SCE 594: Special Topics in Intelligent Automation & Robotics

Topic 3: Fixed-base manipulator modeling

Lecture 11: Forward Kinematics

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Outline

- Motivation
- Exponential coordinates & Exponential map
- Product of Exponentials (PoE) formula
- Examples

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Motivation

- In Topic 3, we will be dealing with the kinematic & dynamic modeling fixed-base open chain manipulators.

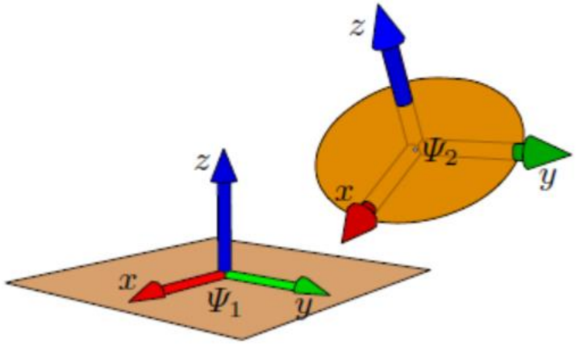
Fixed-base open chain

Fixed-base closed chain

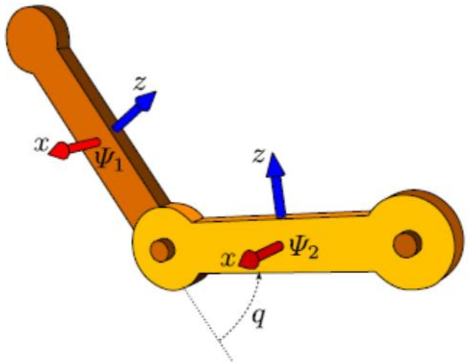
Floating-base open chain

Motivation

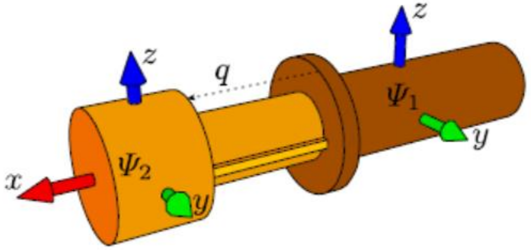
- Majority of robotic systems can be modeled as several rigid bodies connected by ideal joints.
- An ideal joint (aka kinematic pair) is a purely kinematic relation between two rigid bodies restricting the relative twist $\mathcal{V}_1^{*,2}$.
- The degrees of freedom (DoF) of a joint is the number of independent coordinates of $\mathcal{V}_1^{*,2}$.



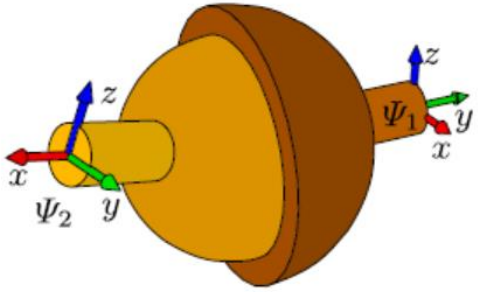
Free motion (6 DoF)



Revolute joint (1 DoF)



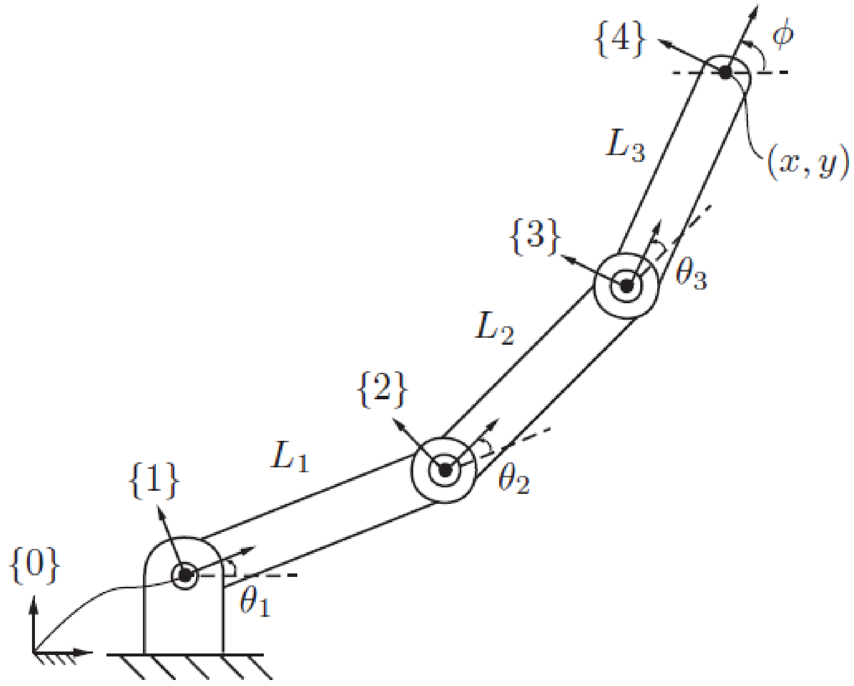
Prismatic joint (1 DoF)



