SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 12: Velocity Kinematics

جامعة الملك فهد للبترول والمعادن
King Fahd University of Petroleum & Minerals

Ramy Rashad
Assistant Professor
CIE Department

Outline

- Recap last lecture
- Examples
- Geometric Jacobian
- Examples

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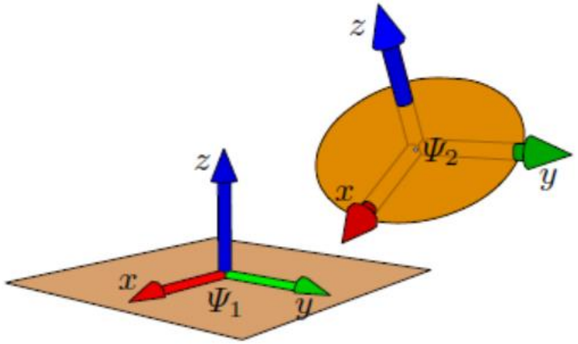
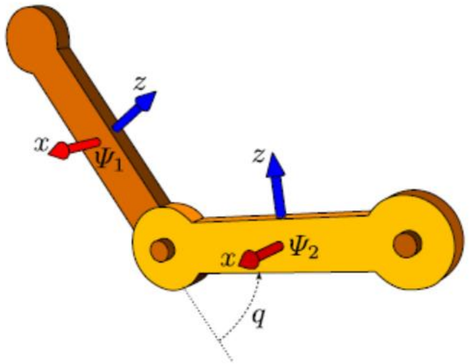
Recap: Motivation

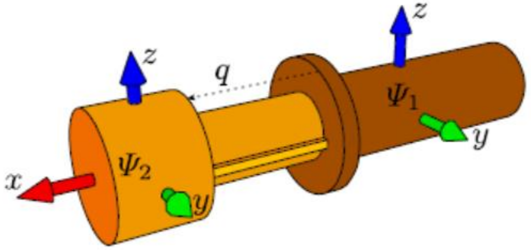
- An ideal joint (aka kinematic pair) is a purely kinematic relation between two rigid bodies restricting the relative twist $\mathcal{V}_1^{*,2}$.
- The degrees of freedom (DoF) of a joint is the number of independent coordinates of $\mathcal{V}_1^{*,2}$.



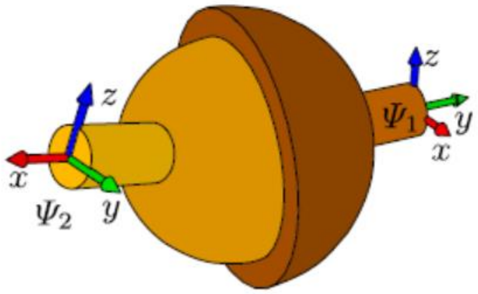
Free motion (6 DoF)



Revolute joint (1 DoF)



Prismatic joint (1 DoF)



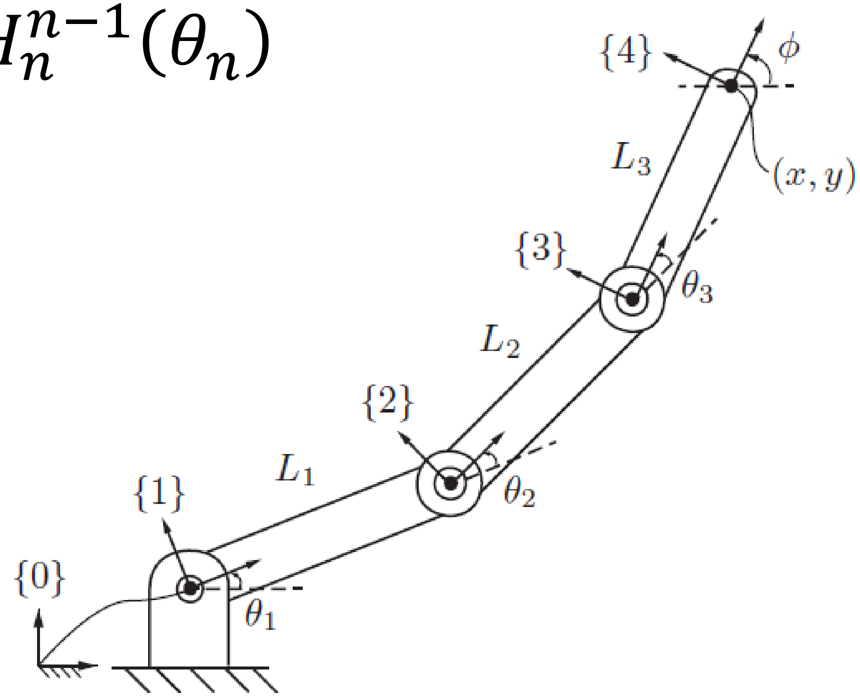
Spherical joint (3 DoF)

