SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 15: Dynamics of Fixed-base Manipulators III

جامعة الملك فهد للبترول والمعادن
King Fahd University of Petroleum & Minerals

Ramy Rashad
Assistant Professor
CIE Department

Outline

- Recap last lecture
- General n-link manipulator dynamics
- Modeling External Wrenches

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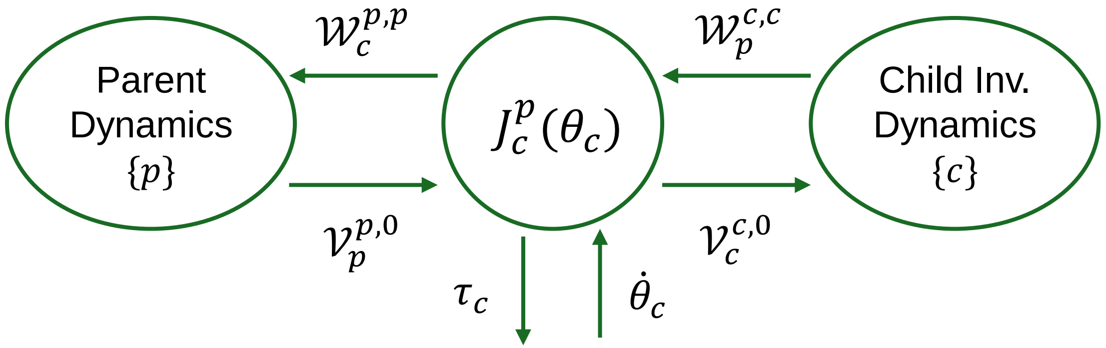
Recap: Parent-Child Dynamic Model

- Relative pose $$H_c^p(\theta_c) = H_c^p(0)e^{\tilde{S}_c^{c,p}\theta_c}$$

$$\boxed{\begin{pmatrix} \mathcal{W}_c^{p,p} \\ \mathcal{V}_c^{c,0} \\ \tau_c \end{pmatrix} = J_c^p(\theta_c)\begin{pmatrix} \mathcal{V}_p^{p,0} \\ \mathcal{W}_p^{c,c} \\ \dot{\theta}_c \end{pmatrix}}$$

- Joint Transformation $$J_c^p(\theta_c) = \begin{pmatrix} 0 & -\mathrm{Ad}^{\top}_{H_p^c(\theta_c)} & 0 \\ \mathrm{Ad}_{H_p^c(\theta_c)} & 0 & S_c^{c,p} \\ 0 & \left(S_c^{c,p}\right)^{\top} & 0 \end{pmatrix}$$



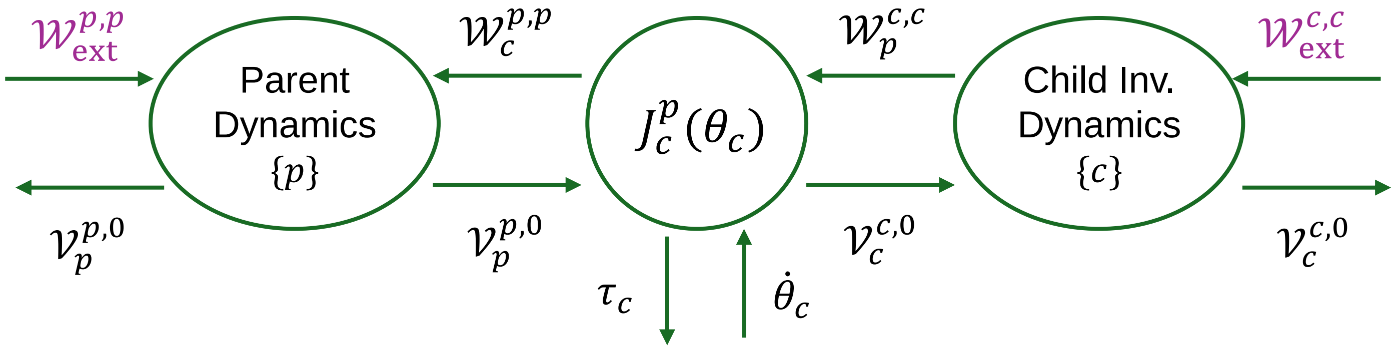
Recap: Parent-Child Dynamic Model + External wrench

- Parent Dynamics

$$\mathfrak{I}^{p,p}\dot{\mathcal{V}}_p^{p,0} = \mathrm{ad}^{\top}_{\mathcal{V}_p^{p,0}}\left(\mathfrak{I}^{p,p}\mathcal{V}_p^{p,0}\right) + \mathcal{W}_c^{p,p} + \mathcal{W}_{\mathrm{ext}}^{p,p}$$

