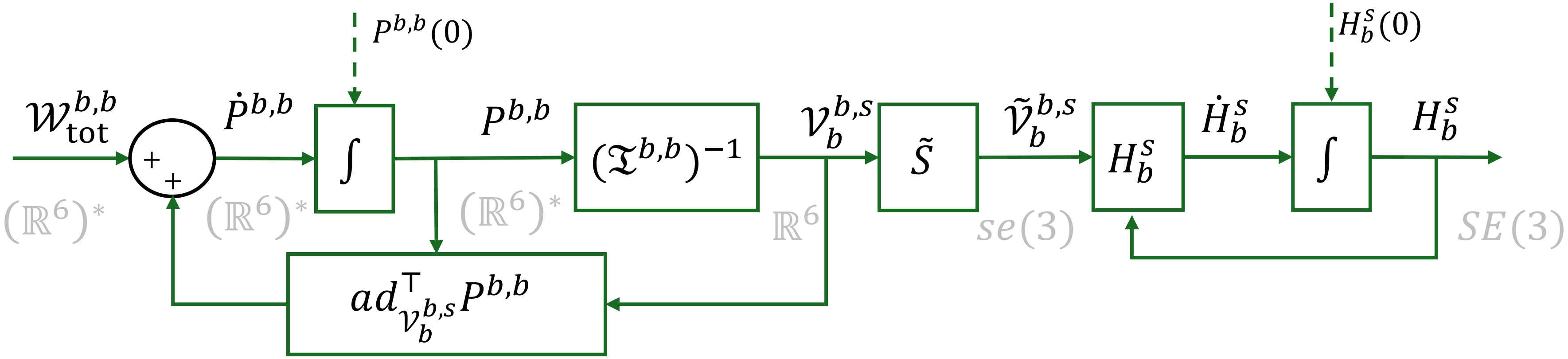
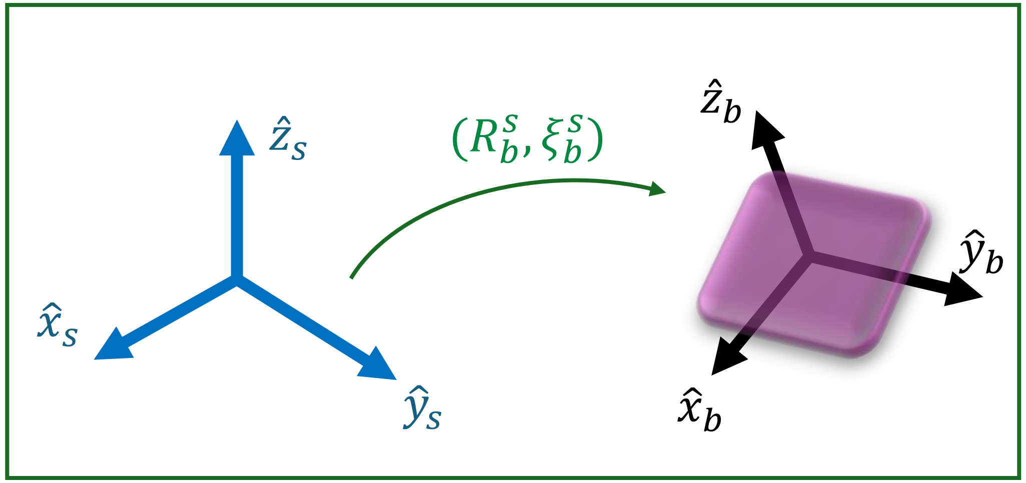
- Case study: Multi-rotor aerial vehicles (MAVs)