Recap: Forward Kinematics

- The forward kinematics of a robot refers to the calculation of the position and orientation of its end-effector frame from its joint coordinates.

$$\theta \coloneqq (\theta_1, \theta_2, \dots, \theta_n) \mapsto H_n^0$$

$$H_n^0(\theta) = H_1^0(\theta_1) H_2^1(\theta_2) \dots H_n^{n-1}(\theta_n)$$



Recap: Axis-angle representation of Angular velocity

- Angular velocity

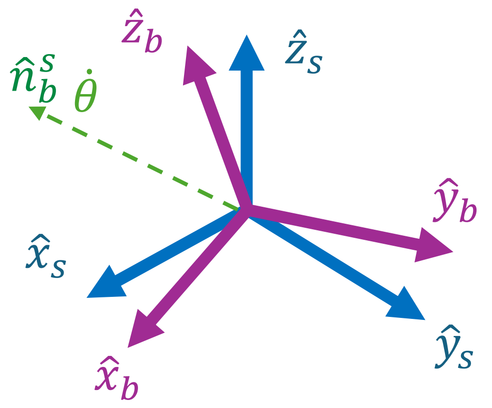
$$\omega_i^{*,j} = \hat{n}_i^{*,j}\dot{\theta}_i \in \mathbb{R}^3$$

- Rotation matrix

$$R_i^j(\theta_i) = e^{\tilde{n}_i^{j,j}\theta_i} R_i^j(0) \in SO(3)$$

- Exponential map

$$\exp: so(3) \to SO(3)$$
$$\tilde{n}_i^{j,j} \mapsto e^{\tilde{n}_i^{j,j}}$$



Rodrigues's formula

$$\boxed{e^{\tilde{n}\theta} = I_3 + \sin\theta\,\tilde{n} + (1-\cos\theta)\tilde{n}^2}$$

Recap: Screw representation of Twist

- Twist

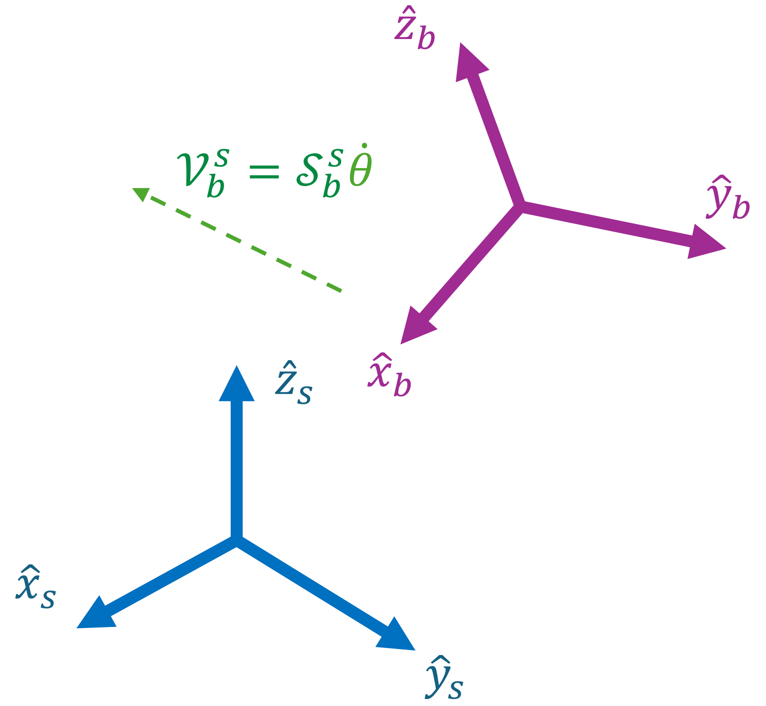
$$\mathcal{V}_i^{*,j} = \mathcal{S}_i^{*,j} \dot{\theta}_i \in \mathbb{R}^6$$

- Homogeneous transformation

$$H_i^j(\theta_i) = e^{\tilde{\mathcal{S}}_i^{j,j} \theta_i} H_i^j(0) \in SE(3)$$

- Exponential map

$$\exp: se(3) \rightarrow SE(3)$$
$$\tilde{\mathcal{S}}_i^{j,j} \mapsto e^{\tilde{\mathcal{S}}_i^{j,j}}$$