Spherical joint (3 DoF)

Forward Kinematics

- The forward kinematics of a robot refers to the calculation of the position and orientation of its end-effector frame from its joint coordinates.
- Consider, the 3R planar open chain below.

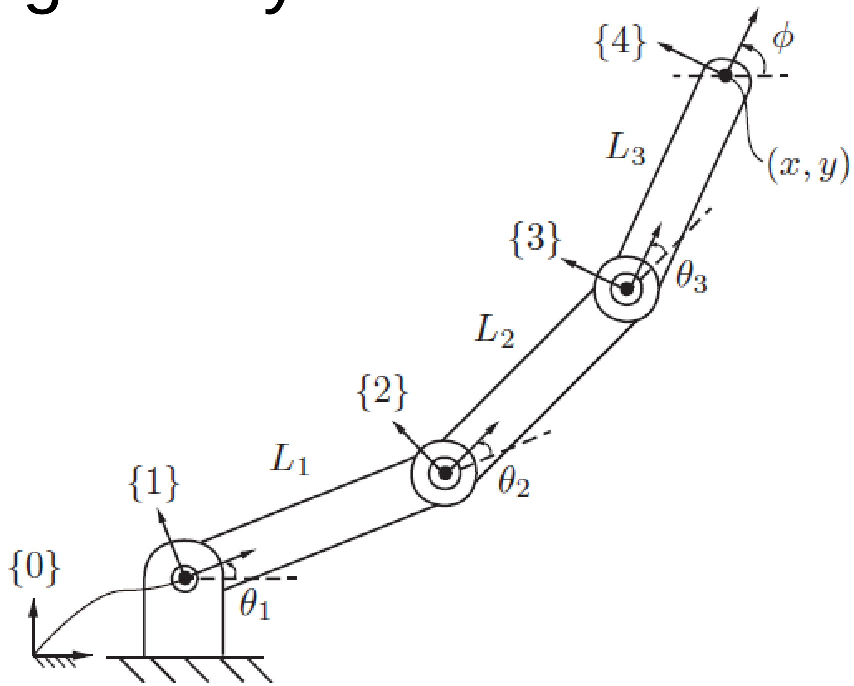


Forward Kinematics

- The forward kinematics of a robot refers to the calculation of the position and orientation of its end-effector frame from its joint coordinates.
- Consider, the 3R planar open chain below.
- The pose of the end-effector's frame $\{4\}$ is given by:

$$H_4^0 = \begin{pmatrix} c_\phi & -s_\phi & 0 & x \\ s_\phi & c_\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) + L_3 \cos(\theta_1+\theta_2+\theta_3), \\ y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) + L_3 \sin(\theta_1+\theta_2+\theta_3), \\ \phi &= \theta_1+\theta_2+\theta_3. \end{aligned}$$