Rigid body dynamics

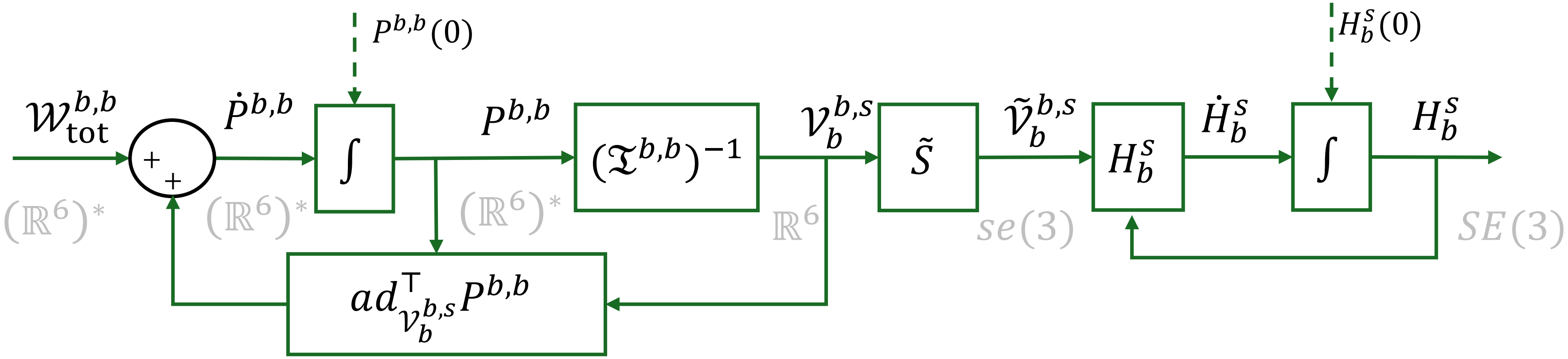


- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$ — Kinematic relation
- $\dot{P}^{s,b} = \mathcal{W}_{\text{tot}}^{s,b}$ — Momentum balance
- $\mathcal{V}_b^{b,s} = (\mathfrak{I}^{b,b})^{-1} P^{b,b}$ — Constitutive relation



Wrenches on moving rigid body

Rigid body dynamics



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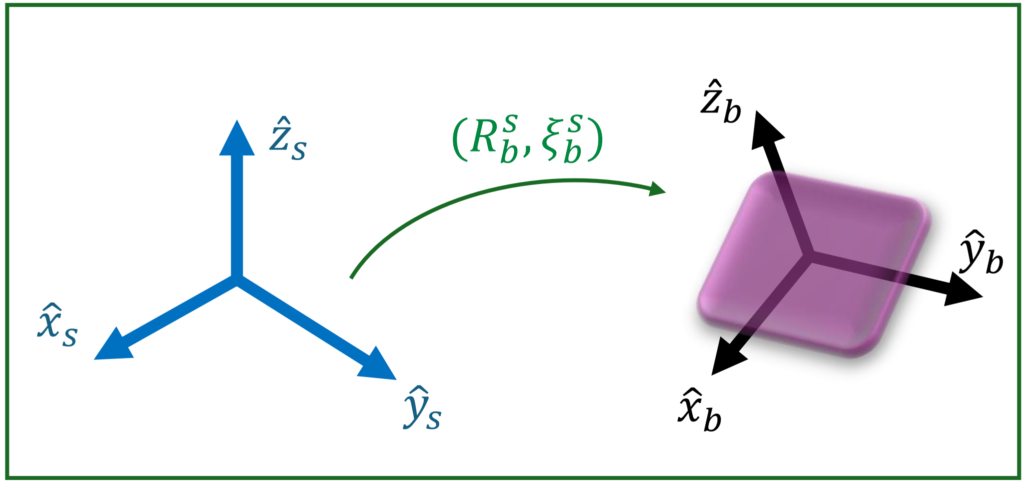
Rewritten as

1

- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$
- $\dot{P}^{b,b} = \mathcal{W}_{\text{tot}}^{s,b} + ad^{\top}_{\mathcal{V}_b^{b,s}} P^{b,b}$
- $\mathcal{V}_b^{b,s} = (\mathfrak{I}^{b,b})^{-1} P^{b,b}$

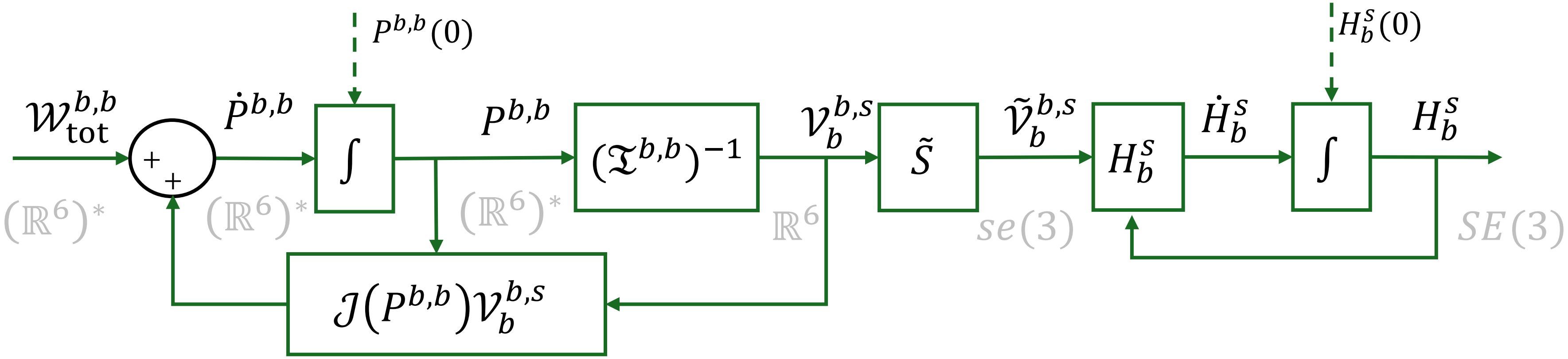
$$\boxed{ad_{\mathcal{V}_b^{b,s}} : \mathbb{R}^6 \to \mathbb{R}^6}$$