Recap: Modeling Ideal Joints

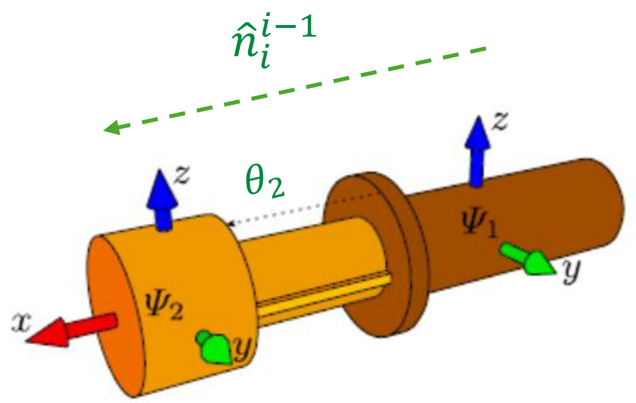
- Joint twist

$$\mathcal{V}_i^{*,i-1} = \mathcal{S}_i^{*,i-1}\dot{\theta}_i \in \mathbb{R}^6$$

- Joint relations

$$H_i^{i-1}(\theta_i) = e^{\tilde{\mathcal{S}}_i^{i-1,i-1}\theta_i} H_i^{i-1}(0) \in SE(3)$$

- Prismatic joints
  - $\mathcal{S}_i^{*,i-1} = \begin{pmatrix} 0 \\ \hat{n}_i^{*,i-1} \end{pmatrix} \in \mathbb{R}^6,$
  - $\hat{n}_i^{*,i-1}$ is the translation axis
  - $\dot{\theta}$ is the linear velocity along the screw axis



Recap: Modeling Ideal Joints

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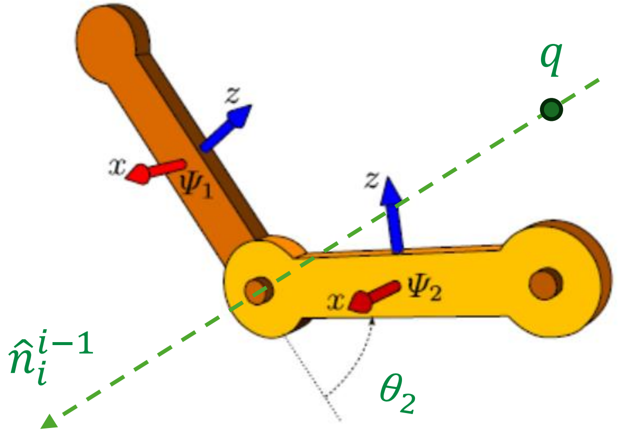
- Joint relations

$$H_i^{i-1}(\theta_i) = e^{\tilde{\mathcal{S}}_i^{i-1,i-1}\theta_i} H_i^{i-1}(0) \in SE(3)$$

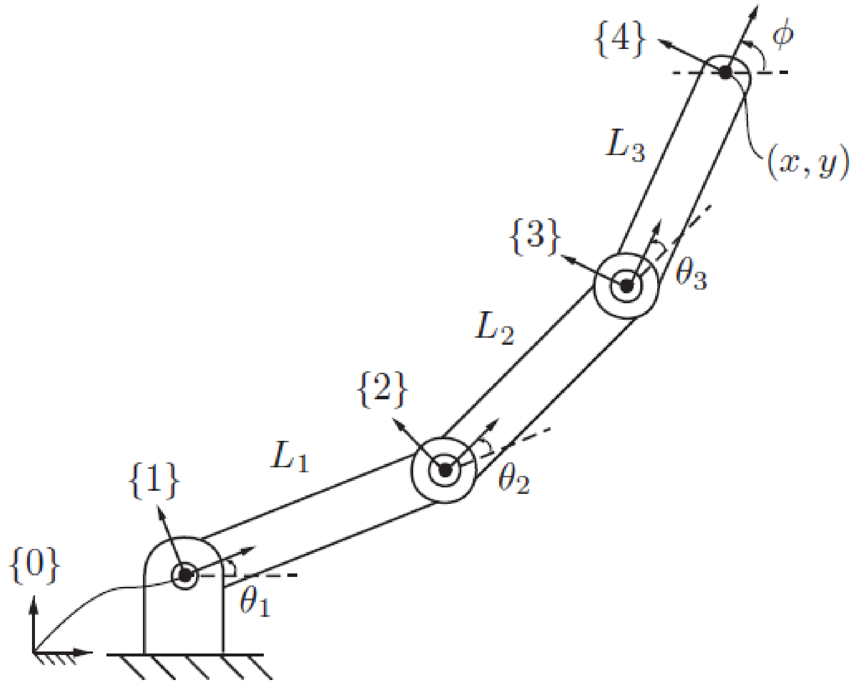
- Revolute joints
  - $\mathcal{S}_i^{*,i-1} = \begin{pmatrix} \hat{n}_i^{*,i-1} \\ -\hat{n}_i^{*,i-1} \wedge q^* \end{pmatrix} \in \mathbb{R}^6,$
  - $\hat{n}_i^{i-1}$ is the rotation axis
  - $\dot{\theta}$ is the angular velocity around the screw axis
  - q is any point on the screw axis



Recap: Product of Exponentials Formula

- We can then compute the forward kinematics using the Product of Exponentials (PoE) formula given by

$$\boxed{H_n^0(\theta) = e^{\tilde{S}_1^{0,0}\theta_1} e^{\tilde{S}_2^{0,1}\theta_2} \dots e^{\tilde{S}_n^{0,n-1}\theta_n} H_n^0(0)}$$



Note that we can use the same formula to compute the pose of the k-th link $H_k^0(\theta)$ not just the end effector !!

