SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 18: Lyapunov’s direct method

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Outline

- Recap last lectures
- Application of Lyapunov’s indirect method
- Lyapunov’s direct method
- La Salle’s invariance principle

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Recap: State Space Model

- A nonlinear dynamic system can be represented by a set of nonlinear differential equations in the form

$$\dot{x} = f(x) + g(x)\,u$$
$$y = h(x)$$

which is called the state space model of the dynamic system.

- We are focusing on analyzing the stability of systems of the form

$$\dot{x} = f(x)$$

<u>Special case:</u> Linear time-invariant systems

$$\dot{x} = Ax + Bu$$
$$y = C\,x$$

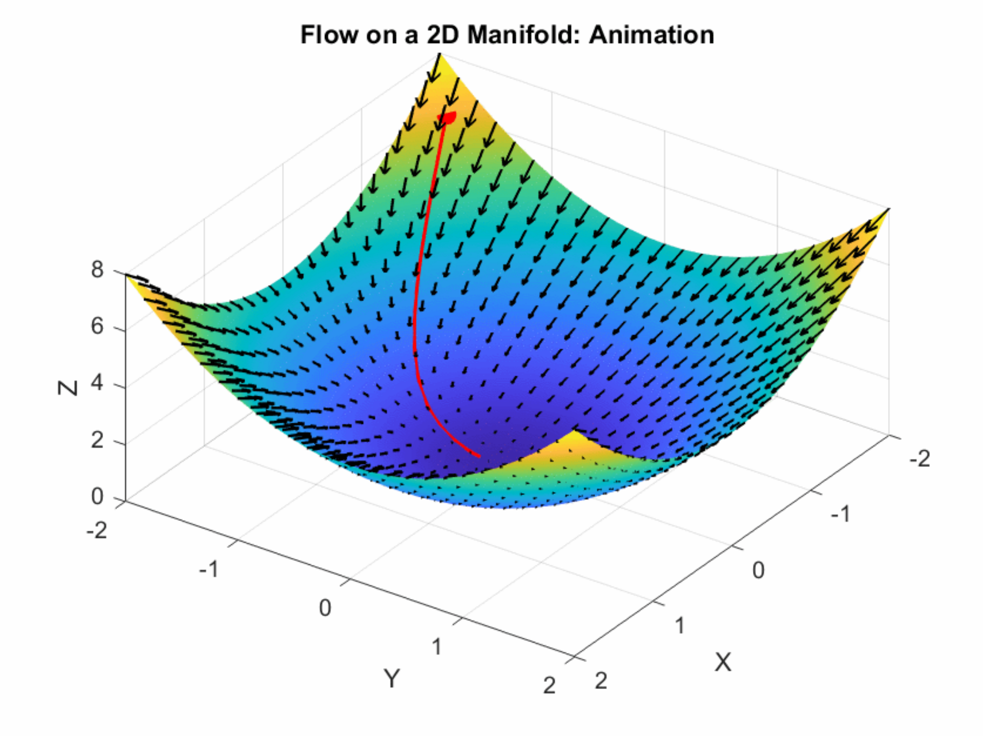
Recap: State Space Models – Mechanical Systems

System	State x	State Space $\mathcal{X}$
Mass-Spring-Damper	$(\xi, \dot{\xi})$	$\mathbb{R} \times \mathbb{R} \cong T\mathbb{R}$
Simple pendulum	$(\theta, \dot{\theta})$	$(-\pi, \pi] \times \mathbb{R} \cong T\mathbb{S}$
n-link Manipulator	$(\theta, \dot{\theta})$	TQ
Satellite	(R, ω)	$SO(3) \times \mathbb{R}^3 \cong TSO(3)$
Multirotor Aerial Vehicle	$(H, \mathcal{V})$	$SE(3) \times \mathbb{R}^6 \cong TSE(3)$

For mechanical systems in general, $x = (q, \dot{q}) \in TQ$, where $q \in Q$ represents a configuration variable of the mechanical system and $\dot{q} \in T_q Q$ denotes a velocity-like variable.

Recap: Geometric Nature of $\dot{x}(t) = f(x(t))$

- Euclidean case $\mathcal{X} = \mathbb{R}^{\boldsymbol{n}}$:
  - $x_t \in \mathbb{R}^{\boldsymbol{n}}$
  - $\dot{x}_t \in \mathbb{R}^{\boldsymbol{n}}$
  - $f: \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}^{\boldsymbol{n}}$
- Non-Euclidean case:
  - $x_t \in \mathcal{X}$
  - $\dot{x}_t \in T_{\boldsymbol{x}}\mathcal{X}$
  - $f: x_t \in \mathcal{X} \mapsto \dot{x}_t \in T_{\boldsymbol{x}}\mathcal{X}$



The solution of the dynamical system $x(t)$ is given by the integral curves of σ_f.

Integral Curves
While $f(x)$ represents the velocity of a particle at every point, the integral curve represents the trajectory of a particle moving along this velocity field.

Recap: Equilibrium Points

- Given $\sigma_f \in \Gamma(T\mathcal{X})$, a point $x_* \in \mathcal{X}$ in the state space is called an *equilibrium point for σ_f* if and only if

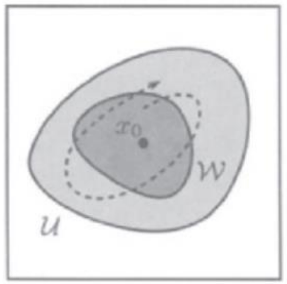
$$f(x_*) = 0$$

- Intuitively, an equilibrium point is a state x_* at which the system state remains for all time, once it reaches it.