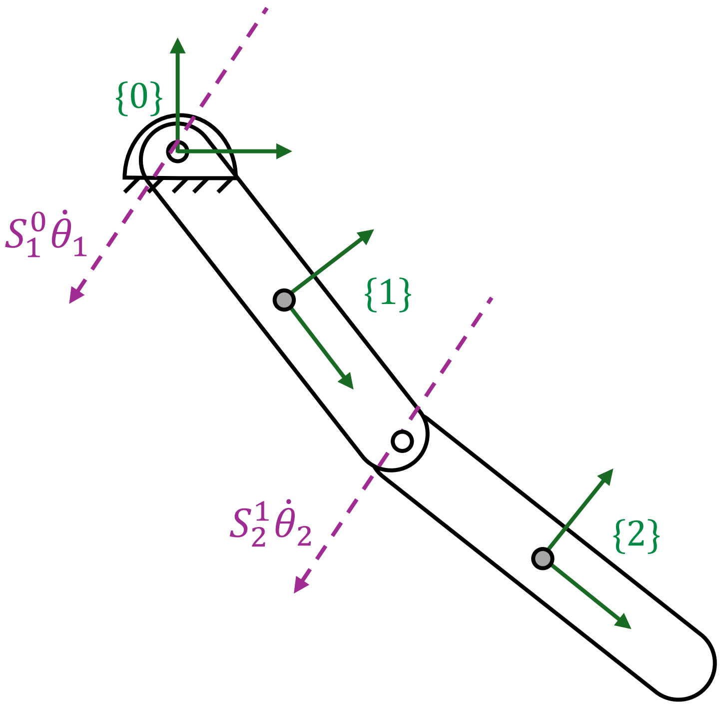
- Child Inverse Dynamics

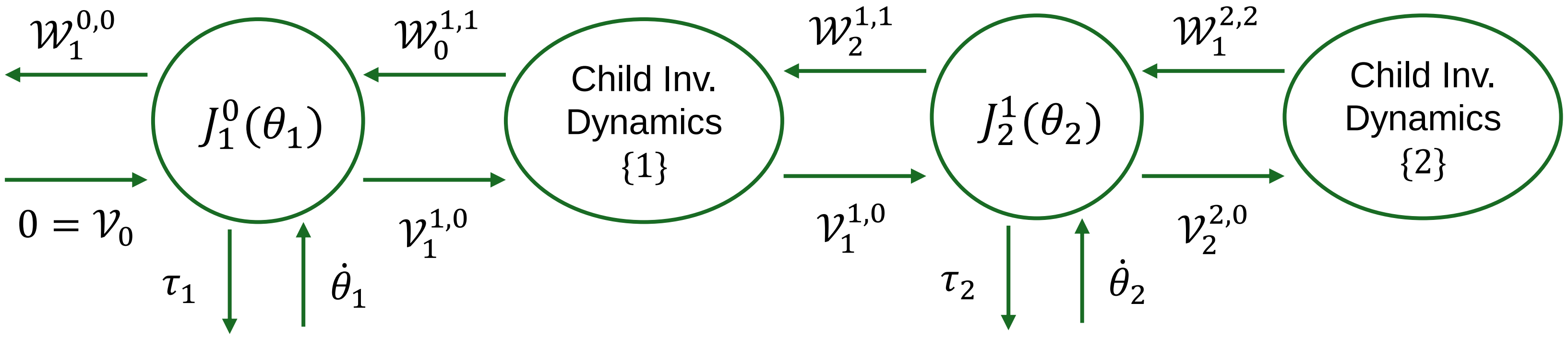
$$\mathcal{W}_p^{c,c} = \mathfrak{I}^{c,c}\dot{\mathcal{V}}_c^{c,0} - \mathrm{ad}^{\top}_{\mathcal{V}_c^{c,0}}\left(\mathfrak{I}^{c,c}\mathcal{V}_c^{c,0}\right) - \mathcal{W}_{\mathrm{ext}}^{c,c}$$



Recap: Two-link manipulator

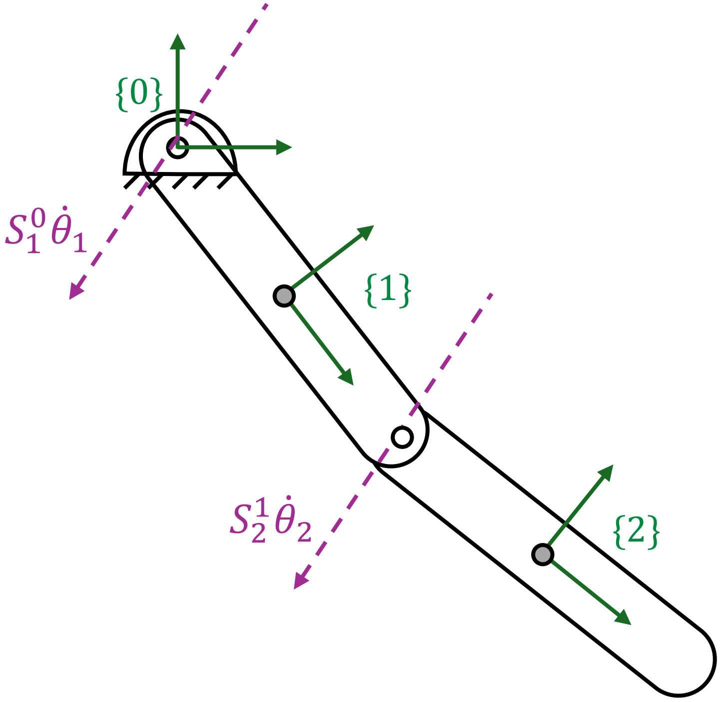
- Newton-Euler Inverse Dynamics Algorithm
  - Forward iterations
    - $H_i^{i-1}(\theta_i) = H_i^{i-1}(0)e^{\tilde{S}_i^{i,i-1}\theta_i}$
    - $\mathcal{V}_i^{i,0} = \mathrm{Ad}_{H_{i-1}^{i}(\theta_i)}\mathcal{V}_{i-1}^{i-1,0} + S_i^{i,i-1}\dot{\theta}_i$
    - $\dot{\mathcal{V}}_i^{i,0} = \mathrm{Ad}_{H_{i-1}^{i}(\theta_i)}\mathcal{V}_{i-1}^{i-1,0} - \mathrm{ad}_{S_i^{i,i-1}\dot{\theta}_i}\left(\mathrm{Ad}_{H_{i-1}^{i}(\theta_i)}\mathcal{V}_{i-1}^{i-1,0}\right) + S_i^{i,i-1}\ddot{\theta}_i$