Exponential map of SE(3)

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \qquad (3.88)$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}. \qquad (3.89)$$

Note: $[\omega] \coloneqq \tilde{\omega} \in so(3)$ is the notation used in the book.

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Example 1

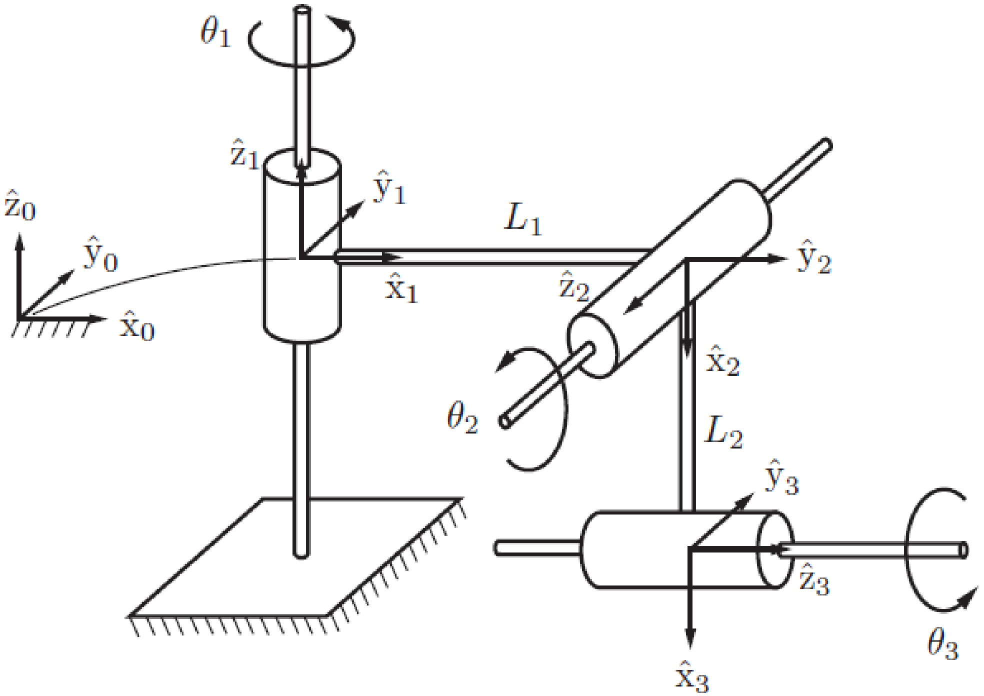

Figure 4.3: A 3R spatial open chain.