Forward Kinematics

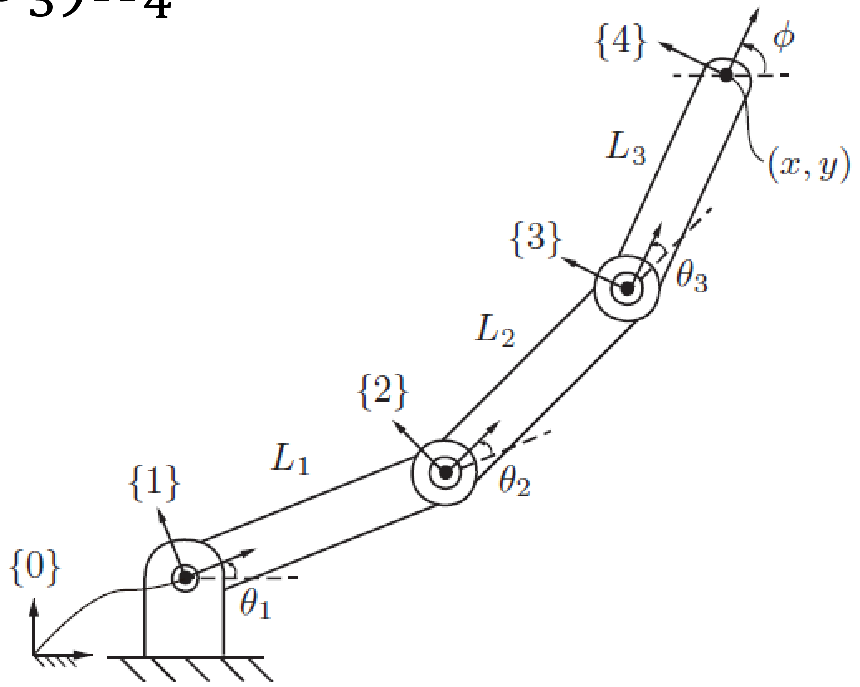
- A systematic way to derive the forward kinematics map

$$(\theta_1, \theta_2, \theta_3) \mapsto H_4^0$$

is to attach a reference frame to each link $\{1\}, \{2\}, \{3\}$.

- Then we have that:

$$H_4^0 = H_1^0(\theta_1) H_2^1(\theta_2) H_3^2(\theta_3) H_4^3$$



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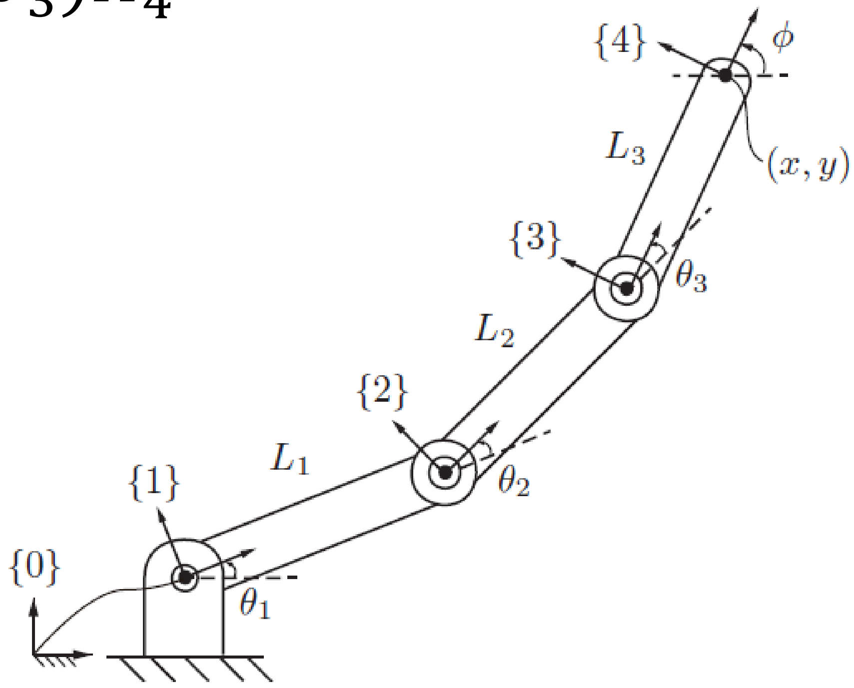


A widely used convention for choosing thee body-fixed frames $\{1\}, \{2\}, \{3\}$ is based on the so-called Denavit–Hartenberg parameters (D–H parameters). We will use a more powerful approach based on the screw-theory interpretation of twists.

