SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 17: Fundamentals of Lyapunov theory

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Outline

- Recap last lecture
- Equilibrium points and Stability notions
- Case study: Pendulum
- Lyapunov’s indirect method

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Recap: State Space Model

- A nonlinear dynamic system can be represented by a set of nonlinear differential equations in the form

$$\dot{x} = f(x) + g(x)\,u$$
$$y = h(x)$$

which is called the state space model of the dynamic system.

- We denote by the
  - State space $\mathcal{X} \ni x$
  - Control space $\mathcal{U} \ni u$
  - Output space $\mathcal{Y} \ni y$

Special case: Linear time-invariant systems

$$\dot{x} = Ax + Bu$$
$$y = C\,x$$

Recap: Control Objectives

- The control input is designed in general as a function of the output $u = \beta(y)$ to achieve:
  - Regulation/Stabilization $x(t) \to x_d$ as $t \to \infty$
  - Tracking $x(t) \to x_d(t), \dot{x}(t) \to \dot{x}_d(t)$ as $t \to \infty$
  - Interaction

- The closed loop system is given by

$$\dot{x} = f_{\mathrm{cl}}(x)$$

where $f_{\mathrm{cl}}(x) \coloneqq f(x) + g(x) \cdot \beta\big(h(x)\big)$

The design of the control system is based on analyzing the stability of the closed-loop system !

Recap: State Space Models – Mechanical Systems

System	State x	State Space $\mathcal{X}$
Mass-Spring-Damper	$(\xi, \dot{\xi})$	$\mathbb{R} \times \mathbb{R} \cong T\mathbb{R}$
Simple pendulum	$(\theta, \dot{\theta})$	$(-\pi, \pi] \times \mathbb{R} \cong T\mathbb{S}$
n-link Manipulator	$(\theta, \dot{\theta})$	TQ
Satellite	(R, ω)	$SO(3) \times \mathbb{R}^3 \cong TSO(3)$
Multirotor Aerial Vehicle	$(H, \mathcal{V})$	$SE(3) \times \mathbb{R}^6 \cong TSE(3)$

For mechanical systems in general, $x = (q, \dot{q}) \in TQ$, where $q \in Q$ represents a configuration variable of the mechanical system and $\dot{q} \in T_q Q$ denotes a velocity-like variable.

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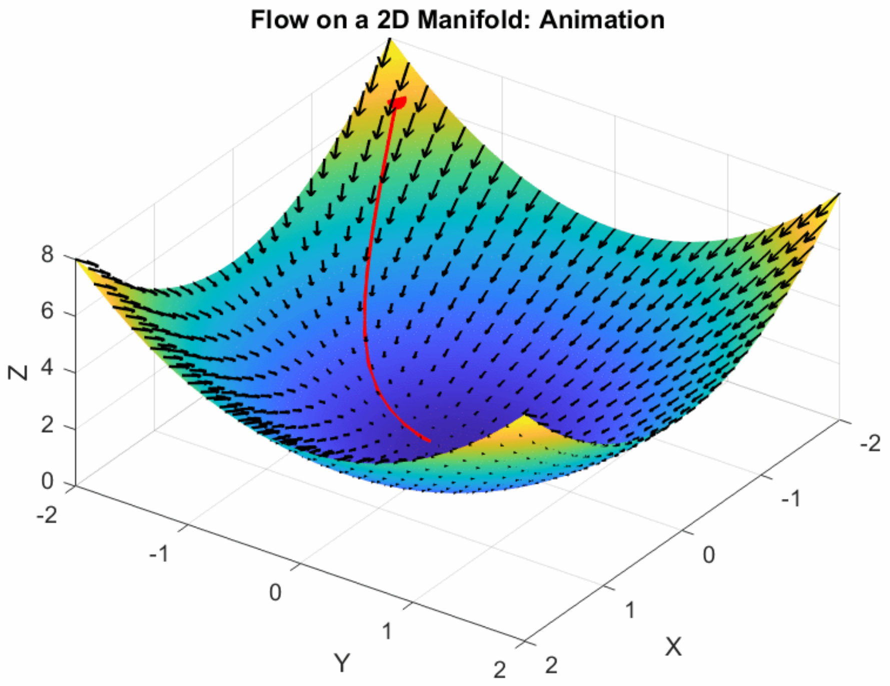
Geometric Nature of $\dot{x}(t) = f(x(t))$