Example 2

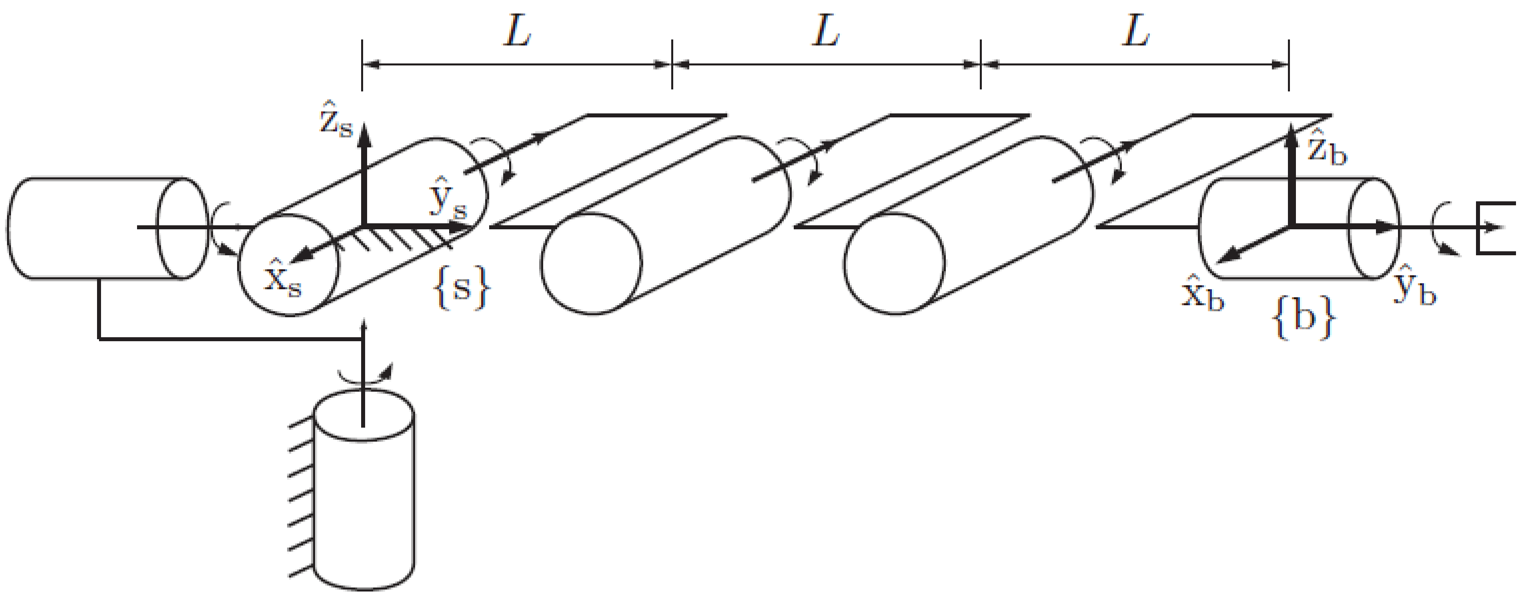

Figure 4.4: PoE forward kinematics for the 6R open chain.

Example 3

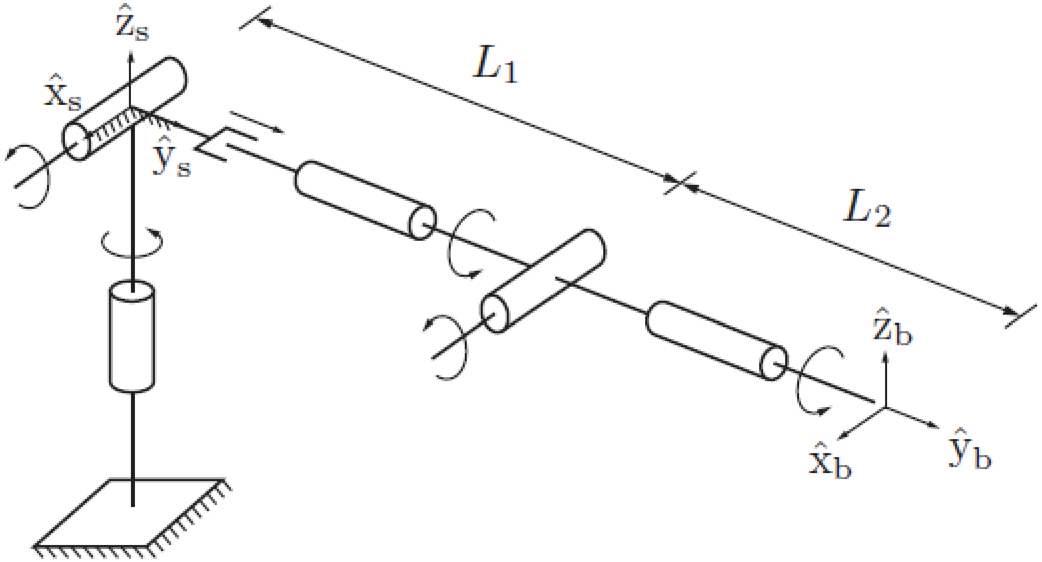

Figure 4.5: The RRPRRR spatial open chain.

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Velocity Kinematics

- The forward kinematics defines a map from the joint angles $\theta \in Q$ to the end effector pose $H_n^0 \in SE(3)$.
- The velocity kinematics is the differential of this map which is a mapping from $\dot{\theta} \in T_\theta Q$ to $\dot{H}_n^0 \in T_{H_n^0} SE(3)$.

Q is known as the joint space which is an n-dimensional manifold

Velocity Kinematics

- The forward kinematics defines a map from the joint angles $\theta \in Q$ to the end effector pose $H_n^0 \in SE(3)$.
- The velocity kinematics is the differential of this map which is a mapping from $\dot{\theta} \in T_\theta Q$ to $\dot{H}_n^0 \in T_{H_n^0} SE(3)$.
- Representing the velocity kinematics as a map from $\dot{\theta} \in T_\theta Q \cong \mathbb{R}^6$ to the end effector's twist $\mathcal{V}_n^{*,0} \in \mathbb{R}^6$ defines the geometric Jacobian

$$\boxed{\mathcal{V}_n^{0,0} = J_0(\theta)\dot{\theta}}$$

Spatial Jacobian

$$\boxed{\mathcal{V}_n^{n,0} = J_n(\theta)\dot{\theta}}$$

Body Jacobian

Q is known as the joint space which is an n-dimensional manifold

$J_*(\theta) \in \mathbb{R}^{6\times n}$

End effector spatial twist

- Recall that $H_n^0 = H_1^0 H_2^1 \dots H_n^{n-1}, \quad (AB)^{-1} = B^{-1}A^{-1}, \quad \left(H_i^k\right)^{-1} = H_k^i$