Outline

- Motivation
- Exponential coordinates & Exponential map
- Product of Exponentials (PoE) formula
- Examples

Recall linear ODE theory

- Consider the scalar linear ordinary differential equation (ODE):

$$\dot{x}(t) = a\, x(t),$$

where $x(t) \in \mathbb{R}$ and $a \in \mathbb{R}$ is a constant.

- The solution to the ODE above is given by

$$x(t) = e^{at} x(0),$$

where the exponential function is defined as the series:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

Recall linear ODE theory

- Similarly, consider the vector linear ODE:

$$\dot{x}(t) = A\,x(t),$$

where $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is constant.

- The solution to the ODE above is given by

$$x(t) = e^{At} x(0),$$

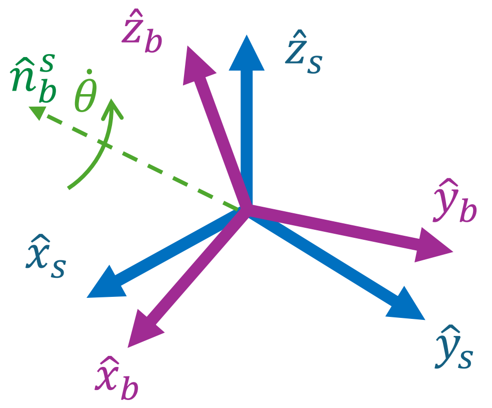
where the matrix exponential function is defined as the series:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

Exponential coordinates of Rotations

- Suppose that a frame $\{\mathrm{b}\}$ with unit axes $\{\hat{x}_b, \hat{y}_b, \hat{z}_b\}$ is attached to a rotating body.
- The angular velocity vector describing the instantaneous rotation of $\{\mathrm{b}\}$ relative to $\{\mathrm{s}\}$ is given by $\omega_b^s := \hat{n}_b^s \dot{\theta} \in \mathbb{R}^3$.
- The instantaneous rate of change of $R_b^s \in SO(3)$ is given by

$$\boxed{\dot{R}_b^s = \widetilde{\omega}_b^{s,s} R_b^s}$$



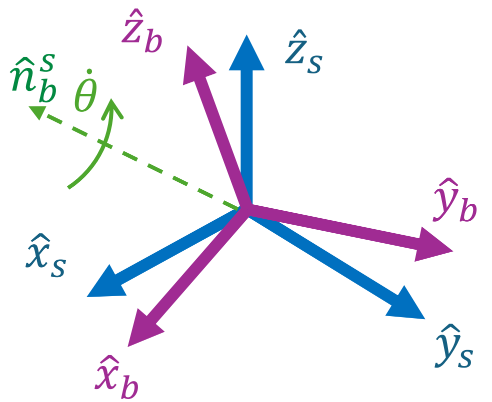
Exponential coordinates of Rotations

- Assume that $\omega_b^{s,s} = \hat{n}_b^{s,s}$ is a constant unit vector with $\dot{\theta} = 1$ and thus $\widetilde{\omega}_b^{s,s}$ is also constant.
- The solution to $\dot{R}_b^s = \widetilde{\omega}_b^{s,s} R_b^s$ then becomes

$$\boxed{R_b^s(t) = e^{\tilde{n}_b^{s,s} t} R_b^s(0)}$$

$$\tilde{n}_b^{s,s} \coloneqq S(\hat{n}_b^{s,s}) \in so(3)$$

- Intuitively, this is the rotation matrix reached from $R_b^s(0)$ by rotating it around $\hat{\omega}$ at a constant 1 rad/s for t seconds.