- Euclidean case $\mathcal{X} = \mathbb{R}^{\boldsymbol{n}}$:
  - $x_t \in \mathbb{R}^{\boldsymbol{n}}$
  - $\dot{x}_t \in \mathbb{R}^{\boldsymbol{n}}$
  - $f: \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}^{\boldsymbol{n}}$
- Non-Euclidean case:
  - $x_t \in \mathcal{X}$
  - $\dot{x}_t \in T_{\boldsymbol{x}}\mathcal{X}$
  - $f: x_t \in \mathcal{X} \mapsto \dot{x}_t \in T_{\boldsymbol{x}}\mathcal{X}$



The map f creates a vector field $\sigma_f \in \Gamma(T\mathcal{X})$ on the state space manifold $\sigma_f: \mathcal{X} \to T\mathcal{X}$ defined by:
$\sigma_f(x) \coloneqq \big(x, f(x)\big) \in T\mathcal{X}$, with $f(x) \in T_{\boldsymbol{x}}\mathcal{X}$

Geometric Nature of $\dot{x}(t) = f(x(t))$

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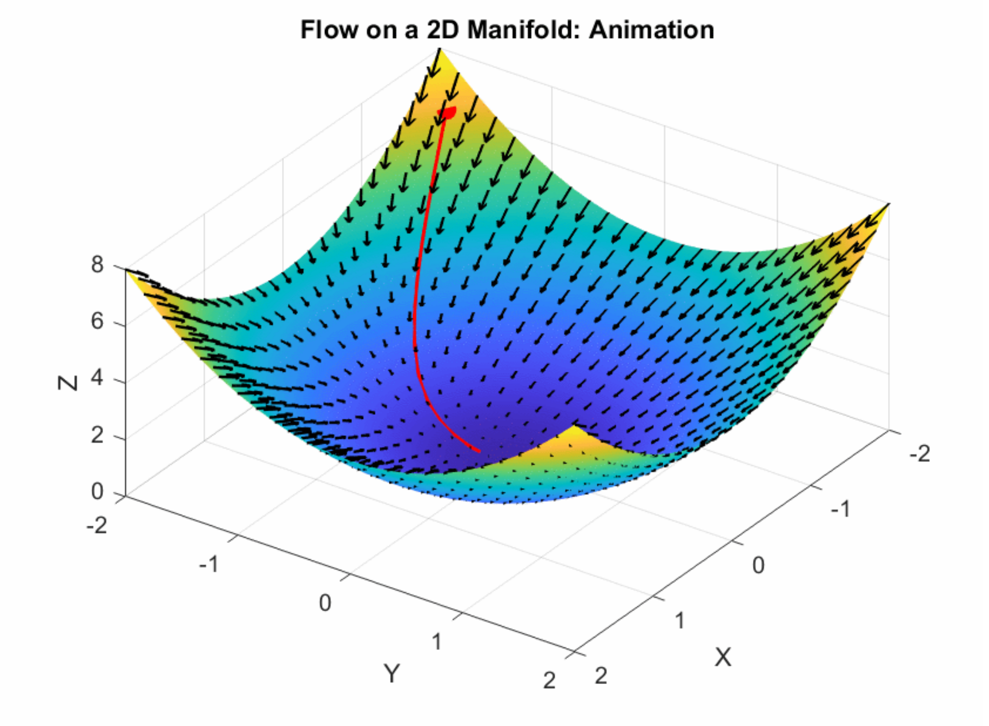

The solution of the dynamical system $x(t)$ is given by the integral curves of σ_f.

Integral Curves

While $f(x)$ represents the velocity of a particle at every point, the integral curve represents the trajectory of a particle moving along this velocity field.