$$\tilde{\mathcal{V}}_n^{0,0} \coloneqq \dot{H}_n^0 H_0^n = \frac{d}{dt}(H_1^0 H_2^1 \dots H_n^{n-1})(H_1^0 H_2^1 \dots H_n^{n-1})^{-1}$$

$$= \frac{d}{dt}(H_1^0 H_2^1 \dots H_n^{n-1}) H_{n-1}^n \dots H_1^2 H_0^1$$

End effector spatial twist

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$$= \dot{H}_1^0 H_2^1 \dots H_n^{n-1} H_{n-1}^n \dots H_1^2 H_0^1 + H_1^0\ \dot{H}_2^1 \dots H_n^{n-1} H_{n-1}^n \dots H_1^2\ H_0^1 + \cdots$$

$$+ H_1^0\ H_2^1 \dots \dot{H}_n^{n-1} H_{n-1}^n \dots H_1^2\ H_0^1$$

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$$= \frac{d}{dt}(H_1^0 H_2^1 \dots H_n^{n-1}) H_{n-1}^n \dots H_1^2 H_0^1$$

$$= \dot{H}_1^0 {\color{red} H_2^1 \dots H_n^{n-1} H_{n-1}^n \dots H_1^2} H_0^1 + H_1^0 \; \dot{H}_2^1 \dots {\color{red} H_n^{n-1} H_{n-1}^n} \dots H_1^2 \; H_0^1 + \cdots$$

$$+ H_1^0 \; H_2^1 \dots \dot{H}_n^{n-1} H_{n-1}^n \dots H_1^2 \; H_0^1$$

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$$= \frac{d}{dt}(H_1^0 H_2^1 \dots H_n^{n-1}) H_{n-1}^n \dots H_1^2 H_0^1$$
$$= \dot{H}_1^0 \; H_0^1 + H_1^0 \dot{H}_2^1 H_1^2 H_0^1 + \cdots + H_{n-1}^0 \dot{H}_n^{n-1} H_{n-1}^n H_0^{n-1}$$

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$$= \dot{H}_1^0 \, H_0^1 + H_1^0 \dot{H}_2^1 H_1^2 H_0^1 + \cdots + H_{n-1}^0 \dot{H}_n^{n-1} H_{n-1}^n H_0^{n-1}$$

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$$
\begin{aligned}
\tilde{\mathcal{V}}_n^{0,0} \coloneqq \dot{H}_n^0 H_0^n &= \frac{d}{dt}(H_1^0 H_2^1 \dots H_n^{n-1})(H_1^0 H_2^1 \dots H_n^{n-1})^{-1} \\
&= \frac{d}{dt}(H_1^0 H_2^1 \dots H_n^{n-1}) H_{n-1}^n \dots H_1^2 H_0^1 \\
&= \dot{H}_1^0\, H_0^1 + H_1^0 \dot{H}_2^1 H_1^2 H_0^1 + \cdots + H_{n-1}^0 \dot{H}_n^{n-1} H_{n-1}^n H_0^{n-1} \\
&= \tilde{\mathcal{V}}_1^{0,0} + H_1^0\, \tilde{\mathcal{V}}_2^{1,1}\, H_0^1 + \cdots + H_{n-1}^0 \tilde{\mathcal{V}}_n^{n-1,n-1} H_0^{n-1}
\end{aligned}
$$

End effector spatial twist

- Recall that $H_n^0 = H_1^0 H_2^1 \dots H_n^{n-1}, \quad (AB)^{-1} = B^{-1}A^{-1}, \quad \left(H_i^k\right)^{-1} = H_k^i$

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&= \dot{H}_1^0\ H_0^1 + H_1^0 \dot{H}_2^1 H_1^2 H_0^1 + \dots + H_{n-1}^0 \dot{H}_n^{n-1} H_{n-1}^n H_0^{n-1} \\
&= \tilde{\mathcal{V}}_1^{0,0} + H_1^0\ \tilde{\mathcal{V}}_2^{1,1}\ H_0^1 + \dots + H_{n-1}^0 \tilde{\mathcal{V}}_n^{n-1,n-1} H_0^{n-1} \\
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$$= \tilde{\mathcal{V}}_1^{0,0} + H_1^0 \, \tilde{\mathcal{V}}_2^{1,1} \, H_0^1 + \cdots + H_{n-1}^0 \tilde{\mathcal{V}}_n^{n-1,n-1} H_0^{n-1}$$

$$= \tilde{\mathcal{V}}_1^{0,0} + \tilde{\mathcal{V}}_2^{0,1} + \cdots + \tilde{\mathcal{V}}_n^{0,n-1}$$

Twists sum like scalars !!!