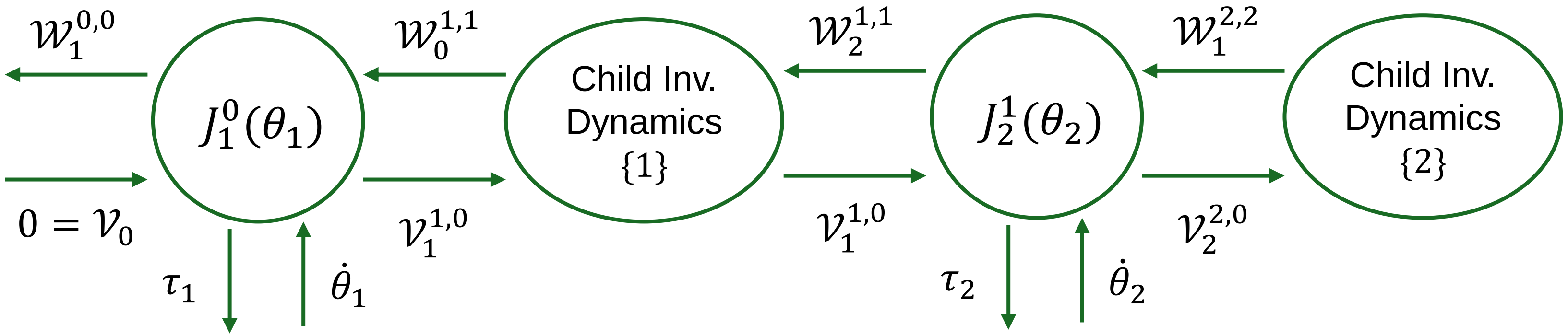




Recap: Two-link manipulator

- Newton-Euler Inverse Dynamics Algorithm
  - Backward iterations
    - $\mathcal{W}_{i-1}^{i,i} = \mathfrak{I}^{i,i}\dot{\mathcal{V}}_i^{i,0} - \mathrm{ad}^{\top}_{\mathcal{V}_i^{i,0}}\left(\mathfrak{I}^{i,i}\mathcal{V}_i^{i,0}\right) + \mathrm{Ad}^{\top}_{H_i^{i+1}(\theta_{i+1})}\mathcal{W}_i^{i+1,i+1}$
    - $\tau_i = \left(S_i^{i,i-1}\right)^{\top}\mathcal{W}_{i-1}^{i,i}$

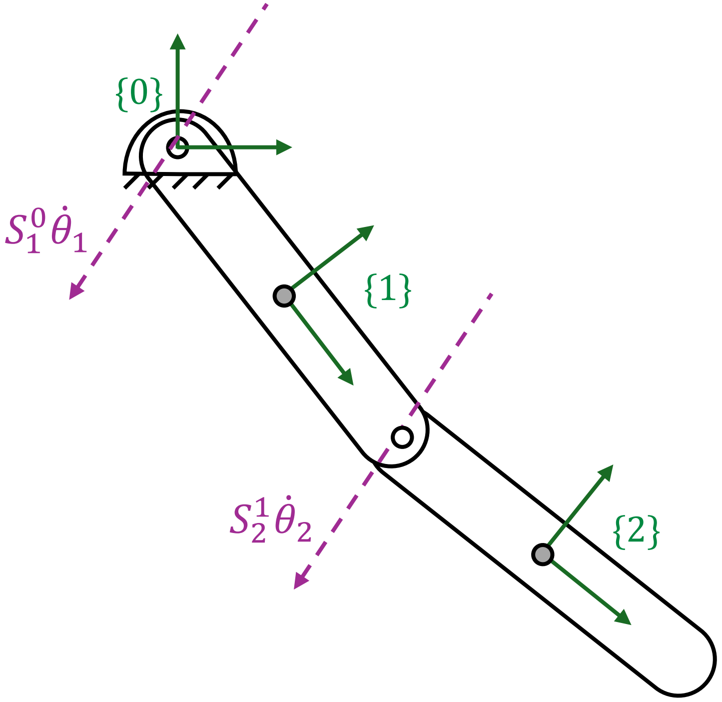


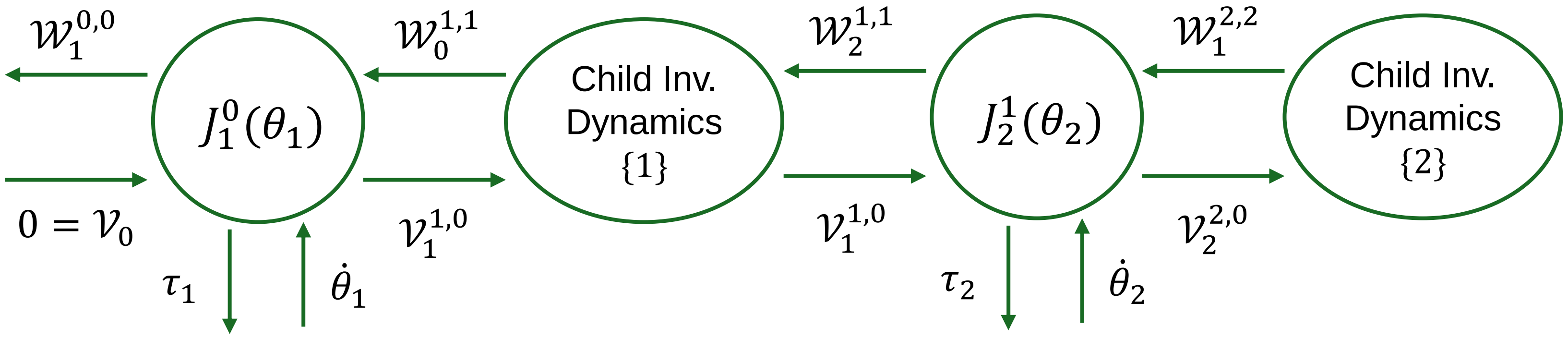


Recap: Two-link manipulator

- Reduced Manipulator Dynamics

$$\boxed{\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix} = M(\theta)\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} + C(\theta,\dot{\theta})\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}}$$





Outline

- Recap last lecture
- General n-link manipulator dynamics
- Modeling External Wrenches