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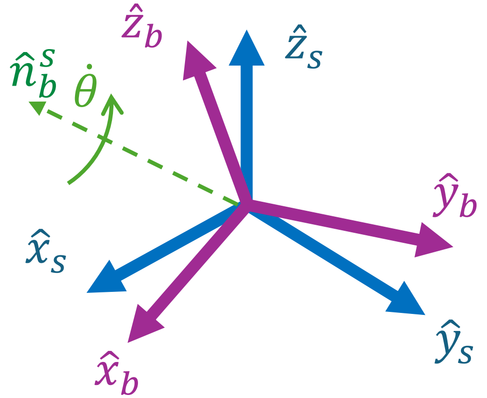
Note that since $\dot{\theta} = 1$, we have that

$$\theta(t) = \int_0^t \dot{\theta}\ dt = \int_0^t dt = t$$

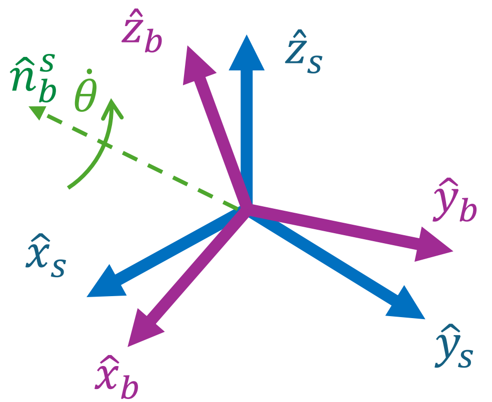


Exponential coordinates of Rotations

- By replacing t with θ, we have that

$$\boxed{R_b^s(\theta) = e^{\tilde{n}_b^{s,s}\theta} R_b^s(0)}$$

which is intuitively the rotation matrix achieved by rotating $R_b^s(0)$ around the axis $\hat{n}_b^{s,s}$ by an angle θ.



The pair $(\hat{n}_b^s, \theta) \in \mathbb{S}^2 \times [-\pi, \pi]$ is called the exponential coordinates or axis-angle representation of a rotation matrix $R \in SO(3)$

Exponential map of SO(3)

- The exponential map

$$\exp: so(3) \rightarrow SO(3)$$
$$\tilde{n}_b^{s,s} \mapsto e^{\tilde{n}_b^{s,s}}$$

maps elements of the Lie algebra to elements of the Lie group.

- It is surjective, but not injective
  - $\forall\, R \in SO(3),\ \ \exists \tilde{n} \in so(3)\ \text{s.t.}\ R = \mathrm{e}^{\tilde{n}}$
  - $\exists \hat{n}_1, \hat{n}_2 \in \mathbb{R}^3$ with $\hat{n}_1 \neq \hat{n}_2$ s.t. $\mathrm{e}^{\tilde{n}_1} = \mathrm{e}^{\tilde{n}_2}$

Exponential map of SO(3)

- The exponential map

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$$\tilde{n}_b^{s,s} \mapsto e^{\tilde{n}_b^{s,s}}$$

maps elements of the Lie algebra to elements of the Lie group.

Given any pair $(\hat{n}, \theta) \in \mathbb{S}^2 \times [-\pi, \pi]$, the matrix exponential $e^{\tilde{n}\theta} \in SO(3)$ of $\tilde{n}\theta \in so(3)$, with $\tilde{n} \coloneqq S(\hat{n})$, is given by:

$$e^{\tilde{n}\theta} = I_3 + \sin\theta\, \tilde{n} + (1 - \cos\theta)\tilde{n}^2$$

Rodrigues's formula

For the proof, see Pg. 83-84 of Lynch & Park

Elementary rotations

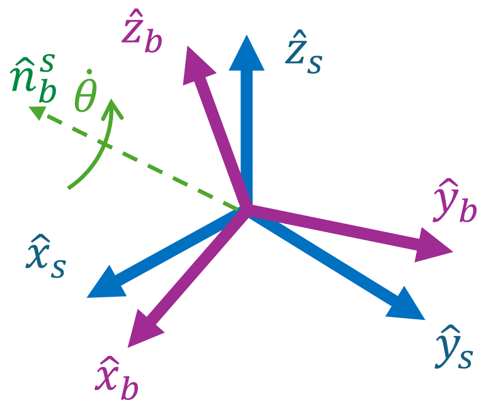
- Rotations about the coordinate frame axes are given by:

$$\mathrm{Rot}(\hat{\mathrm{x}},\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\mathrm{y}},\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\mathrm{z}},\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

