بسم الله الرحمن الرحيم

I. State space Model:

$$\dot{x} = f(x) + g(x)u$$ → O.D.E's. → Control-affine form

$$y = h(x)$$ → output eqn → dual to the input
→ measurement

$x \in X \Rightarrow$ State space
$u \in U \Rightarrow$ Control space
$y \in Y \Rightarrow$ output space

2) Control law / Policy / System.:

$$u = \gamma(x)$$

3) Control Objectives:

a) Regulation / Stabilization

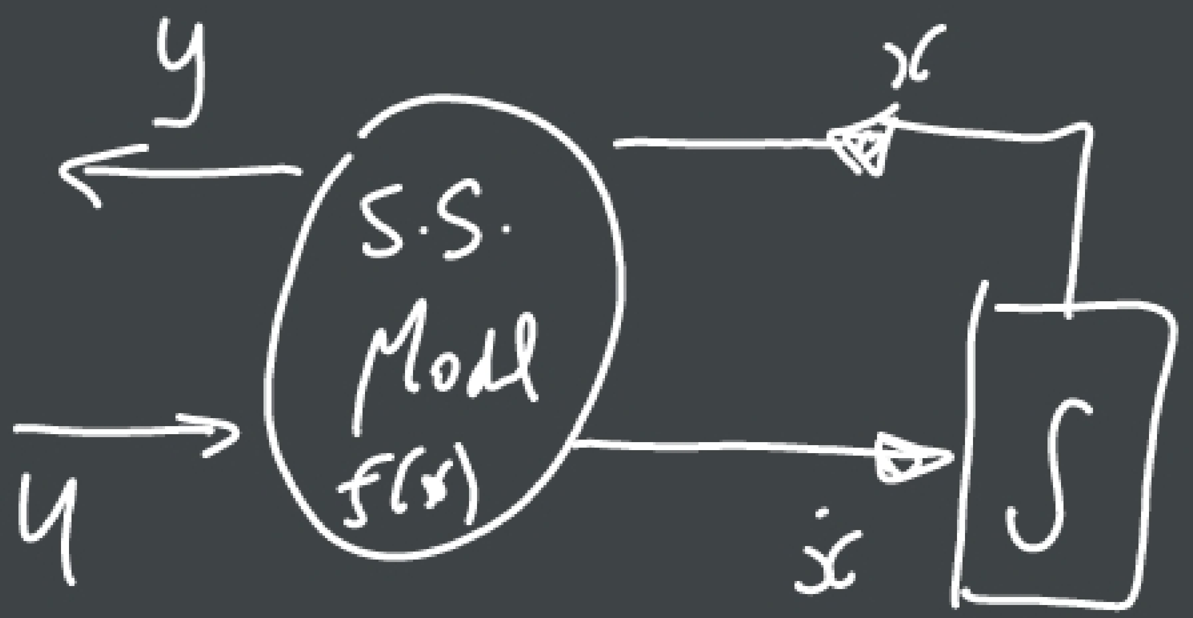
$$x(t) \longrightarrow x_d \quad \text{as} \quad t \to \infty$$