Provided in the homework



Wrenches on moving rigid body

Rigid body dynamics



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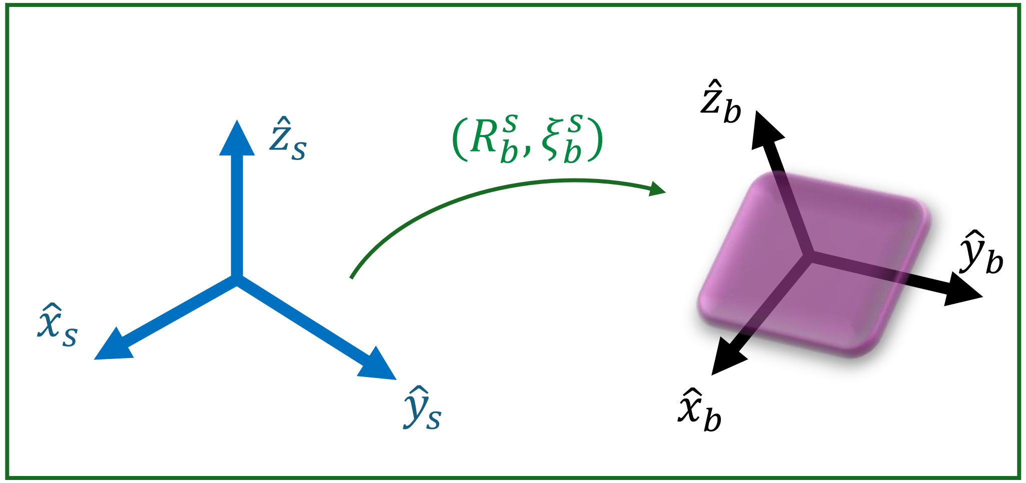
$\mathcal{J}(P^{b,b}): \mathbb{R}^6 \to \mathbb{R}^6$

Provided in the homework

Or rewritten as

2

- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$
- $\dot{P}^{b,b} = \mathcal{W}_{\text{tot}}^{s,b} + \mathcal{J}(P^{b,b})\mathcal{V}_b^{b,s}$
- $\mathcal{V}_b^{b,s} = (\mathfrak{I}^{b,b})^{-1} P^{b,b}$



Wrenches on moving rigid body

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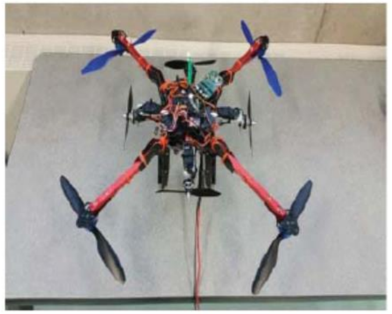
Multi-rotor aerial vehicles

- Multi-rotor aerial vehicles (MAVs) are the most popular choice of aerial robotics platform.
- Usually, they have a simple mechanical structure and few moving parts.
- MAVs are usually modeled as a single rigid body floating in space.





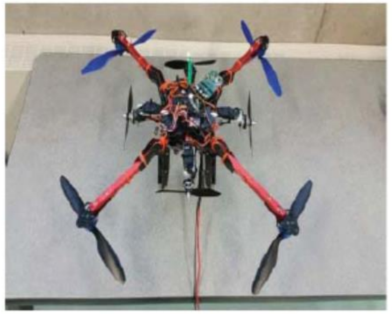



Multi-rotor aerial vehicles

- MAVs are classified based on properties of the map between individual rotor thrusts λ_i and the resultant wrench applied on the MAV's body which is used for control.

$$\boxed{\mathcal{W}_{\text{con}}^{b,b} = \begin{pmatrix} \tau_{\text{con}}^{b,b} \\ f_{\text{con}}^{b,b} \end{pmatrix} = M(t)\lambda}$$








$\lambda = \left(\lambda_1, \cdots, \lambda_{N_p}\right) \in \mathbb{R}^{N_p}$ is the rotors thrust vector and N_p is the number of propellers on the MAV.