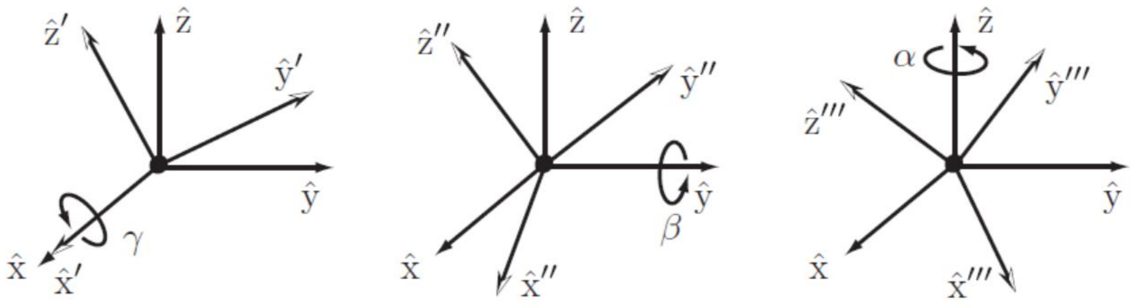
$\boxed{\mathrm{Rot}(\hat{n},\theta) = e^{\tilde{n}\theta} \in SO(3)}$



Roll-Pitch-Yaw Angles

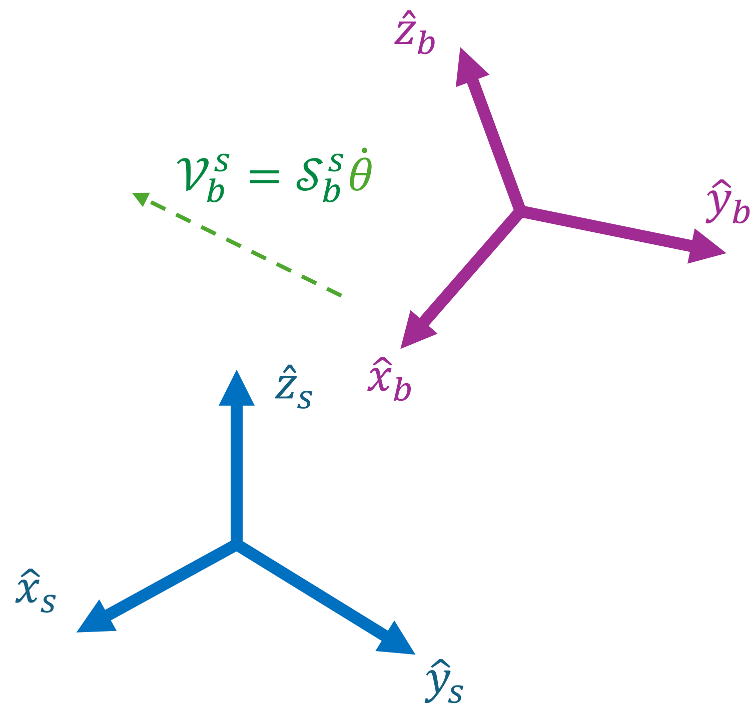
- The roll–pitch–yaw angles refer to the angles in a sequence of rotations about axes fixed in the space frame

$$
\begin{aligned}
R(\alpha,\beta,\gamma) &= \mathrm{Rot}(\hat{\mathrm{z}},\alpha)\mathrm{Rot}(\hat{\mathrm{y}},\beta)\mathrm{Rot}(\hat{\mathrm{x}},\gamma)I \\
&= \begin{bmatrix} \mathrm{c}_\alpha & -\mathrm{s}_\alpha & 0 \\ \mathrm{s}_\alpha & \mathrm{c}_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \mathrm{c}_\beta & 0 & \mathrm{s}_\beta \\ 0 & 1 & 0 \\ -\mathrm{s}_\beta & 0 & \mathrm{c}_\beta \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}_\gamma & -\mathrm{s}_\gamma \\ 0 & \mathrm{s}_\gamma & \mathrm{c}_\gamma \end{bmatrix} I \\
&= \begin{bmatrix} \mathrm{c}_\alpha\mathrm{c}_\beta & \mathrm{c}_\alpha\mathrm{s}_\beta\mathrm{s}_\gamma - \mathrm{s}_\alpha\mathrm{c}_\gamma & \mathrm{c}_\alpha\mathrm{s}_\beta\mathrm{c}_\gamma + \mathrm{s}_\alpha\mathrm{s}_\gamma \\ \mathrm{s}_\alpha\mathrm{c}_\beta & \mathrm{s}_\alpha\mathrm{s}_\beta\mathrm{s}_\gamma + \mathrm{c}_\alpha\mathrm{c}_\gamma & \mathrm{s}_\alpha\mathrm{s}_\beta\mathrm{c}_\gamma - \mathrm{c}_\alpha\mathrm{s}_\gamma \\ -\mathrm{s}_\beta & \mathrm{c}_\beta\mathrm{s}_\gamma & \mathrm{c}_\beta\mathrm{c}_\gamma \end{bmatrix}.
\end{aligned}
$$