Geometric Nature of $\dot{x}(t) = f(x(t))$

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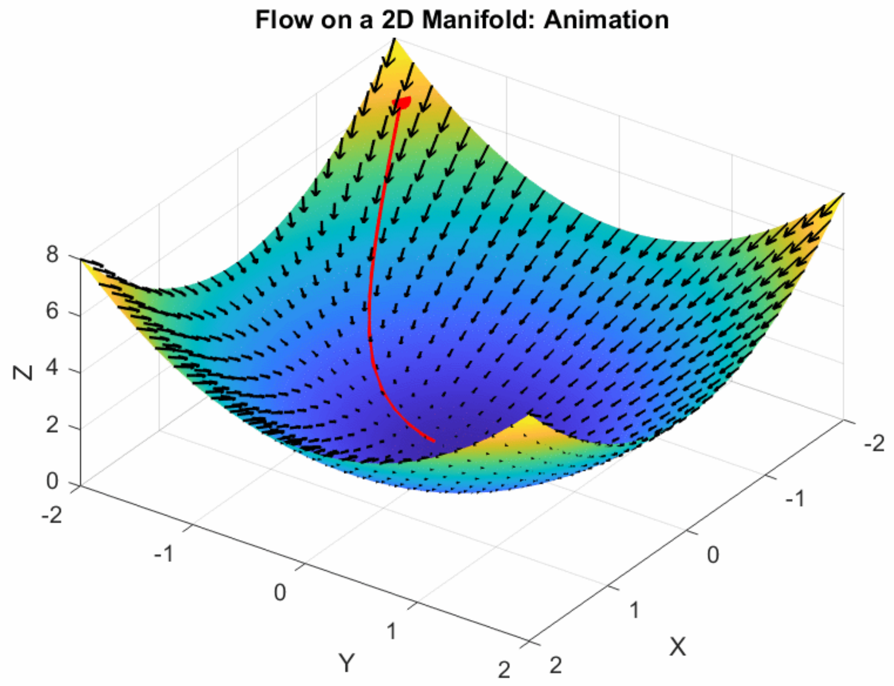


Stability analysis

By analyzing properties of f or σ_f, we can infer how the system's state $x(t)$ will evolve with time, without explicitly computing the solution as a function of time.

Equilibrium Points

- Given $\sigma_f \in \Gamma(T\mathcal{X})$, a point $x_* \in \mathcal{X}$ in the state space is called an *equilibrium point for σ_f* if and only if

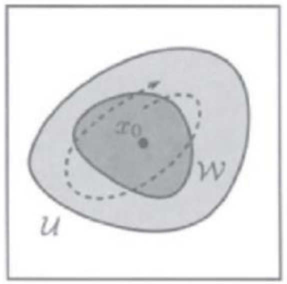
$$f(x_*) = 0$$

- Intuitively, an equilibrium point is a state x_* at which the system state remains for all time, once it reaches it.

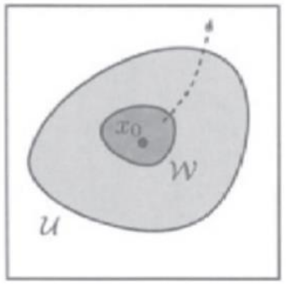
- Equilibrium points are also referred to as "fixed points" or "critical points" of the system $\dot{x}(t) = f(x(t))$.

Stability of Equilibrium Points

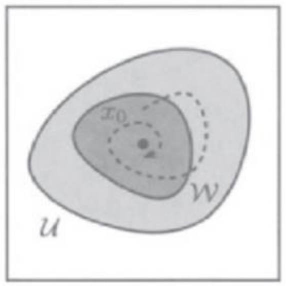
- An equilibrium point x_* of σ_f is said to be:
    - Stable: If integral curves of σ_f stay “close” to x_*
    - Unstable: If it is not stable
    - Locally asymptotically stable: If it is stable and integral curves of σ_f converge to x_* only within a region $\mathrm{U} \subset \mathcal{X}$
    - Globally asymptotically stable: If it is stable and integral curves of σ_f converge to x_* for all $x \in \mathcal{X}$