General n-link manipulator dynamics

- In general, the recursive inverse dynamics algorithm can be assembled into the following set of matrix equations:
  - $\mathcal{V} = \mathcal{L}(\theta)\, \mathcal{S}\, \dot{\theta}$
  - $\dot{\mathcal{V}} = \mathcal{L}(\theta)\, \mathcal{S}\, \ddot{\theta} - \mathcal{L}(\theta)\, \mathrm{ad}_{\mathcal{S}\,\dot{\theta}}\, \mathcal{A}(\theta)\, \mathcal{V}$
  - $\mathcal{W} = \mathcal{L}^{\top}(\theta)\, \mathfrak{I}\, \dot{\mathcal{V}} - \mathcal{L}^{\top}(\theta)\, \mathrm{ad}_{\mathcal{V}}^{\top}\, \mathfrak{I}\, \mathcal{V}$
  - $\tau = \mathcal{S}^{\top} \mathcal{W}$

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_1^{1,0} \\ \vdots \\ \mathcal{V}_n^{n,0} \end{pmatrix} \in \mathbb{R}^{6n}, \qquad \dot{\mathcal{V}} = \begin{pmatrix} \dot{\mathcal{V}}_1^{1,0} \\ \vdots \\ \dot{\mathcal{V}}_n^{n,0} \end{pmatrix} \in \mathbb{R}^{6n}, \qquad \mathcal{W} = \begin{pmatrix} \mathcal{W}_0^{1,1} \\ \vdots \\ \mathcal{W}_{n-1}^{n,n} \end{pmatrix} \in (\mathbb{R}^{6n})^*$$

$$\dot{\theta} = \begin{pmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{pmatrix} \in \mathbb{R}^{n}, \qquad \ddot{\theta} = \begin{pmatrix} \ddot{\theta}_1 \\ \vdots \\ \ddot{\theta}_n \end{pmatrix} \in \mathbb{R}^{n}, \qquad \tau = \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_n \end{pmatrix} \in (\mathbb{R}^{n})^*$$

General n-link manipulator dynamics

- In general, the recursive inverse dynamics algorithm can be assembled into the following set of matrix equations:
  - $\mathcal{V} = \mathcal{L}(\theta)\, \mathcal{S}\, \dot{\theta}$
  - $\dot{\mathcal{V}} = \mathcal{L}(\theta)\, \mathcal{S}\, \ddot{\theta} - \mathcal{L}(\theta)\, \mathrm{ad}_{\mathcal{S}\, \dot{\theta}}\, \mathcal{A}(\theta)\, \mathcal{V}$
  - $\mathcal{W} = \mathcal{L}^{\top}(\theta)\, \mathfrak{T}\, \dot{\mathcal{V}} - \mathcal{L}^{\top}(\theta)\, \mathrm{ad}_{\mathcal{V}}^{\top}\, \mathfrak{T}\, \mathcal{V}$
  - $\tau = \mathcal{S}^{\top} \mathcal{W}$

$$\mathcal{S} = \begin{pmatrix} S_1^{1,0} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & S_n^{n,n-1} \end{pmatrix} \in \mathbb{R}^{6n \times n}, \qquad \mathfrak{T} = \begin{pmatrix} \mathfrak{T}^{1,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathfrak{T}^{n,n} \end{pmatrix} \in \mathbb{R}^{6n \times 6n}$$

$$\mathrm{ad}_{\mathcal{S}\, \dot{\theta}} = \begin{pmatrix} \mathrm{ad}_{S_1^{1,0} \dot{\theta}_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathrm{ad}_{S_n^{n,n-1} \dot{\theta}_n} \end{pmatrix} \in \mathbb{R}^{6n \times 6n}, \qquad \mathrm{ad}_{\mathcal{V}} = \begin{pmatrix} \mathrm{ad}_{\mathcal{V}_1^{1,0}} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathrm{ad}_{\mathcal{V}_n^{n,0}} \end{pmatrix} \in \mathbb{R}^{6n \times 6n}$$

General n-link manipulator dynamics

- In general, the recursive inverse dynamics algorithm can be assembled into the following set of matrix equations:
  - $\mathcal{V} = \mathcal{L}(\theta)\,\mathcal{S}\,\dot{\theta}$
  - $\dot{\mathcal{V}} = \mathcal{L}(\theta)\,\mathcal{S}\,\ddot{\theta} - \mathcal{L}(\theta)\,\mathrm{ad}_{\mathcal{S}\,\dot{\theta}}\,\mathcal{A}(\theta)\,\mathcal{V}$
  - $\mathcal{W} = \mathcal{L}^\top(\theta)\,\mathfrak{I}\,\dot{\mathcal{V}} - \mathcal{L}^\top(\theta)\,\mathrm{ad}_{\mathcal{V}}^\top\,\mathfrak{I}\,\mathcal{V}$
  - $\tau = \mathcal{S}^\top \mathcal{W}$

$$\mathcal{A}(\theta) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathrm{Ad}_{H_1^2(\theta_2)} & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{Ad}_{H_2^3(\theta_3)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{Ad}_{H_{n-1}^n(\theta_n)} & 0 \end{pmatrix} \in \mathbb{R}^{6n\times 6n}, \qquad \mathcal{L}(\theta) = \begin{pmatrix} I & 0 & \cdots & \cdots & 0 \\ \mathrm{Ad}_{H_1^2(\theta_2)} & I & \cdots & \cdots & 0 \\ \mathrm{Ad}_{H_1^3(\theta)} & \mathrm{Ad}_{H_2^3(\theta_3)} & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathrm{Ad}_{H_1^n(\theta)} & \mathrm{Ad}_{H_2^n(\theta)} & \mathrm{Ad}_{H_3^n(\theta)} & \cdots & I \end{pmatrix} \in \mathbb{R}^{6n\times 6n}$$

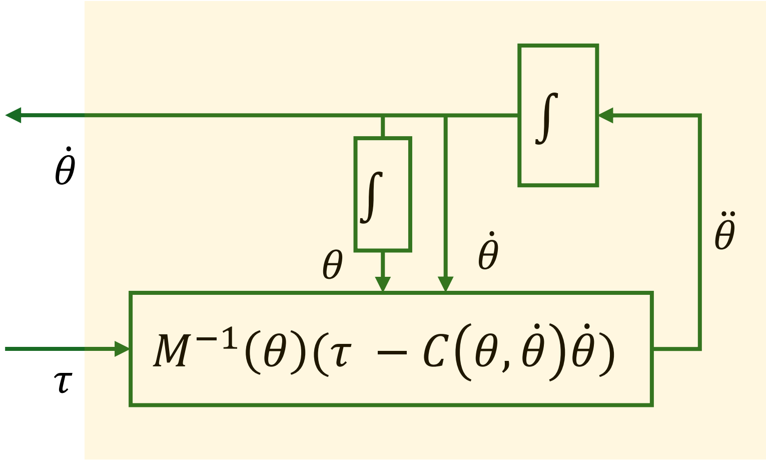
Recall $\mathcal{L}(\theta) \coloneqq [I - \mathcal{A}(\theta)]^{-1}$