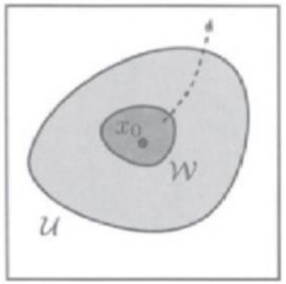
Recap: Stability of Equilibrium Points

- An equilibrium point x_* of σ_f is said to be:

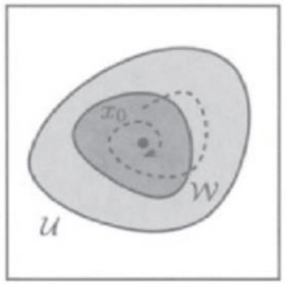
• Stable	If integral curves of σ_f stay "close" to x_*
• Unstable	If it is not stable
• Locally asymptotically stable	If it is stable and integral curves of σ_f converge to x_* only within a region $\mathrm{U} \subset \mathcal{X}$
• Globally asymptotically stable	If it is stable and integral curves of σ_f converge to x_* for all $x \in \mathcal{X}$



Stable x_0



Unstable x_0



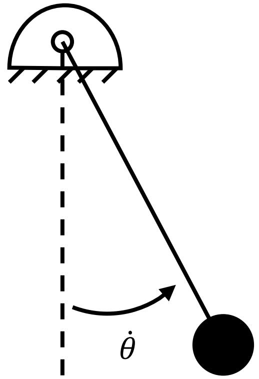
Locally asym. stable x_0

Recap: Equilibrium points of pendulum

- Consider the state-space model of the pendulum given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 \end{pmatrix} =: f(x)$$

where $x_1 = \theta, x_2 = \dot{\theta}$.



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where $x_1 = \theta, x_2 = \dot{\theta}$.

- To find the equilibrium points x_*, we set $f(x) = 0$, which simplifies to

$$x_2 = 0 \ , \ \ \sin x_1 = 0$$

  - Solving for $x_1 = \theta \in (-\pi, \pi] \cong \mathbb{S}^1$, we get:

$$\sin x_1 = 0 \quad \Rightarrow \quad x_1 = 0, \text{ or } x_1 = \pi$$

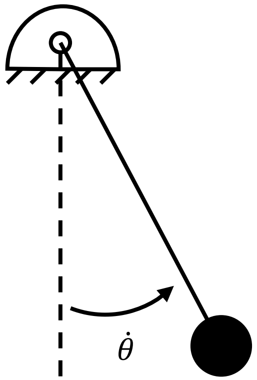
  - Thus, the system has only two equilibrium points

$$\boxed{x_* = (0,0)}$$

Downward position

$$\boxed{x_* = (\pi, 0)}$$

Upward position

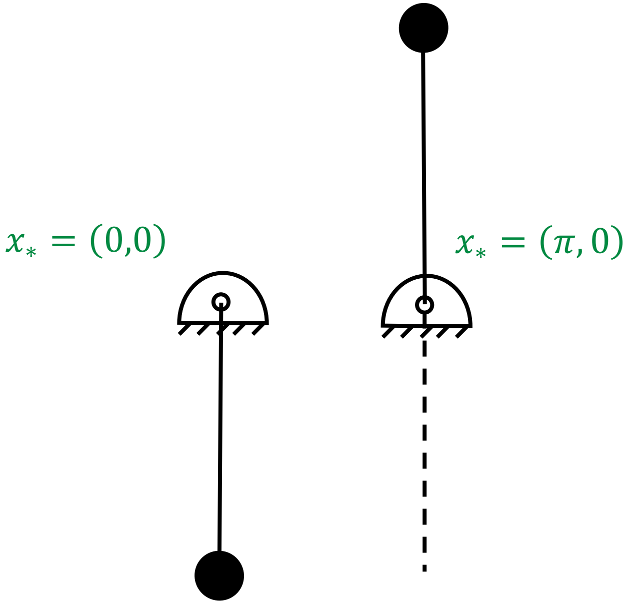


Recap: Stability of equilibrium points of pendulum

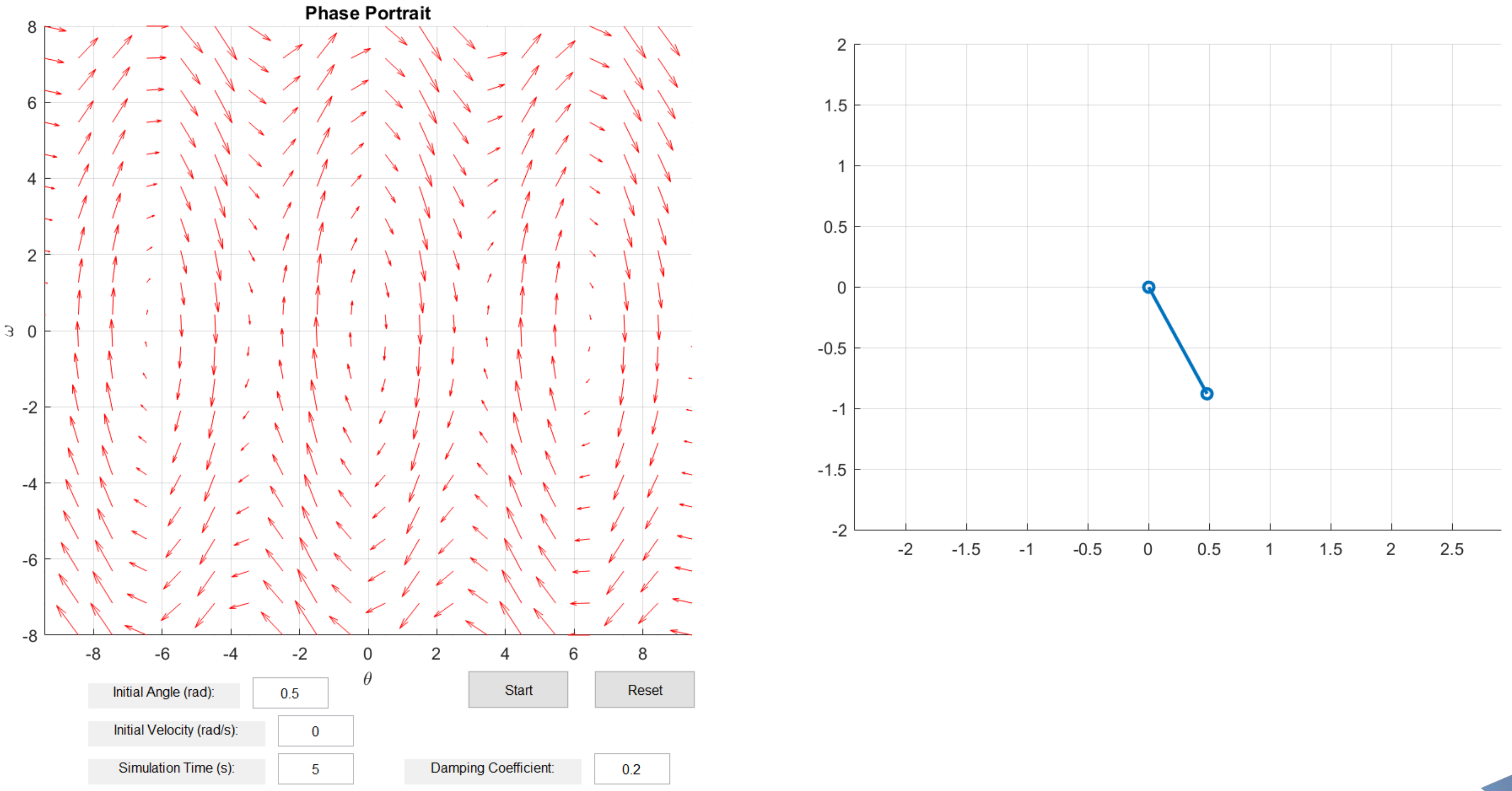
- In summary, the state space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 \end{pmatrix}$$