Screw interpretation of Twists

- Just as an angular velocity ω_b^s can be viewed as $\hat{n}_b^s\dot{\theta} \in \mathbb{R}^3$ where $\hat{n}_b^s$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about it,
- A twist $\mathcal{V}_b^s$ can interpreted in terms of a screw axis $\mathcal{S}_b^s \in \mathbb{R}^6$ and a velocity $\dot{\theta}$ about the screw axis



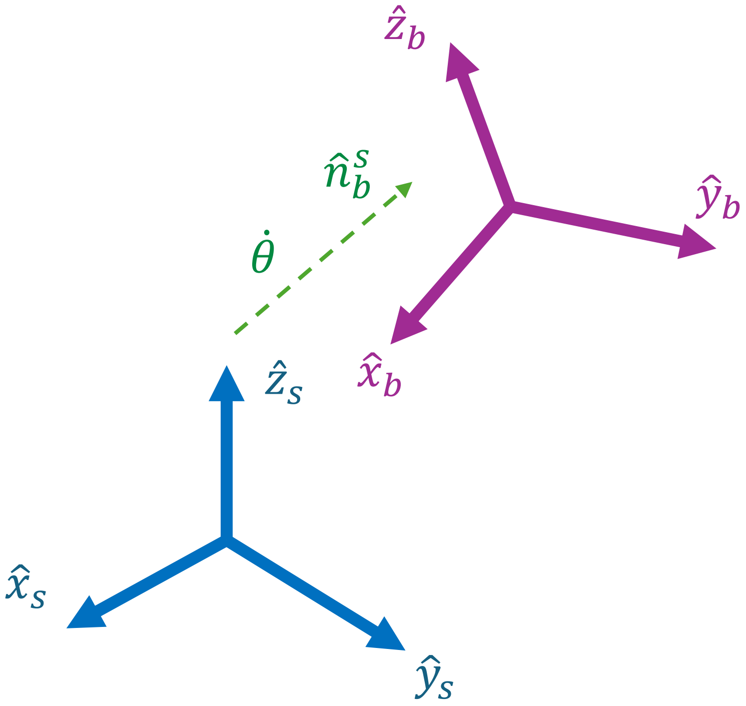
Screw interpretation of Twists

- We define the screw axis $\mathcal{S}_b^s$ and $\dot{\theta}$ in the following manner:
    a) Pure translation

$$\mathcal{S}_b^s = \begin{pmatrix} 0 \\ \hat{n}_b^s \end{pmatrix} \in \mathbb{R}^6,$$