General n-link manipulator dynamics

- Solving the above equations together yields finally:

$$\boxed{\tau = M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta}}$$

$$M(\theta) := \mathcal{S}^\top \mathcal{L}^\top(\theta)\, \mathfrak{I}\, \mathcal{L}(\theta)\, \mathcal{S} \quad \in \mathbb{R}^{n\times n}$$
$$C\big(\theta,\dot{\theta}\big) := \mathcal{S}^\top \mathcal{L}^\top(\theta)\, [-\mathfrak{I}\mathcal{L}(\theta)\mathrm{ad}_{\mathcal{S}\,\dot{\theta}}\, \mathcal{A}(\theta) \; - \mathrm{ad}^\top_{\mathcal{L}(\theta)\,\mathcal{S}\,\dot{\theta}}\mathfrak{I}]\, \mathcal{L}(\theta)\, \mathcal{S} \quad \in \mathbb{R}^{n\times n}$$



MATLAB Implementation

- Forward Kinematics:
  - $H_i^{i-1}(\theta_i) = H_i^{i-1}(0)e^{\tilde{S}_i^{i,i-1}\theta_i}$

- Recursive Algorithm:
  - $\mathcal{V} = \mathcal{L}(\theta)\,\mathcal{S}\,\dot{\theta}$
  - $\dot{\mathcal{V}} = \mathcal{L}(\theta)\,\mathcal{S}\,\ddot{\theta} - \mathcal{L}(\theta)\,\mathrm{ad}_{\mathcal{S}\,\dot{\theta}}\,\mathcal{A}(\theta)\,\mathcal{V}$
  - $\mathcal{W} = \mathcal{L}^{\top}(\theta)\,\mathfrak{I}\,\dot{\mathcal{V}} - \mathcal{L}^{\top}(\theta)\,\mathrm{ad}_{\mathcal{V}}^{\top}\,\mathfrak{I}\,\mathcal{V}$
  - $\tau = \mathcal{S}^{\top}\mathcal{W}$

```
%% Forward Kinematics
syms th1 th2 real
S_1_0_1 = Adjoint(TransInv(H_0_1_home))*S_0_0_1;
S_2_1_2 = Adjoint(TransInv(H_1_2_home))*S_1_1_2;

H_0_1 = simplify(H_0_1_home*Exp_SE3(S_1_0_1,th1));
H_1_2 = simplify(H_1_2_home*Exp_SE3(S_2_1_2,th2));
```

```
%% Recursive Inverse Dynamics Algorithm
S = blkdiag(S_1_0_1,S_2_1_2);
I = blkdiag(I_1,I_2);

A_th = [zeros(6,6),zeros(6,6);Adjoint(H_2_1),zeros(6,6)];
L_th = [eye(6),zeros(6,6);Adjoint(H_2_1),eye(6)];

syms th1dt th2dt real
ad_S_thdt = blkdiag(adjoint_se3(S_1_0_1*th1dt),adjoint_se3(S_2_1_2*th2dt));

V = L_th*S*[th1dt;th2dt];
V_1_0_1 = V(1:6,:);
V_2_0_2 = V(7:12,:);
ad_V = blkdiag(adjoint_se3(V_1_0_1),adjoint_se3(V_2_0_2));
```

MATLAB Implementation

- Reduced Dynamics:
  - $\tau = M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta}$
- Input parameters:
  - $H_i^{i-1}(0)$
  - $S_i^{i,i-1}$
  - $\mathfrak{I}^{i,i}$

```
%% Reduced Forward Dynamics

M_th = S'*L_th' * I * L_th * S;
M_th = simplify(M_th)

C_th = S'*L_th' * (-I*L_th*ad_S_thdt*A_th - (ad_V)'*I) * L_th * S;
C_th = simplify(C_th)
```

```
%% Home configuration and Constant Transformations
syms L1 L2 real
R_0_1 = eye(3);
xi_0_1 = [L1;0;0];
H_0_1_home = RpToTrans(R_0_1,xi_0_1);
R_1_2 = eye(3);
xi_1_2 = [L2;0;0];
H_1_2_home = RpToTrans(R_1_2,xi_1_2);
```

```
%% Joint Parameters
S_0_0_1 = [0;0;1;0;0;0];
S_1_1_2 = [0;0;1;0;0;0];
```

```
%% Inertial Parameters
syms m1 m2
I_1 = [zeros(3,3) zeros(3,3);
    zeros(3,3) m1*eye(3)];
I_2 = [zeros(3,3) zeros(3,3);
    zeros(3,3) m2*eye(3)];
```

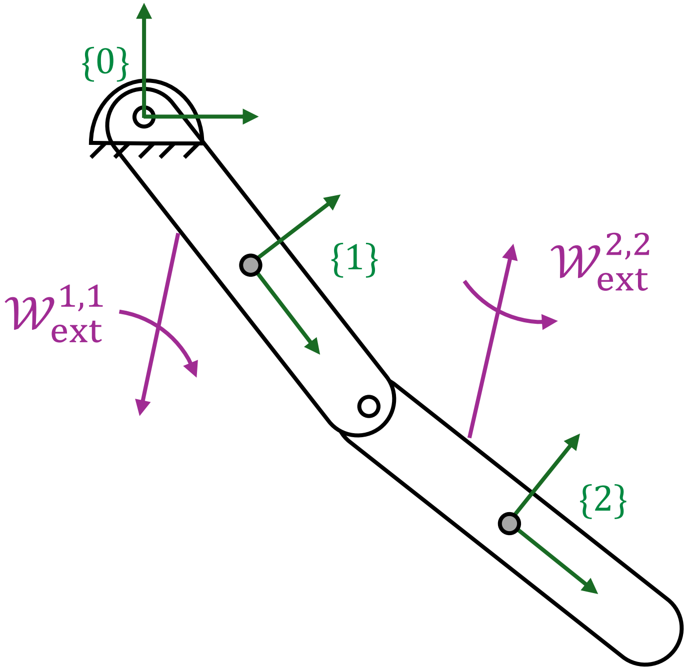
Outline

- Recap last lecture
- General n-link manipulator dynamics
- Modeling External Wrenches