$$\boxed{\begin{aligned} \mathcal{V}_n^{0,0} &= \mathcal{V}_1^{0,0} + \mathcal{V}_2^{0,1} + \cdots + \mathcal{V}_n^{0,n-1} \\ &= \mathcal{V}_1^{0,0} + Ad_{H_1^0(\theta)} \mathcal{V}_2^{1,1} + \cdots + Ad_{H_{n-1}^0(\theta)} \mathcal{V}_n^{n-1,n-1} \end{aligned}}$$

Spatial Jacobian

- Recall that an ideal joint is defined by $\mathcal{V}_i^{*,i-1} = \mathcal{S}_i^{*,i-1}\dot{\theta}_i$
- Therefore,

$$\mathcal{V}_n^{0,0} = \mathcal{V}_1^{0,0} + Ad_{H_1^0(\theta)}\mathcal{V}_2^{1,1} + \cdots + Ad_{H_{n-1}^0(\theta)}\mathcal{V}_n^{n-1,n-1}$$

$$= \mathcal{S}_1^{0,0}\dot{\theta}_1 + Ad_{H_1^0(\theta)}\mathcal{S}_2^{1,1}\dot{\theta}_2 + \cdots + Ad_{H_{n-1}^0(\theta)}\mathcal{S}_n^{n-1,n-1}\dot{\theta}_n$$

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$$= \mathcal{S}_1^{0,0}\dot{\theta}_1 + Ad_{H_1^0(\theta)}\mathcal{S}_2^{1,1}\dot{\theta}_2 + \cdots + Ad_{H_{n-1}^0(\theta)}\mathcal{S}_n^{n-1,n-1}\dot{\theta}_n$$

$$= \mathcal{S}_1^{0,0}\dot{\theta}_1 + \mathcal{S}_2^{0,1}(\theta)\dot{\theta}_2 + \cdots + \mathcal{S}_n^{0,n-1}(\theta)\dot{\theta}_n$$

$$= \left(\underset{\text{1st column}}{\mathcal{S}_1^{0,0}}\;,\; \underset{\text{2nd column}}{\mathcal{S}_2^{0,1}(\theta)}\;, \cdots ,\underset{n\text{-th column}}{\mathcal{S}_n^{0,n-1}(\theta)}\right)\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \vdots\\ \dot{\theta}_n\end{pmatrix} = J_0(\theta)\dot{\theta}$$

The Jacobian expressed in the stationary frame or simply the spatial Jacobian

Summary

- The forward kinematics of an n-link open chain manipulator is expressed by

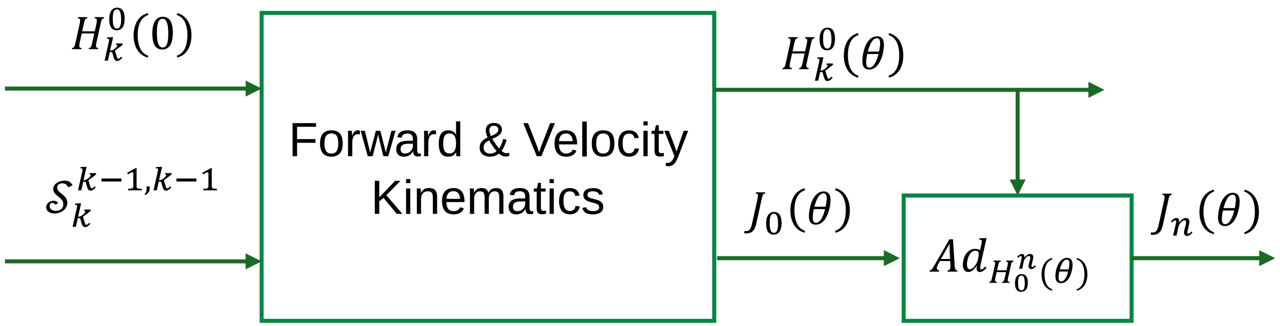
$$\boxed{H_n^0(\theta) = e^{\tilde{\mathcal{S}}_1^{0,0}\theta_1} e^{\tilde{\mathcal{S}}_2^{0,1}\theta_2} \ldots e^{\tilde{\mathcal{S}}_n^{0,n-1}\theta_n} H_n^0(0)}$$

- The spatial Jacobian $J_0(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rates $\dot{\theta} \in \mathbb{R}^n$ to the spatial end effector's twist $\mathcal{V}_n^{0,0} \in \mathbb{R}^6$.
- The k-th column of $J_0(\theta)$ is given by

$$\boxed{\mathcal{S}_k^{0,k-1}(\theta) = \mathrm{Ad}_{H_{k-1}^0(\theta)} \mathcal{S}_k^{k-1,k-1}}$$

Summary

- The above approach is highly systematic and can be easily programmable.
- The only inputs needed are the initial poses $H_k^0(0)$ and constant screw axes $\mathcal{S}_k^{k-1,k-1}$ for all links and joints.
- Note that we can compute $\mathcal{S}_k^{0,k-1} = \mathrm{Ad}_{H_{k-1}^0(0)} \mathcal{S}_k^{k-1,k-1}$ to be used in the PoE formula.
- Since each column of $J_0(\theta)$ is a twist expressed in $\{0\}$, we can compute the body Jacobian $J_n(\theta)$ simply by the Adjoint transformation $Ad_{H_0^n(\theta)}$.