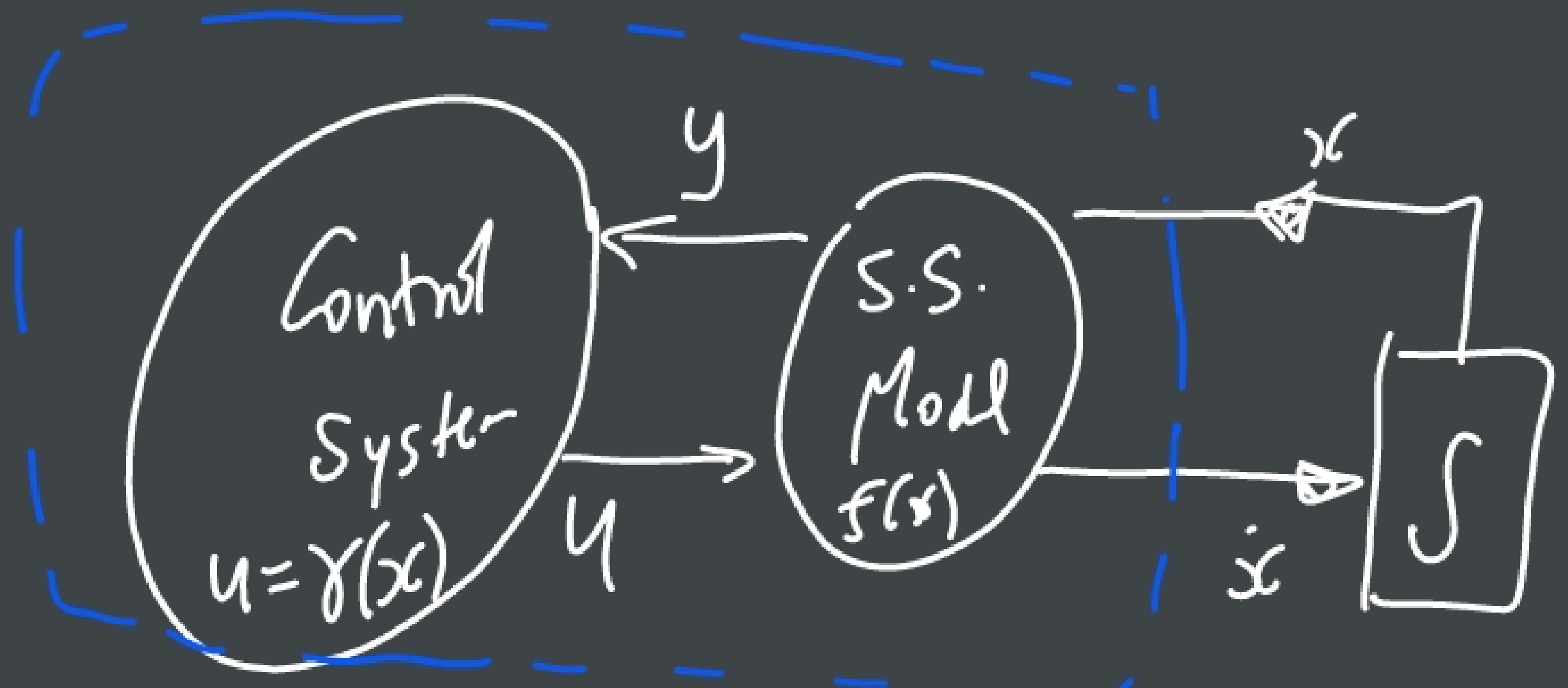
b) Tracking

$$x(t) \longrightarrow x_d(t) \quad \text{as} \quad t \to \infty$$

$$\dot{x}(t) \longrightarrow \dot{x}_d(t)$$

c) Interaction





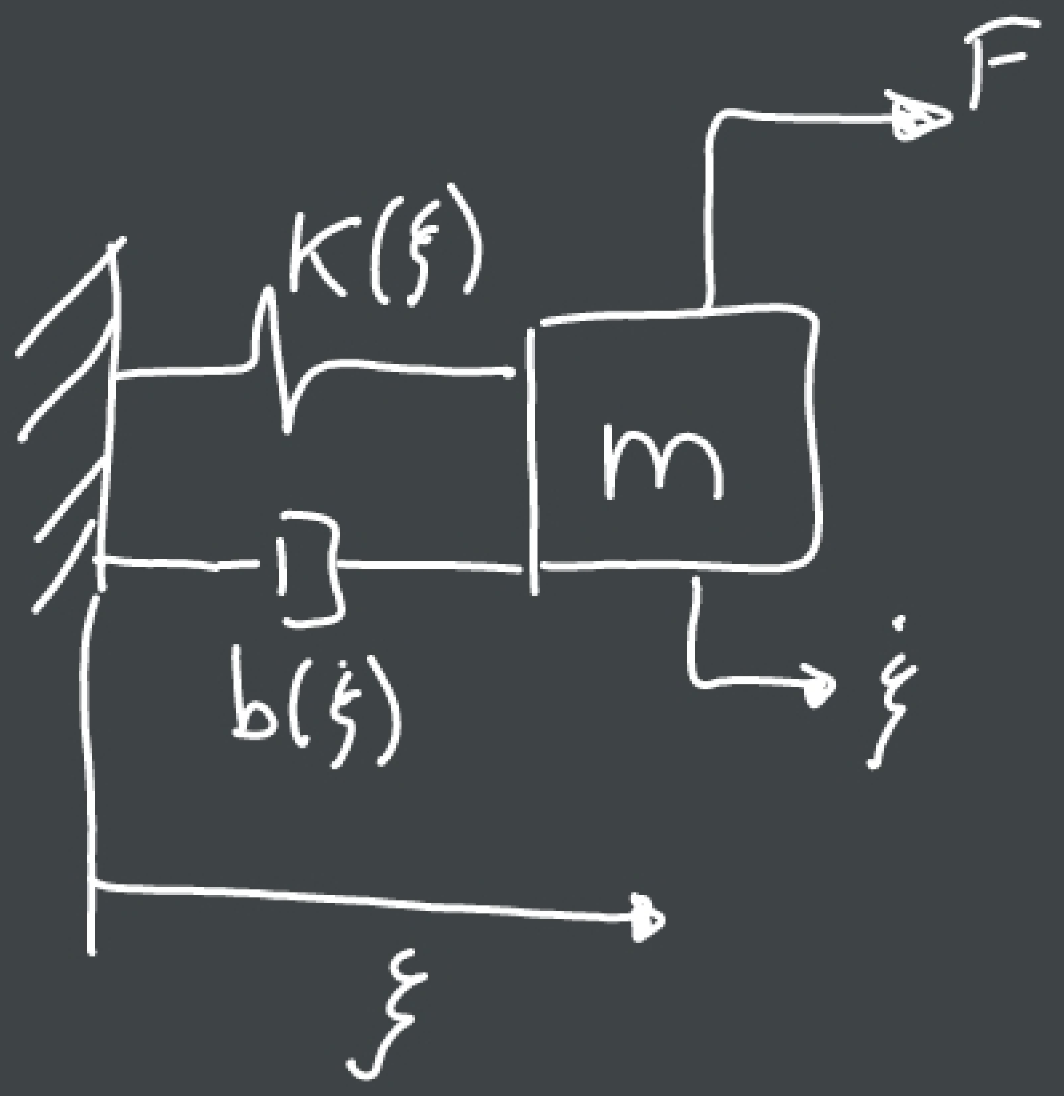
Closed loop system

$$\boxed{\dot{x} = f_{cl}(x)} \to x(t)$$

The Design of the Control system is based on analyzing the stability of the C.L. System

II. State space model of Mech Systems:

1) Mass-spring-damper System:



$$\ddot\xi = \frac{1}{m}\left(F - K(\xi)\,\xi - b(\dot\xi)\,\dot\xi\right)$$

a) $(x_1, x_2) = (\xi, \dot\xi) \in \mathbb{R}\times\mathbb{R} \cong \mathbb{R}^2$

$$\begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{K(x_1)}{m}x_1 - \frac{b(x_2)}{m}x_2\end{pmatrix} + \begin{pmatrix}0\\ 1/m\end{pmatrix}F$$

$$y = x_2$$

b)

$$\begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -k/m & -b/m\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1/m\end{pmatrix}F$$