Stable x_0



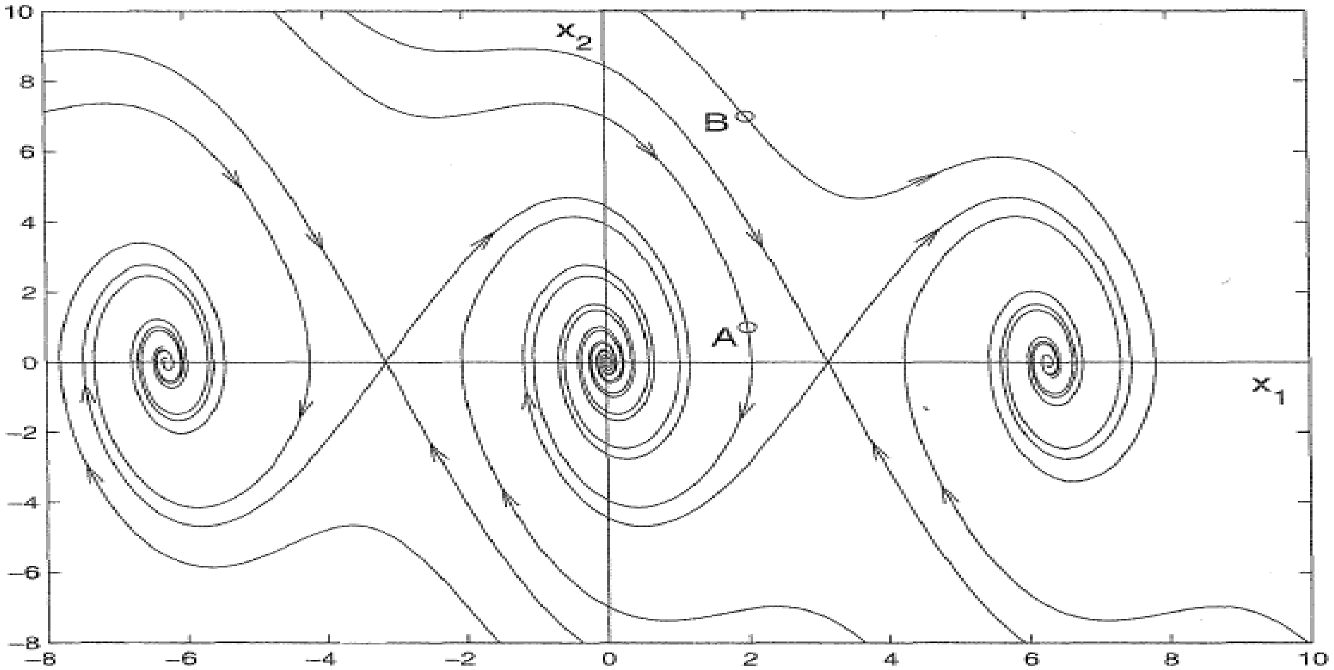
Unstable x_0



Locally asym. stable x_0

Phase Portrait

- A phase portrait is a visual representation of the trajectories of a dynamical system in phase space.
- It helps illustrate the behavior of integral curves and the stability of equilibrium points.



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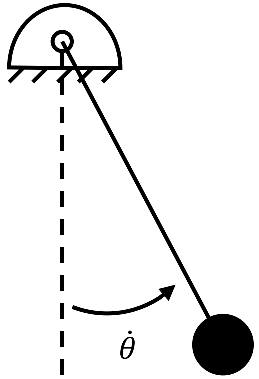
Equilibrium points of pendulum

- Consider the state-space model of the pendulum given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 \end{pmatrix} =: f(x)$$

where $x_1 = \theta, x_2 = \dot{\theta}$.



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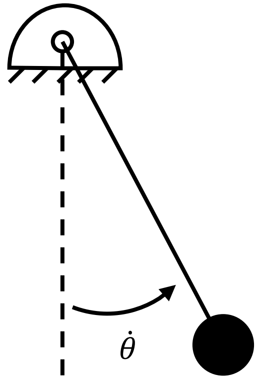
- To find the equilibrium points x_*, we set $f(x) = 0$:

$$x_2 = 0 \quad , \quad -\frac{b}{mL^2}x_2 - \frac{g}{L}\sin x_1 = 0,$$



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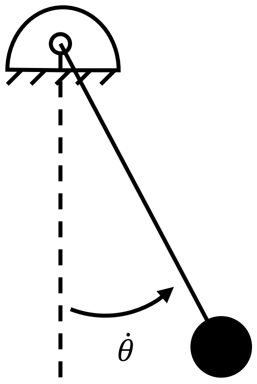
- To find the equilibrium points x_*, we set $f(x) = 0$, which simplifies to

$$x_2 = 0 \ , \ \ \sin x_1 = 0$$

  - Solving for $x_1 \in \mathbb{R}$, we get:

$$\sin x_1 = 0 \quad \Rightarrow \quad x_1 = k\pi, \qquad k \in \mathbb{Z}$$



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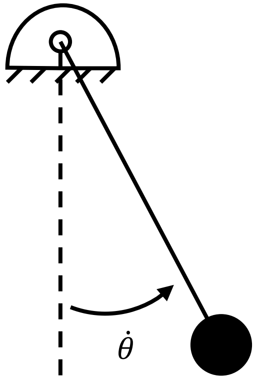
  - However, recall that $x_1 = \theta \in (-\pi, \pi] \cong \mathbb{S}^1$, thus the system has only two equilibrium points