has an asymptotically stable equilibrium at $x_* = (0,0)$ and an unstable equilibrium at $x_* = (\pi, 0)$.



Recap: MATLAB Code for Phase Portrait



Recap: Lyapunov stability

- The most useful and general approach for studying the stability of nonlinear dynamical systems is the theory introduced in the late 19th century by the Russian mathematician Lyapunov.

- His work introduced two methods:
  - **Indirect Method:** studies nonlinear local stability around an equilibrium point x_* from stability properties of its linear approximation.
  - **Direct Method:** not restricted to local motion. Stability of nonlinear system is studied by proposing a scalar "energy-like" function $V: \mathcal{X} \to \mathbb{R}$ for the system and examining its time variation

Aleksandr Mikhailovich Lyapunov (1857-1918) was a Russian mathematician, mechanic and physicist.
https://en.wikipedia.org/wiki/Aleksandr_Lyapunov

Recap: Lyapunov's indirect method

- Consider the nonlinear system $\dot{x} = f(x)$ with equilibrium point x_*. The linearization of this system around the equilibrium point x_* is given by:

$$\dot{z} = A\, z$$

where $z \coloneqq x - x_* \;\; \in \mathbb{R}^n$ and $A \coloneqq J_f(x_*)$ is the Jacobian of $f(x)$ evaluated at the equilibrium points.

$$J_f(x) = \begin{pmatrix} df_1(x) \\ \vdots \\ df_n(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{pmatrix}$$

Recap: Lyapunov's indirect method

- Consider the nonlinear system $\dot{x} = f(x)$ with equilibrium point x_*. The linearization of this system around the equilibrium point x_* is given by:

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- **Stability Conditions:**

Compute $eig(A) \coloneqq (\lambda_1, \cdots \lambda_n)$, with $\lambda_i \in \mathbb{C}$

- Is there any λ_i with $\mathrm{Re}(\lambda_i) > 0$? → x_* is **unstable** for the nonlinear system.
- Is there any λ_i with $\mathrm{Re}(\lambda_i) = 0$? → Lyapunov's indirect method is inconclusive
- Do all λ_i have $\mathrm{Re}(\lambda_i) < 0$? → x_* is **locally asymptotically stable** for the nonlinear system.

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Example 1: Pendulum

- The Jacobian of the pendulum dynamics is given by

$$J_f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c_1 \cos x_1 & -c_2 \end{pmatrix}$$

<u>Pendulum dynamics:</u>

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 \end{pmatrix} =: \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

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- Let $\dot{z}_1 = A_1 \, z_1$ and $\dot{z}_2 = A_2 \, z_2$ denote the linearization of the nonlinear system around $x_{*,1} \coloneqq (0,0)$ and $x_{*,2} \coloneqq (\pi, 0)$, respectively:

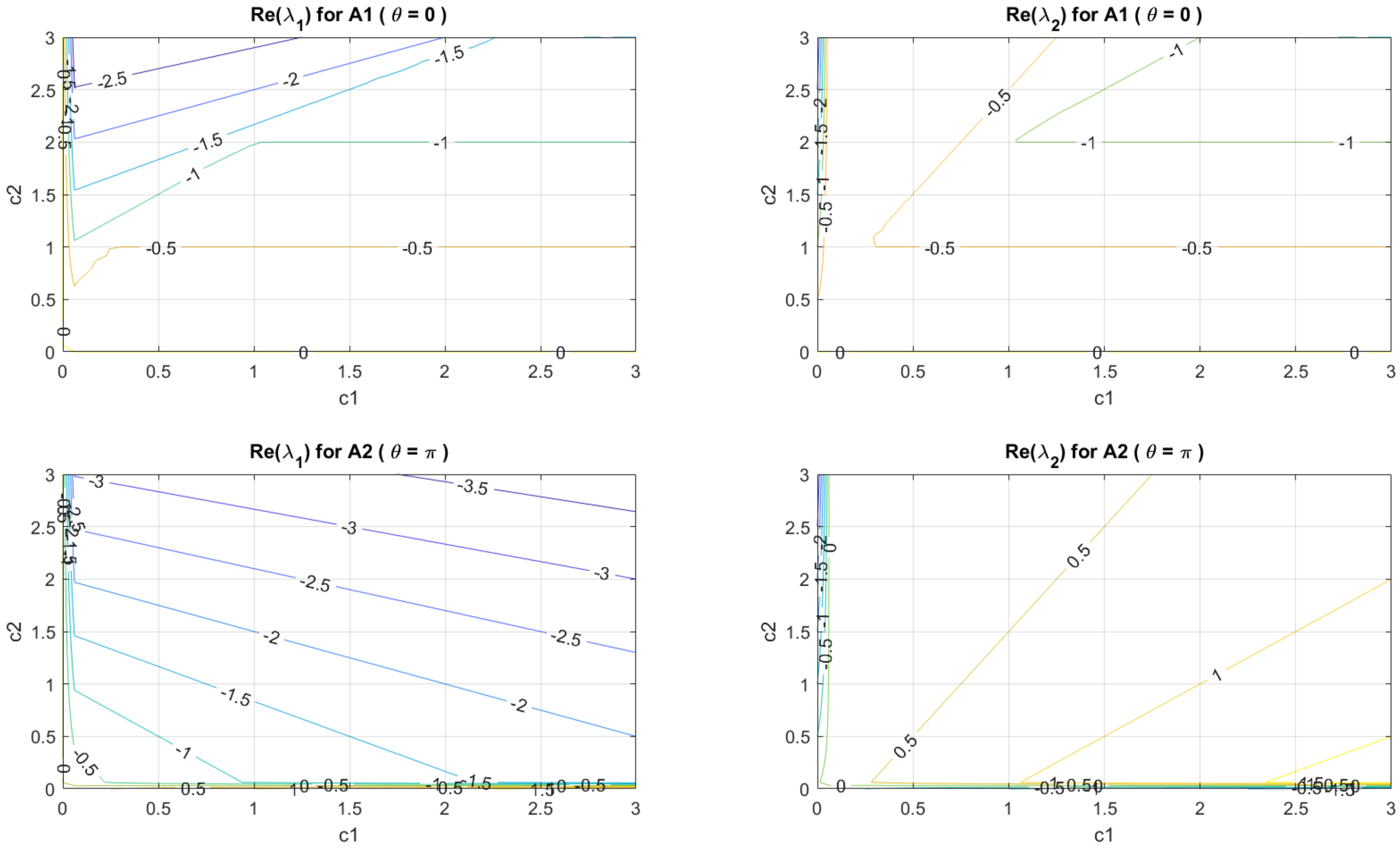
$$A_1 \coloneqq J_f(x_{*,1}) = \begin{pmatrix} 0 & 1 \\ -c_1 & -c_2 \end{pmatrix} \qquad A_2 \coloneqq J_f(x_{*,2}) = \begin{pmatrix} 0 & 1 \\ c_1 & -c_2 \end{pmatrix}$$

Pendulum dynamics:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 \end{pmatrix} =: \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

Example 1: Pendulum

- Below we show **four contour plots** that show how the real parts of the eigenvalues of A_1 and A_2 vary as a function of (c_1, c_2).



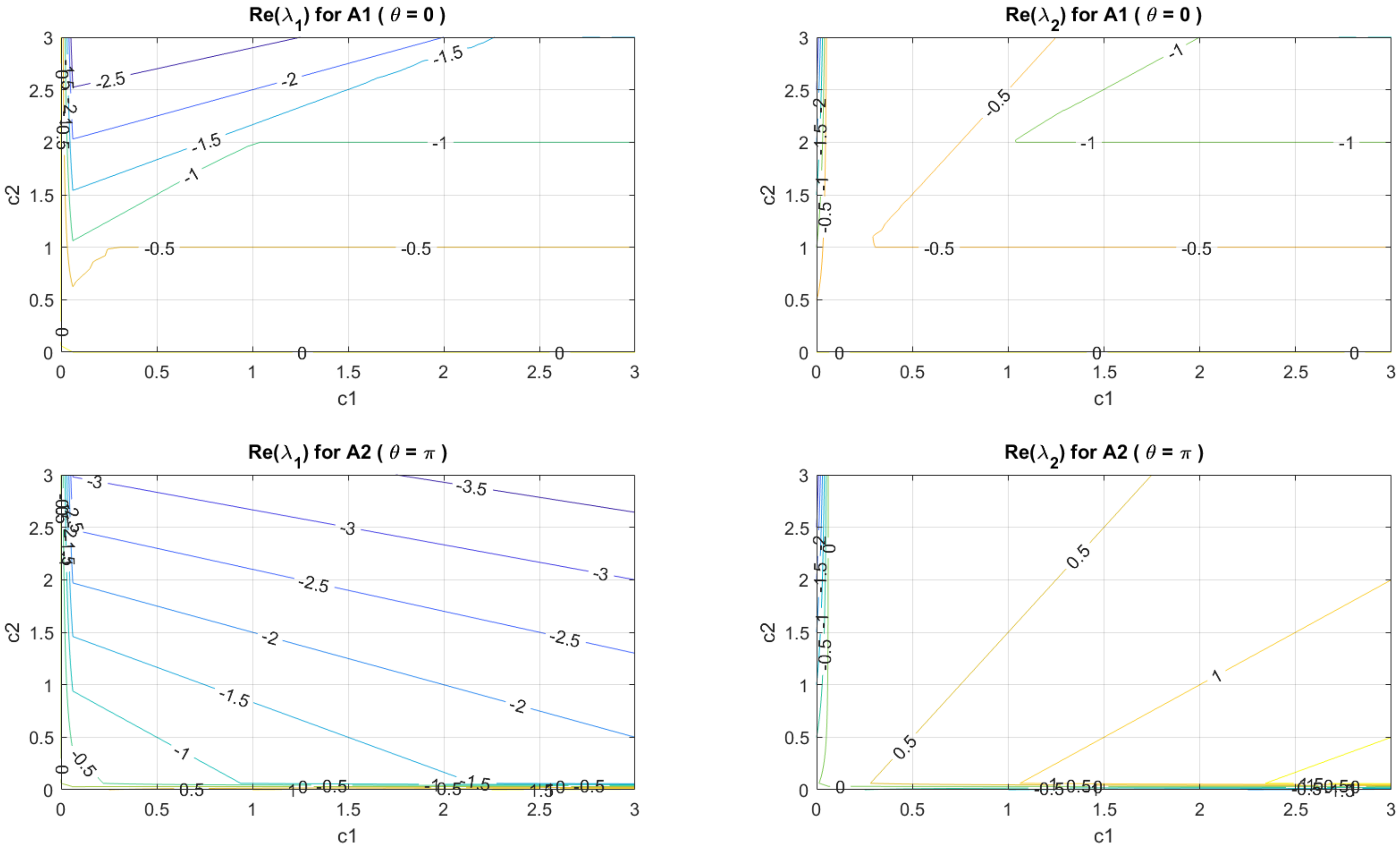
MATLAB code is provided on Blackboard

Example 1: Pendulum

- Below we show **four contour plots** that show how the real parts of the eigenvalues of A_1 and A_2 vary as a function of (c_1, c_2).