$$y = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$\Longrightarrow$

$$\boxed{\begin{aligned}\dot x &= Ax + Bu\\ y &= Cx\end{aligned}}$$

$x \in \mathbb{R}^d$

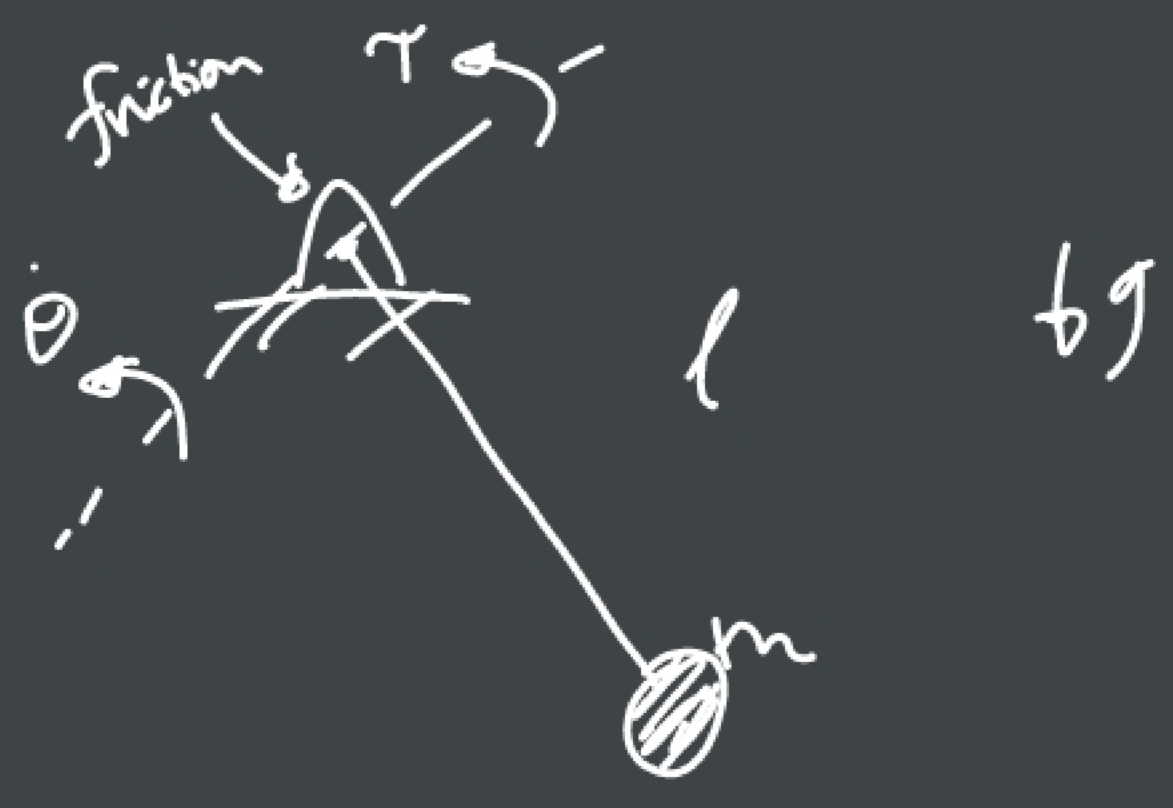
$u, y \in \mathbb{R}^n$

2) Simple Pendulum with friction:

$$(x_1, x_2) = (\theta, \dot{\theta}) \in (-\pi, \pi] \times \mathbb{R} \cong S^1 \times \mathbb{R}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{b}{ml^2}x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/ml^2 \end{pmatrix}\tau$$

$$y = x_2$$



$$\boxed{ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = \tau}$$

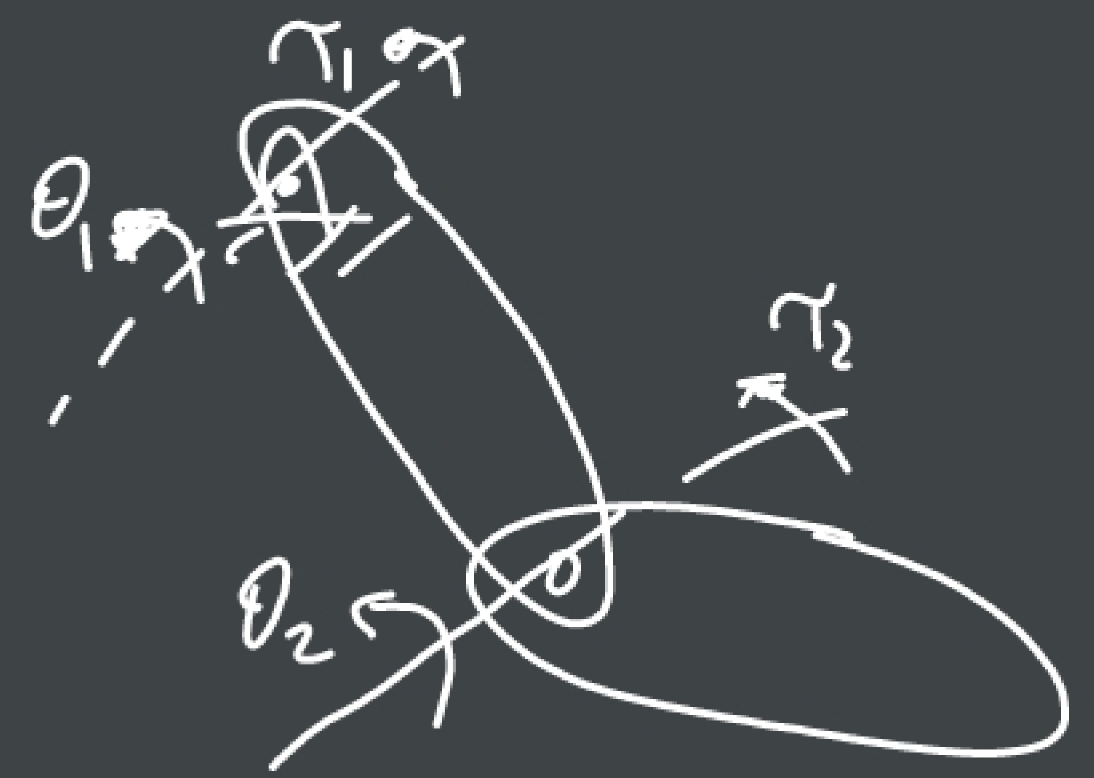
3) n-link Manipulator:

$$(x_1, x_2) = (\theta, \dot{\theta}) \in TQ$$

$$\theta \in Q \quad , \quad \dot{\theta} \in T_\theta Q$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)\left[C(x_1, x_2) + g(x_1) \right] \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} \tau$$

$$y = x_2$$

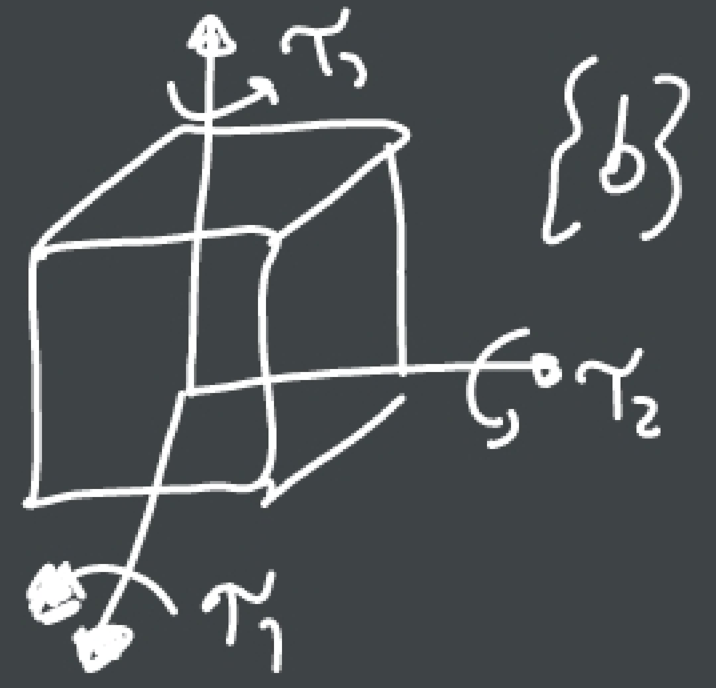


$$\boxed{M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + g(\theta) = \tau}$$

Euler-Lagrange eqns

4) Sattelite:

$$(x_1, x_2) = (R, \omega) \in SO(3) \times \mathbb{R}^3 \cong TSO(3)$$



$$\begin{pmatrix} \dot x_1 \\ \dot x_2 \end{pmatrix} = \begin{pmatrix} \beta_{x_1}(x_2) \\ -J^{-1}\tilde{x}_2 J x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ J^{-1} \end{pmatrix} \tau$$