Adding External Wrenches

- If we now consider the case that there is an external wrench $\mathcal{W}_{\text{ext}}^{i,i}$ applied to each link, then the dynamic equations would be:
  - $\mathcal{V} = \mathcal{L}(\theta)\, \mathcal{S}\, \dot{\theta}$
  - $\dot{\mathcal{V}} = \mathcal{L}(\theta)\, \mathcal{S}\, \ddot{\theta} - \mathcal{L}(\theta)\, \mathrm{ad}_{\mathcal{S}\,\dot{\theta}}\, \mathcal{A}(\theta)\, \mathcal{V}$
  - $\mathcal{W} = \mathcal{L}^\top(\theta)\, \mathfrak{I}\, \dot{\mathcal{V}} - \mathcal{L}^\top(\theta)\, \mathrm{ad}_{\mathcal{V}}^\top\, \mathfrak{I}\, \mathcal{V} - \mathcal{L}^\top(\theta) \mathcal{W}_{\text{ext}}$
  - $\tau = \mathcal{S}^\top \mathcal{W}$

where $\mathcal{W}_{\text{ext}} := \begin{pmatrix} \mathcal{W}_{\text{ext}}^{1,1} \\ \vdots \\ \mathcal{W}_{\text{ext}}^{n,n} \end{pmatrix} \in (\mathbb{R}^{6n})^*$.

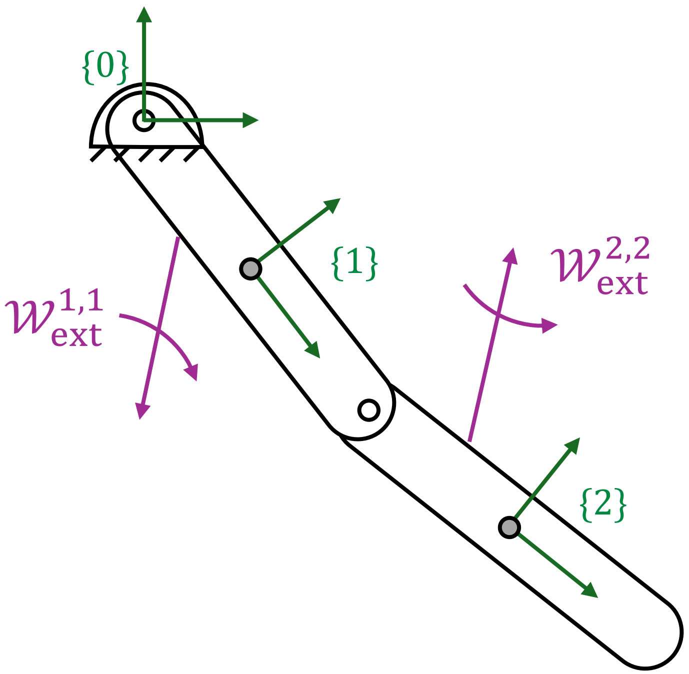


Note that $\mathcal{W}_{\text{ext}}^{i,i}$ should be expressed in the CoM frame of link i !

Adding External Wrenches

- The reduced dynamics then take the form
  - $\tau = M(\theta)\ddot{\theta} + C\big(\theta, \dot{\theta}\big)\dot{\theta} - \mathcal{J}^{\top}(\theta)\mathcal{W}_{\text{ext}}$
  - $\mathcal{V} = \mathcal{J}(\theta)\,\dot{\theta}$

where $\mathcal{J}(\theta) \coloneqq \mathcal{L}(\theta)\mathcal{S} \in \mathbb{R}^{6n\times n}$ and the second equation is an (extra) output equation.

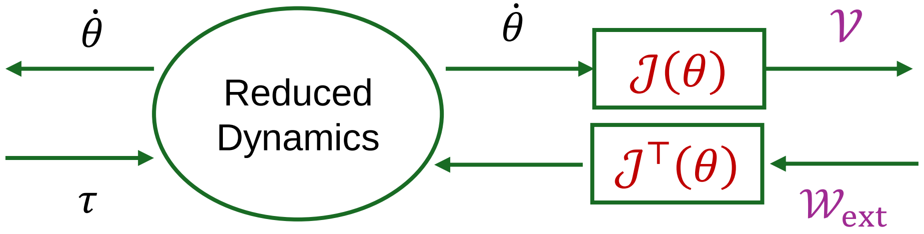


Geometric Jacobian

- If we examine carefully the mapping $\mathcal{J}(\theta)$ one can see:

$$\mathcal{J}(\theta)=\begin{pmatrix} S_1^{1,0} & 0 & \cdots & \cdots & 0 \\ \mathrm{Ad}_{H_1^2(\theta_2)}S_1^{1,0} & S_2^{2,1} & \cdots & \cdots & 0 \\ \mathrm{Ad}_{H_1^3(\theta)}S_1^{1,0} & \mathrm{Ad}_{H_2^3(\theta_3)}S_2^{2,1} & S_3^{3,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathrm{Ad}_{H_1^n(\theta)}S_1^{1,0} & \mathrm{Ad}_{H_2^n(\theta)}S_2^{2,1} & \mathrm{Ad}_{H_3^n(\theta)}S_3^{3,2} & \cdots & S_n^{n,n-1} \end{pmatrix}=\begin{pmatrix} S_1^{1,0} & 0 & \cdots & \cdots & 0 \\ S_1^{2,0}(\theta) & S_2^{2,1} & \cdots & \cdots & 0 \\ S_1^{3,0}(\theta) & S_2^{3,1}(\theta) & S_3^{3,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ S_1^{n,0}(\theta) & S_2^{n,1}(\theta) & S_3^{n,2}(\theta) & \cdots & S_n^{n,n-1} \end{pmatrix}=\begin{pmatrix} \mathcal{J}_1^{1,0} \\ \mathcal{J}_2^{2,0}(\theta) \\ \mathcal{J}_3^{3,0}(\theta) \\ \vdots \\ \mathcal{J}_n^{n,0}(\theta) \end{pmatrix}$$