$\hat{n}_b^s$ is the translation axis

$\dot{\theta}$ is the linear velocity along the screw axis



Screw interpretation of Twists

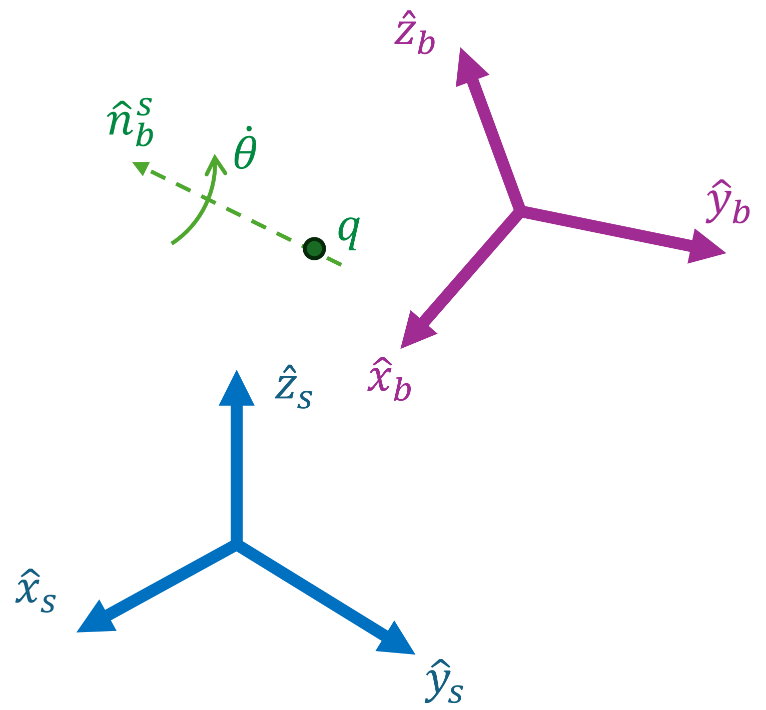
- We define the screw axis $\mathcal{S}_b^s$ and $\dot{\theta}$ in the following manner:
  b) Pure rotation

$$\mathcal{S}_b^s = \begin{pmatrix} \hat{n}_b^s \\ -\hat{n}_b^s \wedge q \end{pmatrix} \in \mathbb{R}^6,$$

$\hat{n}_b^s$ is the rotation axis

$\dot{\theta}$ is the angular velocity around the screw axis

q is any point on the screw axis



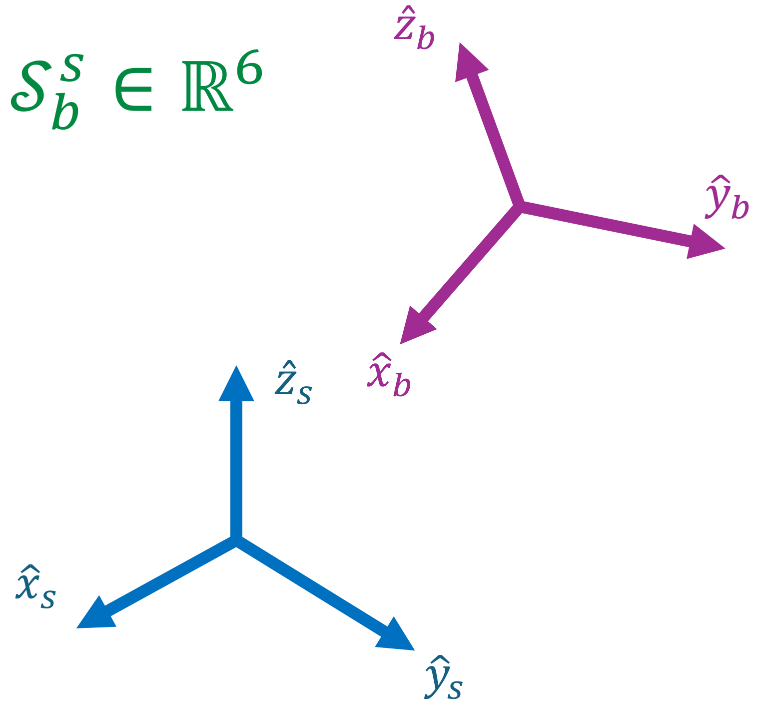
For a third type (screw motion), see Pg. 102-04 of Lynch & Park

Exp. coordinates of homogeneous transformations

- By analogy to the exponential coordinates $(\hat{n}_b^s, \theta)$ for rotations, we can define the six-dimensional exponential coordinates of a homogeneous transformation H_b^s by $\mathcal{S}_b^s\theta \in \mathbb{R}^6$ such that

$$\boxed{H_b^s(\theta) = e^{\tilde{\mathcal{S}}_b^{s,s}\theta} H_b^s(0)}$$

where $\tilde{\mathcal{S}}_b^{s,s} \in se(3)$ is the matrix representation of $\mathcal{S}_b^s \in \mathbb{R}^6$



Exponential map of SE(3)

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