$$y = x_2$$

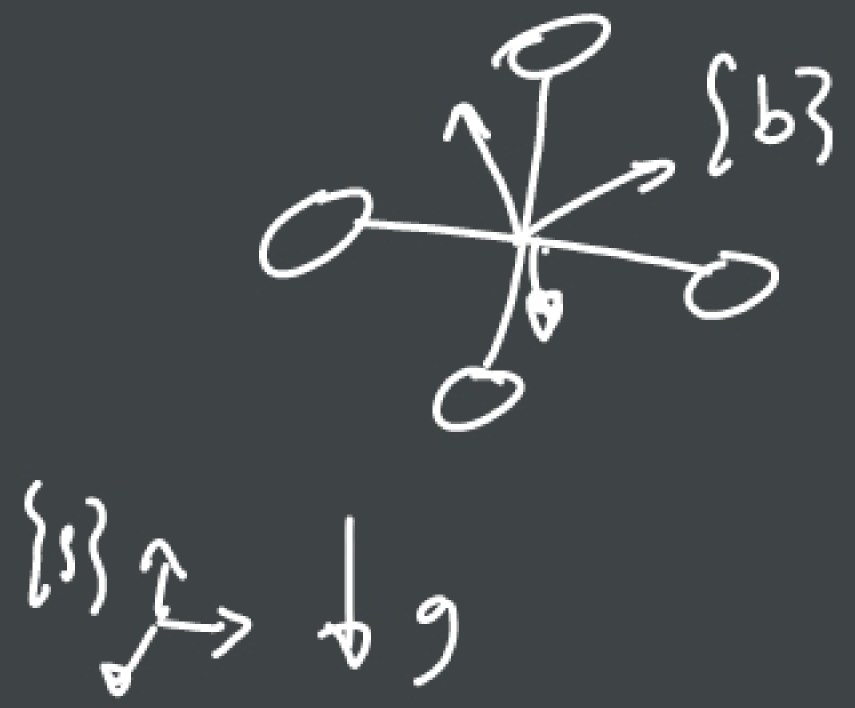
$$\beta_R : \mathbb{R}^3 \to T_R SO(3)$$
$$\omega \mapsto R\tilde{\omega}$$

$$\boxed{\begin{aligned} \dot R &= R\tilde{\omega} \\ J\dot{\omega} &= -\tilde{\omega} J \omega + \tau \end{aligned}}$$

Euler-Poincare on so(3)

5) MAV:

$$(x_1, x_2) = (H, v) \in SE(3) \times \mathbb{R}^6 \cong TSE(3)$$



$$\begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}\beta_{x_1}(x_2) \\ -\mathbb{I}^{-1}\, \mathrm{ad}^T_{x_2}(\mathbb{I} x_2)\end{pmatrix} + \begin{pmatrix}0 \\ \mathbb{I}^{-1} G\end{pmatrix} w_c$$

$$\boxed{\begin{aligned}\dot{H} &= H \tilde{v} \\ \mathbb{I}\dot{v} &= \mathrm{ad}^T_v \mathbb{I} v + G w_c\end{aligned}}$$

Euler-Poincaré eqn on $se(3)$

$$y = G^T v$$

$$\boxed{\beta_H(v) := H \tilde{v}}$$

6) Summary:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$x_1 \rightarrow$ Configuration $\in Q \implies \dim(Q) = n$

$x_2 \rightarrow$ velocity-like $\in T_{x_1} Q$

$$\uparrow$$

$$X = TQ$$

$$\Downarrow$$

$$\dim(TQ) = 2n$$

III. Stability Notions:

$\dot{x}(t) = f(x(t))$

1) Geometric Nature of $\dot{x}_t = f(x_t)$

a) Euclidean Case:

$x_t \in \mathbb{R}^n$

$\dot{x}_t \in \mathbb{R}^n$

$f : \mathbb{R}^n \to \mathbb{R}^n$

b) non-Euclidean Case:

$x_t \in X$

$\dot{x}_t \in T_{x_t} X$

$f : x \in X \to \dot{x} \in T_x X$

f create a vector field σ_f on X

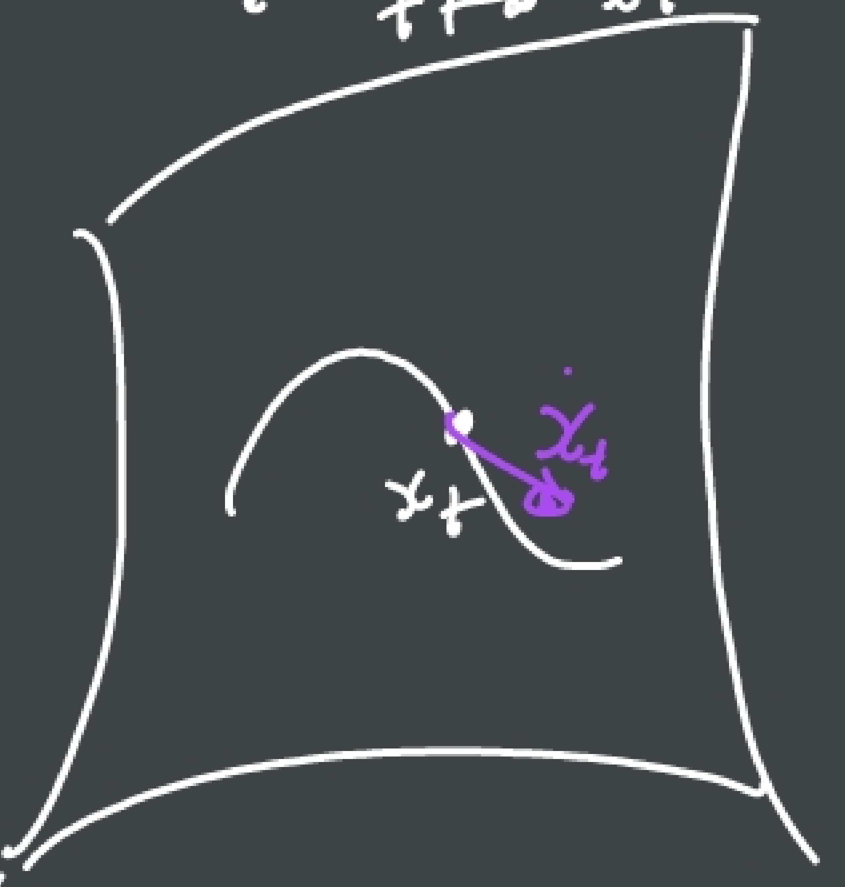
$\sigma_f : X \to TX$

s.t it maps any $x \in X$ to

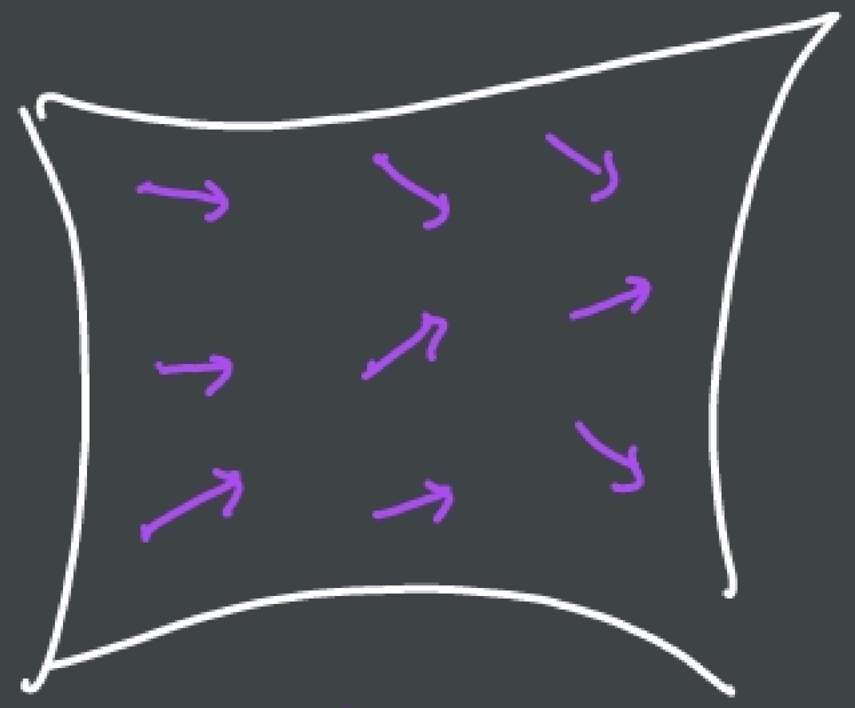
$TX \ni \sigma_f(x) = (x, f(x))$

where $f(x) \in T_x X$.

$c_x : I \to X$
$t \mapsto x_t$



X



$\sigma_f \in \Gamma(TM)$

sections of the tangent bundle.