For $\mathrm{A}_1, \mathrm{Re}(\lambda_i) < 0$,
$\therefore$ $x_{*,1} \coloneqq (0,0)$ is locally asymptotically stable.

For $\mathrm{A}_2, \mathrm{Re}(\lambda_2) > 0$
$\therefore$ $x_{*,2} \coloneqq (\pi, 0)$ is unstable.



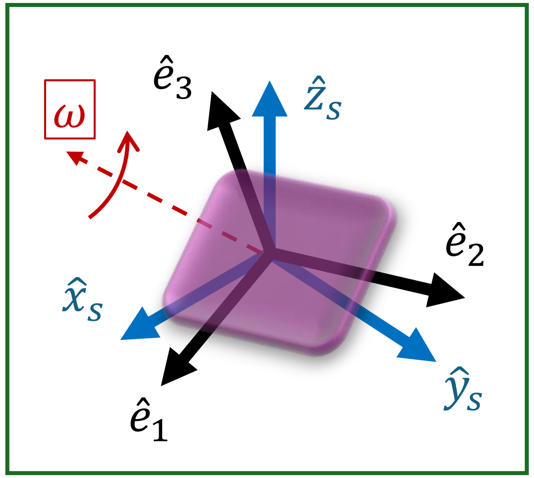
Example 2: Satellite

- Consider a rigid body with principal moments of inertia and angular velocity components

$$J = \mathrm{diag}(J_1, J_2, J_3), \quad \omega = (\omega_1, \omega_2, \omega_3)$$

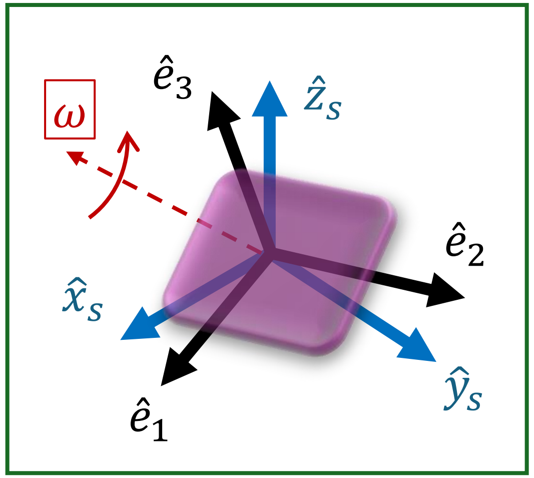
expressed in the principal axes.

- In the absence of external torques, The governing equations are given by
  - $\dot{R} = R\,\tilde{\omega}$
  - $\dot{\omega} = -J^{-1}(\omega \wedge J\omega)$



Example 2: Satellite

- In the absence of external torques, The governing equations are given by
  - $\dot{R} = R\,\widetilde{\omega}$
  - $\dot{\omega} = -J^{-1}(\omega \wedge J\omega)$
- Note that the second equation is independent of the configuration R, which implies that we can analyze the stability of the system by only considering the momentum balance equation.

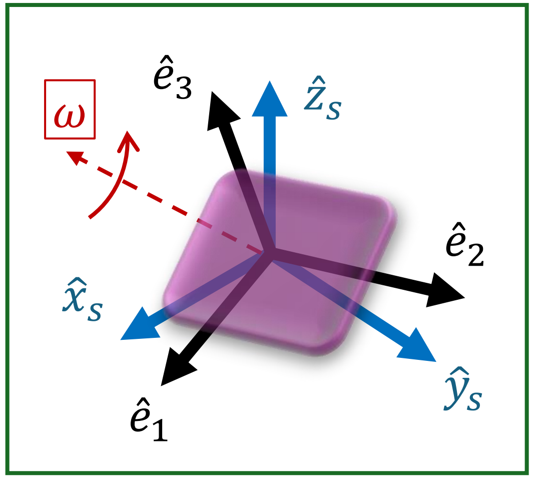


Example 2: Satellite

- We express the momentum balance $\dot{\omega} = -J^{-1}(\omega \wedge J\omega)$ as

$$\dot{\omega}_1 = \frac{J_2 - J_3}{J_1}\omega_2\omega_3$$
$$\dot{\omega}_2 = \frac{J_3 - J_1}{J_2}\omega_3\omega_1$$
$$\dot{\omega}_3 = \frac{J_1 - J_2}{J_3}\omega_1\omega_2$$

- An equilibrium point $\omega_* \in \mathbb{R}^3$ for the system is one for which $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$.



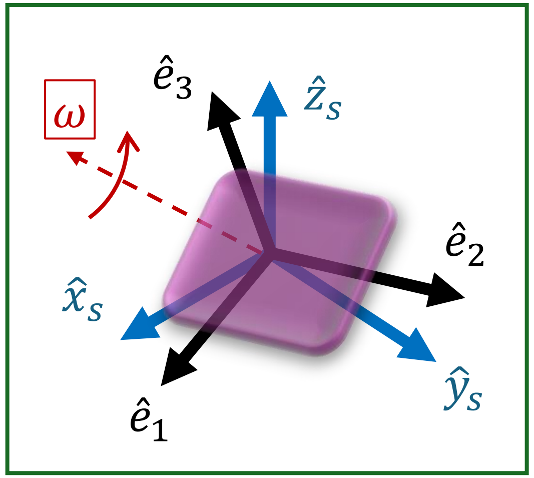
Example 2: Satellite

- The four equilibrium points of the nonlinear system

$$\dot{\omega}_1 = \frac{J_2 - J_3}{J_1}\omega_2\omega_3$$
$$\dot{\omega}_2 = \frac{J_3 - J_1}{J_2}\omega_3\omega_1$$
$$\dot{\omega}_3 = \frac{J_1 - J_2}{J_3}\omega_1\omega_2$$

are:

- **Trivial equilibrium**: $\omega_{*,a} = (0,0,0)$
- **Spin about $\hat{e}_1$ only**: $\omega_{*,b} = (\Omega, 0, 0), \quad \Omega \neq 0$
- **Spin about $\hat{e}_2$ only**: $\omega_{*,c} = (0, \Omega, 0), \quad \Omega \neq 0$
- **Spin about $\hat{e}_3$ only**: $\omega_{*,d} = (0, 0, \Omega), \quad \Omega \neq 0$



Example 2: Satellite

- To analyze the stability of these equilibrium points, we will assume that $J_1 > J_2 > J_3 > 0$, and thus we can rewrite the system as:

$$\boxed{\begin{aligned} \dot{\omega}_1 &= \frac{J_2 - J_3}{J_1}\omega_2\omega_3 \\ \dot{\omega}_2 &= \frac{J_3 - J_1}{J_2}\omega_3\omega_1 \\ \dot{\omega}_3 &= \frac{J_1 - J_2}{J_3}\omega_1\omega_2 \end{aligned}} \longrightarrow \boxed{\begin{aligned} \dot{\omega}_1 &= c_1\omega_2\omega_3 \\ \dot{\omega}_2 &= c_2\omega_3\omega_1 \\ \dot{\omega}_3 &= c_3\omega_1\omega_2 \end{aligned}}$$

with $c_1, c_3 > 0$, and $c_2 < 0$

Example 2: Satellite

- The Jacobian of the system is given by

$$\begin{aligned} \dot{\omega}_1 &= c_1\omega_2\omega_3 \\ \dot{\omega}_2 &= c_2\omega_3\omega_1 \\ \dot{\omega}_3 &= c_3\omega_1\omega_2 \end{aligned}$$

$$J_f(\omega) = \begin{pmatrix} \frac{\partial f_1}{\partial \omega_1}(\omega) & \frac{\partial f_1}{\partial \omega_2}(\omega) & \frac{\partial f_1}{\partial \omega_3}(\omega) \\ \frac{\partial f_2}{\partial \omega_1}(\omega) & \frac{\partial f_2}{\partial \omega_2}(\omega) & \frac{\partial f_2}{\partial \omega_3}(\omega) \\ \frac{\partial f_3}{\partial \omega_1}(\omega) & \frac{\partial f_3}{\partial \omega_2}(\omega) & \frac{\partial f_3}{\partial \omega_3}(\omega) \end{pmatrix} = \begin{pmatrix} 0 & c_1\omega_3 & c_1\omega_2 \\ c_2\omega_3 & 0 & c_2\omega_1 \\ c_3\omega_2 & c_3\omega_1 & 0 \end{pmatrix}$$

Example 2: Satellite

- By evaluating the Jacobian at each equilibrium point, we get the linearized systems

$$J_f(\omega) = \begin{pmatrix} 0 & c_1\omega_3 & c_1\omega_2 \\ c_2\omega_3 & 0 & c_2\omega_1 \\ c_3\omega_2 & c_3\omega_1 & 0 \end{pmatrix}$$

$$z_a := \omega - \omega_{*,a}$$
$$\dot{z}_a = J_f(\omega_{*,a}) z_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z_a$$

$$z_b := \omega - \omega_{*,b}$$
$$\dot{z}_b = J_f(\omega_{*,b}) z_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_2\Omega \\ 0 & c_3\Omega & 0 \end{pmatrix} z_b$$

$$z_c := \omega - \omega_{*,c}$$
$$\dot{z}_c = J_f(\omega_{*,c}) z_c = \begin{pmatrix} 0 & 0 & c_1\Omega \\ 0 & 0 & 0 \\ c_3\Omega & 0 & 0 \end{pmatrix} z_c$$

$$z_d := \omega - \omega_{*,d}$$
$$\dot{z}_d = J_f(\omega_{*,d}) z_d = \begin{pmatrix} 0 & c_1\Omega & 0 \\ c_2\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z_d$$

Example 2: Satellite

Recall $c_1, c_3 > 0$, and $c_2 < 0$

- By evaluating the Jacobian at each equilibrium point, we get the linearized systems

$$J_f(\omega) = \begin{pmatrix} 0 & c_1\omega_3 & c_1\omega_2 \\ c_2\omega_3 & 0 & c_2\omega_1 \\ c_3\omega_2 & c_3\omega_1 & 0 \end{pmatrix}$$

$$z_a \coloneqq \omega - \omega_{*,a}$$
$$\dot{z}_a = J_f(\omega_{*,a}) z_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z_a$$

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0$$

$$z_b \coloneqq \omega - \omega_{*,b}$$
$$\dot{z}_b = J_f(\omega_{*,b}) z_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_2\Omega \\ 0 & c_3\Omega & 0 \end{pmatrix} z_b$$

$$\lambda_1 = 0, \quad \lambda_2 = \Omega\sqrt{c_2 c_3}, \quad \lambda_3 = -\Omega\sqrt{c_2 c_3}$$

$$z_c \coloneqq \omega - \omega_{*,c}$$
$$\dot{z}_c = J_f(\omega_{*,c}) z_c = \begin{pmatrix} 0 & 0 & c_1\Omega \\ 0 & 0 & 0 \\ c_3\Omega & 0 & 0 \end{pmatrix} z_c$$

$$\lambda_1 = 0, \quad \lambda_2 = \Omega\sqrt{c_1 c_3}, \quad \lambda_3 = -\Omega\sqrt{c_1 c_3}$$

$$z_d \coloneqq \omega - \omega_{*,d}$$
$$\dot{z}_d = J_f(\omega_{*,d}) z_d = \begin{pmatrix} 0 & c_1\Omega & 0 \\ c_2\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z_d$$

$$\lambda_1 = 0, \quad \lambda_2 = \Omega\sqrt{c_1 c_2}, \quad \lambda_3 = -\Omega\sqrt{c_1 c_2}$$