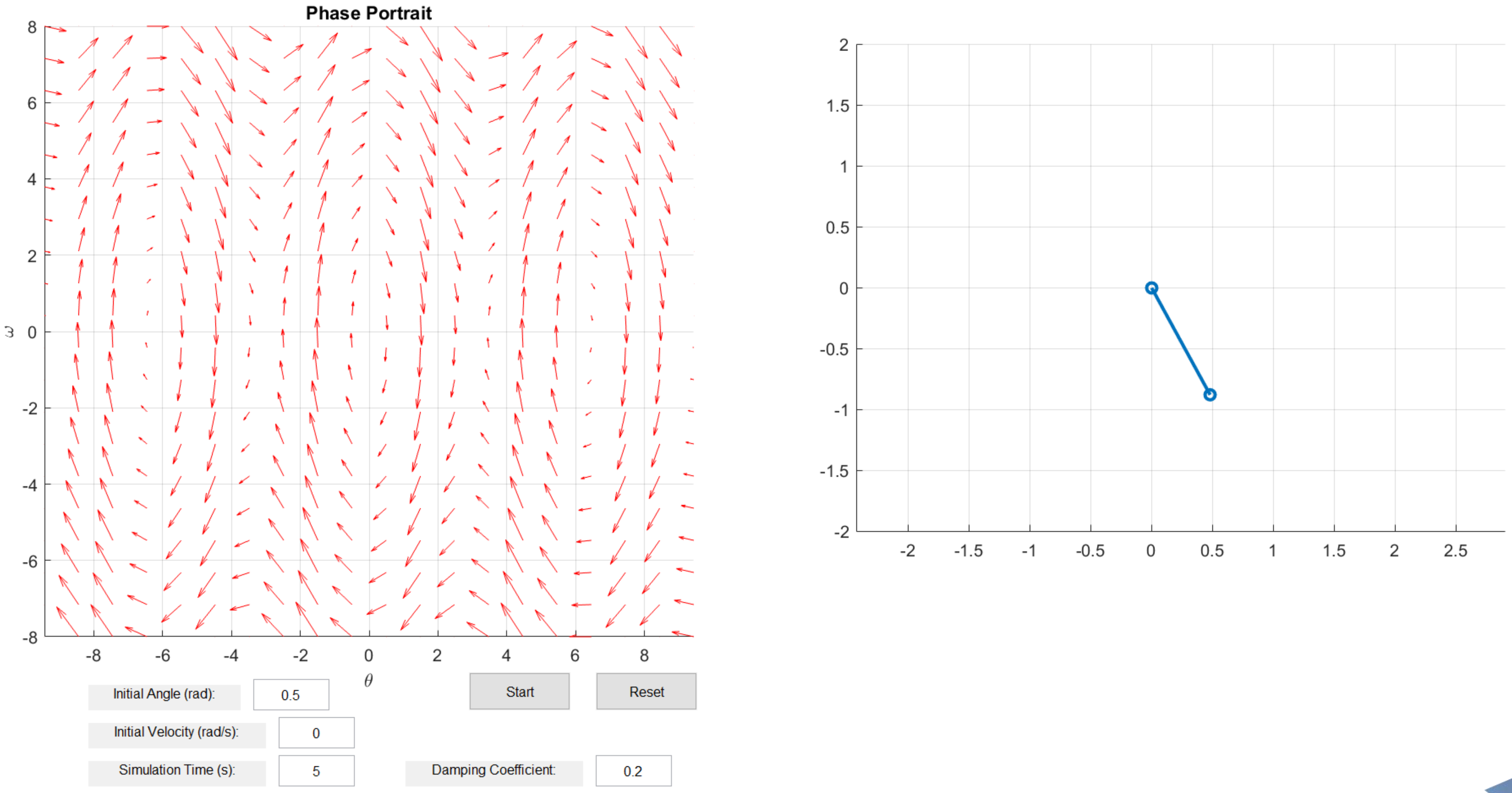
$$\boxed{x_* = (0,0)}$$

Downward position

$$\boxed{x_* = (\pi, 0)}$$

Upward position

MATLAB Code

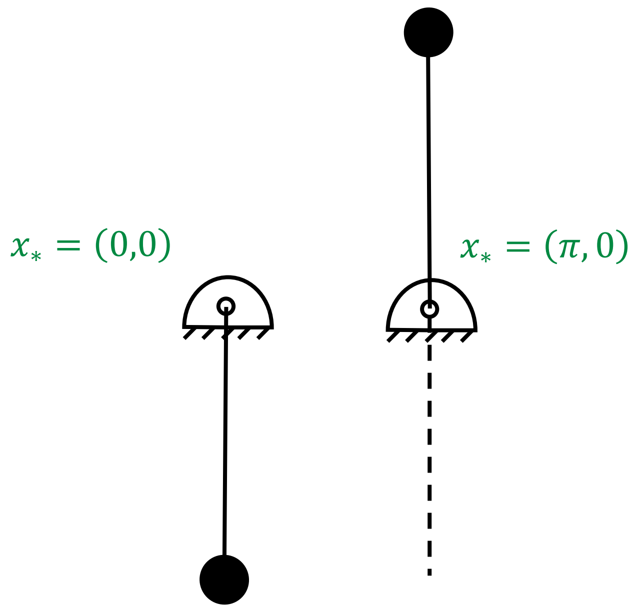

Phase Portrait
ω
θ
Initial Angle (rad):
0.5
Initial Velocity (rad/s):
0
Simulation Time (s):
5
Damping Coefficient:
0.2
Start
Reset

Equilibrium points of pendulum

- In summary, the state space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 \end{pmatrix}$$

has an asymptotically stable equilibrium at $x_* = (0,0)$ and an unstable equilibrium at $x_* = (\pi, 0)$.



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Phase Portrait Limitation

- A phase portrait has several useful properties:
  - It allows us to visualize what goes on in a nonlinear system starting from various initial conditions, without having to solve the nonlinear equations analytically.
  - It is not restricted to small or smooth nonlinearities, but applies equally well to strong nonlinearities.
- However, its main disadvantage is:
  - It is only restricted to 2^{nd} order dynamical systems (which correspond to 1^{st} order mechanical systems !)

Lyapunov stability

- The most useful and general approach for studying the stability of nonlinear dynamical systems is the theory introduced in the late 19^{th} century by the Russian mathematician Lyapunov.

- His work introduced two methods:
  - **Indirect Method:** studies nonlinear local stability around an equilibrium point x_* from stability properties of its linear approximation.
  - **Direct Method:** not restricted to local motion. Stability of nonlinear system is studied by proposing a scalar "energy-like" function $V : \mathcal{X} \to \mathbb{R}$ for the system and examining its time variation

Aleksandr Mikhailovich Lyapunov (1857-1918) was a Russian mathematician, mechanic and physicist.