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Example 1

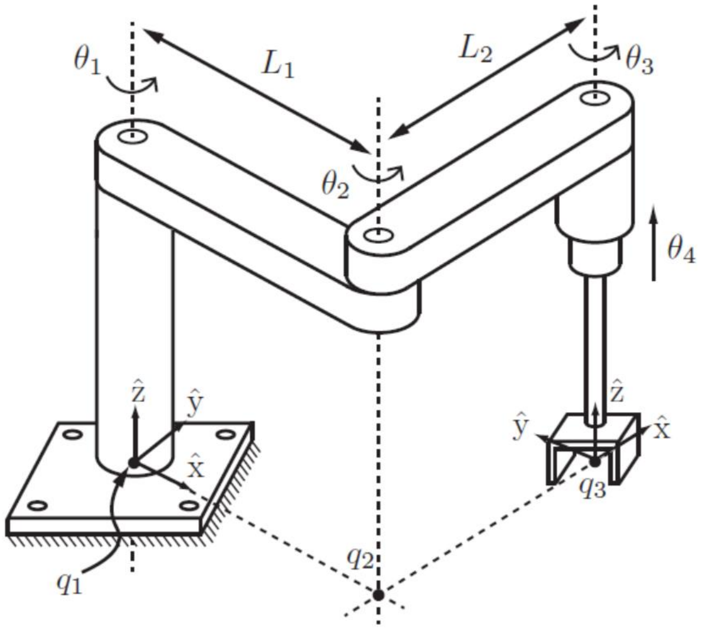

Figure 5.7: Space Jacobian for a spatial RRRP chain.

Example 2

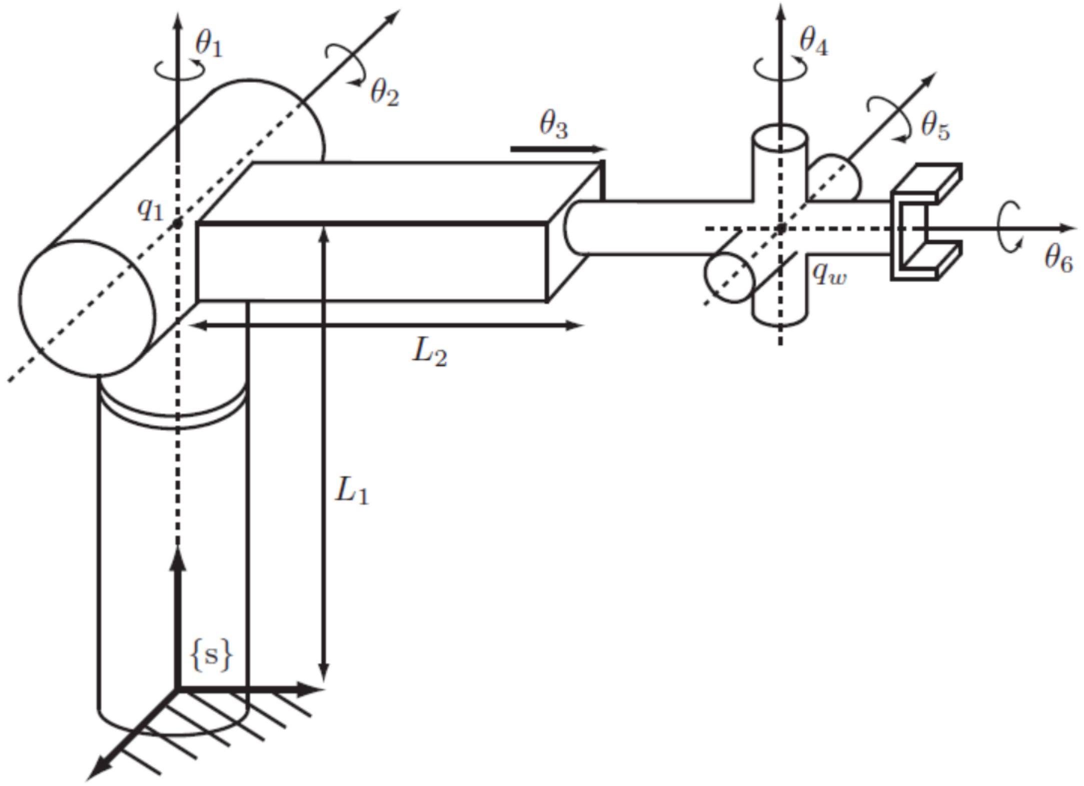

Figure 5.8: Space Jacobian for the spatial RRPRRR chain.