- Which implies that $\mathcal{J}(\theta)$ is a collection of n geometric Jacobians, with its i-th block equivalent to the geometric body* Jacobian of the i-th link.

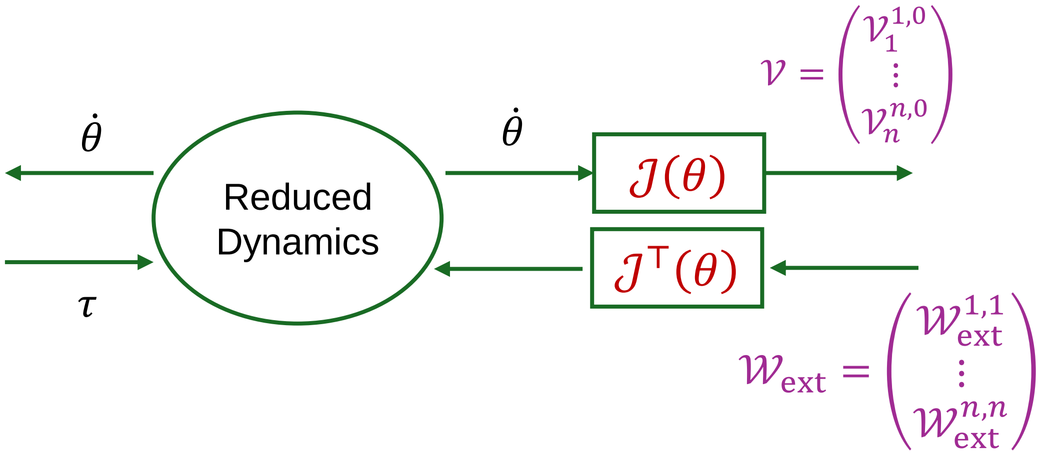


* See for reference Lect. 12, slides 31 and 29.

Adding External Wrenches

- The reduced dynamics can be equivalently then expressed as
  - $\tau = M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} - \left[\mathcal{J}_1^{1,0\,\top}(\theta)\mathcal{W}_{\text{ext}}^{1,1} + \mathcal{J}_2^{2,0\,\top}(\theta)\mathcal{W}_{\text{ext}}^{2,2} + \cdots + \mathcal{J}_n^{n,0\,\top}(\theta)\mathcal{W}_{\text{ext}}^{n,n}\right]$
  - $\mathcal{V}_1^{1,0} = \mathcal{J}_1^{1,0}(\theta)\dot{\theta}$
  - $\mathcal{V}_2^{2,0} = \mathcal{J}_2^{2,0}(\theta)\dot{\theta}$

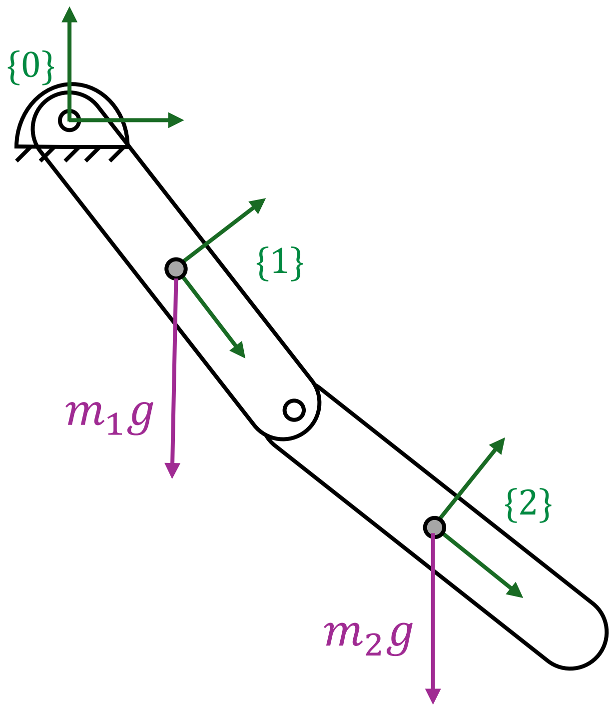
    $\vdots$
  - $\mathcal{V}_n^{n,0} = \mathcal{J}_n^{n,0}(\theta)\dot{\theta}$



Adding Gravity

- One can use the above formulation to add gravity:
  - $\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} - \left[\mathcal{J}_1^{1,0^\top}(\theta)\mathcal{W}_{\text{grv}}^{1,1} + \mathcal{J}_2^{2,0^\top}(\theta)\mathcal{W}_{\text{grv}}^{2,2}\right]$
  - $\mathcal{V}_1^{1,0} = \mathcal{J}_1^{1,0}(\theta)\dot{\theta}$
  - $\mathcal{V}_2^{2,0} = \mathcal{J}_2^{2,0}(\theta)\dot{\theta}$

where $\mathcal{W}_{\text{grv}}^{i,i} = \begin{pmatrix} 0 \\ m_i R_0^i g^0 \end{pmatrix} \in (\mathbb{R}^6)^*$.