https://en.wikipedia.org/wiki/Aleksandr_Lyapunov

Recall: Taylor series expansion

- Given a smooth function $f: \mathbb{R}^n \to \mathbb{R}$, the **Taylor series expansion** of order k around a point $x_0 \in \mathbb{R}^n$ expresses $f(x)$ as a sum of k derivatives evaluated at x_0.

- Case $k = 1$:

$$f(x) \approx f(x_0) + df(x_0)(x - x_0),$$

where $df \ := \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$ is a "covector field"

$$df: \mathbb{R}^n \to (\mathbb{R}^n)^*$$

defined such that $df(x) = \left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x)\right) \in (\mathbb{R}^n)^*$ is a covector.

Recall: Taylor series expansion

- Now suppose $f: \mathbb{R}^n \to \mathbb{R}^n$, the **Taylor series expansion** of order 1 around a point $x_0 \in \mathbb{R}^n$ is given by

$$f(x) \approx f(x_0) + J_f(x_0)(x - x_0),$$

where $J_f(x_0) \in \mathbb{R}^{n\times n}$ is the Jacobian matrix given by

$$J_f(x) = \begin{pmatrix} df_1(x) \\ \vdots \\ df_n(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{pmatrix}$$

Lyapunov's indirect method

- Consider the nonlinear system $\dot{x} = f(x)$ with equilibrium point x_*. The linearization of this system around the equilibrium point x_* is given by:

$$\dot{z} = A\, z$$

where $z \coloneqq x - x_* \in \mathbb{R}^n$, $A \coloneqq J_f(x_*)$.

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- **Stability Conditions**
  - **Asymptotic Stability**: If all eigenvalues of A have **negative real parts**, then x_* is **locally asymptotically stable** for the nonlinear system.
  - **Instability**: If at least one eigenvalue of A has a **positive real part**, then x_* is **unstable** for the nonlinear system.
  - **Inconclusive Case**: If some eigenvalues of A have **zero real parts**, Lyapunov's Indirect Method does not provide a definite answer, and higher-order nonlinear terms must be considered.