Example 2: Satellite

Recall $c_1, c_3 > 0$, and $c_2 < 0$

- By evaluating the Jacobian at each equilibrium point, we get the linearized systems

$$J_f(\omega) = \begin{pmatrix} 0 & c_1\omega_3 & c_1\omega_2 \\ c_2\omega_3 & 0 & c_2\omega_1 \\ c_3\omega_2 & c_3\omega_1 & 0 \end{pmatrix}$$

$$z_a := \omega - \omega_{*,a}$$
$$\dot{z}_a = J_f(\omega_{*,a}) z_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z_a$$

$$\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = 0$$

Inconclusive

$$z_b := \omega - \omega_{*,b}$$
$$\dot{z}_b = J_f(\omega_{*,b}) z_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_2\Omega \\ 0 & c_3\Omega & 0 \end{pmatrix} z_b$$

$$\lambda_1 = 0 \quad \lambda_2 = \Omega\sqrt{c_2 c_3} \quad \lambda_3 = -\Omega\sqrt{c_2 c_3}$$

Inconclusive

$$z_c := \omega - \omega_{*,c}$$
$$\dot{z}_c = J_f(\omega_{*,c}) z_c = \begin{pmatrix} 0 & 0 & c_1\Omega \\ 0 & 0 & 0 \\ c_3\Omega & 0 & 0 \end{pmatrix} z_c$$

$$\lambda_1 = 0 \quad \lambda_2 = \Omega\sqrt{c_1 c_3} \quad \lambda_3 = -\Omega\sqrt{c_1 c_3}$$

Unstable

$$z_d := \omega - \omega_{*,d}$$
$$\dot{z}_d = J_f(\omega_{*,d}) z_d = \begin{pmatrix} 0 & c_1\Omega & 0 \\ c_2\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z_d$$

$$\lambda_1 = 0 \quad \lambda_2 = \Omega\sqrt{c_1 c_2} \quad \lambda_3 = -\Omega\sqrt{c_1 c_2}$$

Inconclusive

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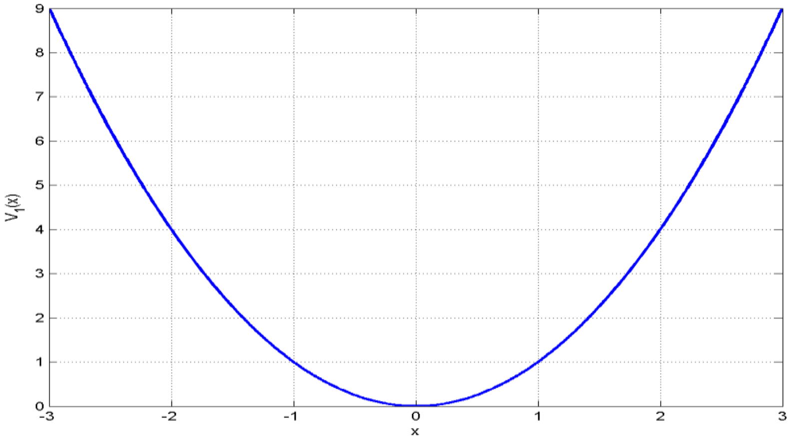
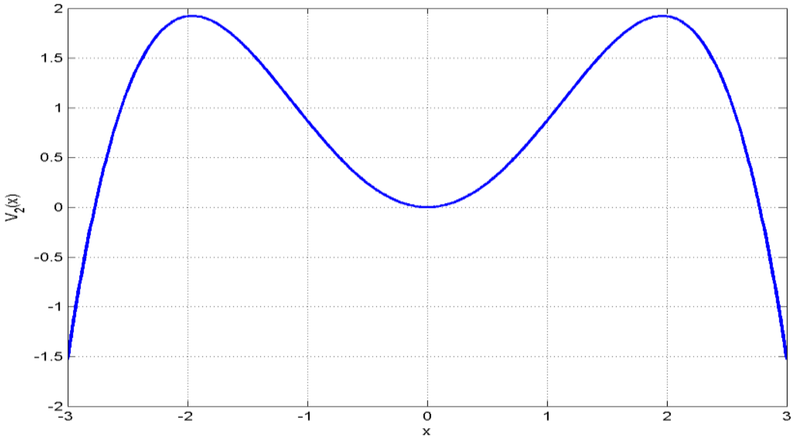
Positive Definite Functions

- Before discussing Lyapunov's direct method, we introduce the concept of a positive definite function.
- A function $V(\boldsymbol{x})$ is said to be positive definite it
  - $V(\mathbf{0}) = 0$
  - $V(\boldsymbol{x}) > 0 \quad \forall \boldsymbol{x} \neq \mathbf{0}$
- Examples $x \in \mathbb{R}$:
  - $V_1(x) = x^2$
  - $V_2(x) = x^2$ - 0.13 x^4

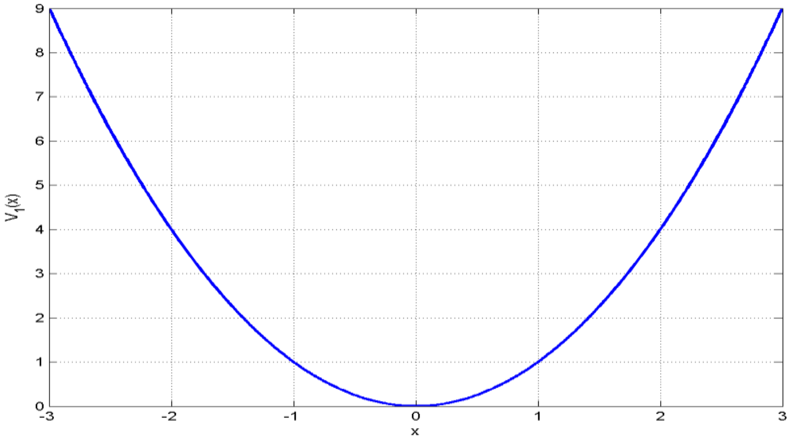


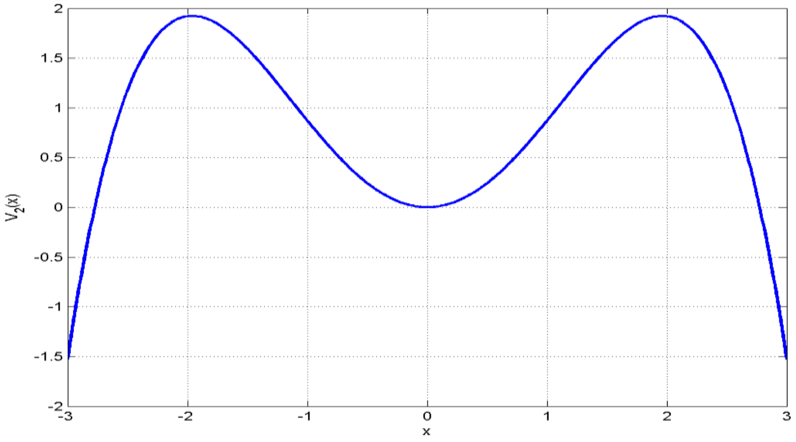


Positive Definite Functions

- Examples $x \in \mathbb{R}$:
  - $V_1(x) = x^2$
  - $V_1(0) = 0$
  - $V_1(x) > 0 \quad \forall x \neq 0$
  - Therefore $V_1(x)$ is positive definite **globally**