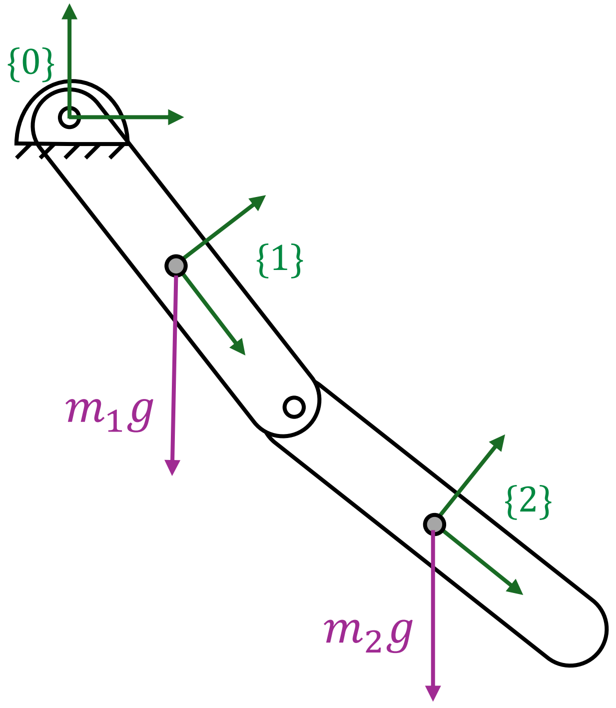


Adding Gravity

- A popular form of the reduced dynamics is

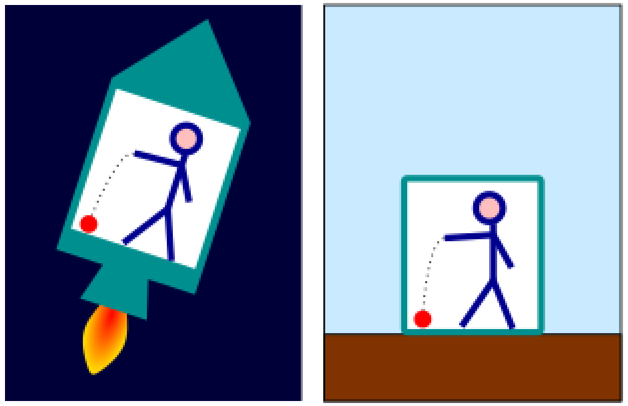
$$\tau = M(\theta)\ddot{\theta} + C\big(\theta, \dot{\theta}\big)\dot{\theta} + g(\theta)$$

where $g(\theta) \coloneqq -\left[\mathcal{J}_1^{1,0\,\top}(\theta)\mathcal{W}_{\mathrm{grv}}^{1,1} + \mathcal{J}_2^{2,0\,\top}(\theta)\mathcal{W}_{\mathrm{grv}}^{2,2}\right] \in \mathbb{R}^n$ are the torques due to gravity.



Einstein’s equivalence principle

- An alternative way of modeling gravity is to consider the stationary frame “free-falling”.
- Einstein’s equivalence principle can be summarized as follows:
  - *“No local experiment can distinguish a uniform gravitational field from an accelerating reference frame.”*
- *Consequently,* All physical laws in a freely falling laboratory take the same form as in an inertial frame with no gravity.

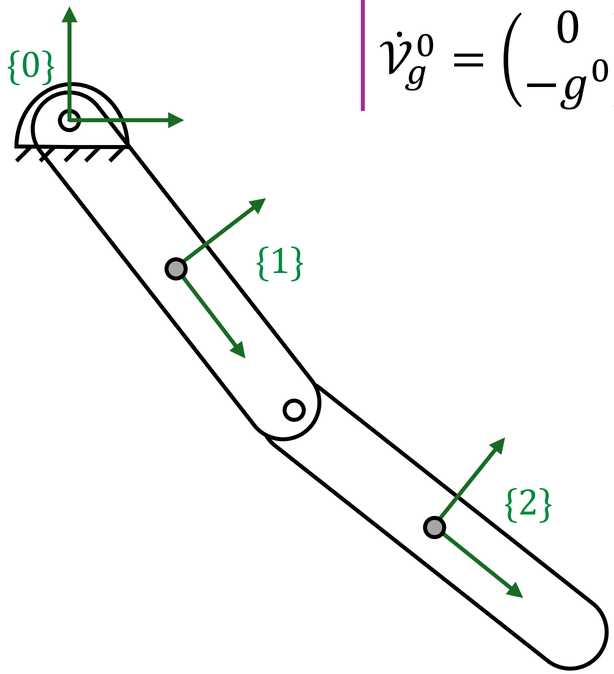


A falling object behaves exactly the same on Earth or in an equivalent accelerating frame of reference.

Adding Gravity

- By considering the stationary frame free-falling, let $\dot{\mathcal{V}}_g$ denote the acceleration of the "world" relative to the frame $\{0\}$.
- One has elegantly that the gravity torques are computed by

$$\boxed{g(\theta) \coloneqq \mathcal{S}^\top \mathcal{L}^\top(\theta)\, \mathfrak{I} \mathcal{L}(\theta) \dot{\mathcal{V}}_{\text{base}}}$$



$$\dot{\mathcal{V}}_g^0 = \begin{pmatrix} 0 \\ -g^0 \end{pmatrix}$$

where $\dot{\mathcal{V}}_{\text{base}} \coloneqq \begin{pmatrix} \mathrm{Ad}_{H_0^1(\theta_1)} \dot{\mathcal{V}}_g^0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{6n}$

Adding End-effector wrench

- Let $\mathcal{W}_{\text{tip}}^{e,e}$ to be the wrench applied to the environment by the end-effector of a two-link manipulator.
- The dynamics then take the form:

  - $\tau = M(\theta)\ddot{\theta} + C\left(\theta, \dot{\theta}\right)\dot{\theta} + J_{\mathrm{e}}^{\top}(\theta)\mathcal{W}_{\text{tip}}^{e,e}$
  - $\mathcal{V}_{e}^{e,0} = J_{e}(\theta)\dot{\theta}$

where $J_e(\theta) \coloneqq Ad_{H_2^e}\, \mathcal{J}_2^{2,0}(\theta)$