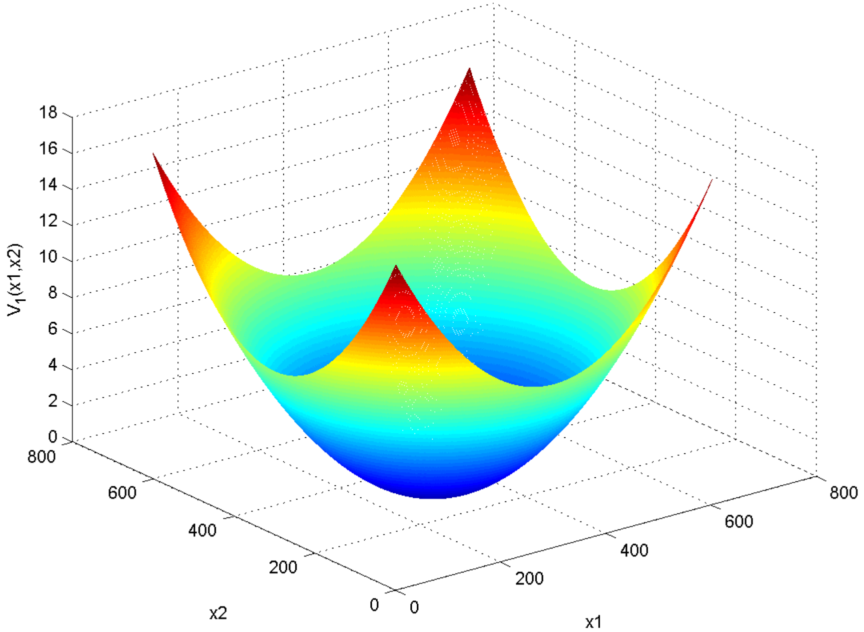
  - $V_2(x) = x^2$ - 0.13 x^4
  - $V_2(0) = 0$
  - $V_2(x) > 0 \quad (\text{but not } \forall x \neq 0)$
  - Therefore $V_2(x)$ is positive definite only **locally** in the region $|x| < 2.77$.

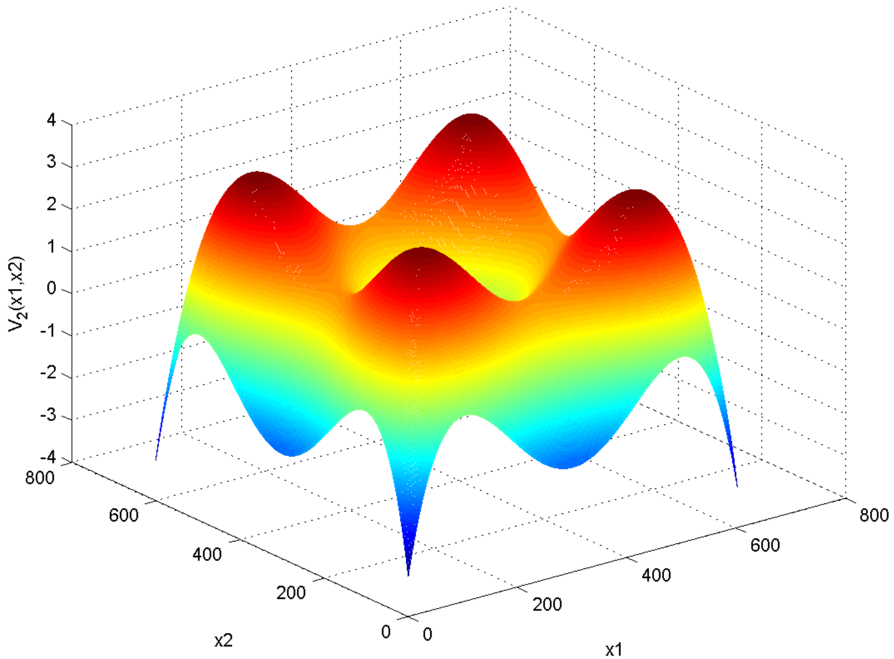




Positive Definite Functions

- Examples $x \in \mathbb{R}^2$:
  - $V_1(x) = x_1^2 + x_2^2$
  - $V_2(x) = x_1^2 + x_2^2 - 0.13(x_1^4 + x_2^4)$





Lyapunov's Direct Method

$$\dot{x} = f(x)$$

x^* is an Eq. point of the system

Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) \leq 0$

$\dot{V}(x)$ is negative semi-definite

Asymptotically Stable

Find a Lyapunov function $V(x)$ such that:

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$

$\dot{V}(x)$ is negative definite

Globally Asymptotically Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$
- $V(x) \to \infty$ as $\|x - x^*\| \to \infty$

$V(x)$ is radially unbounded

Outline

- Recap last lectures
- Application of Lyapunov’s indirect method
- Lyapunov’s direct method
- La Salle’s invariance principle

Motivation Example: Pendulum

- Consider the Pendulum dynamics with all parameters set to 1 for simplicity:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_2 - \sin x_1 \end{pmatrix}, \qquad x_1 = \theta, x_2 = \dot{\theta}$$

- Consider the Lyapunov function given by the total energy of the system:
  - $V(x) = \underbrace{\frac{1}{2}\dot{\theta}^2}_{\text{Kinetic Energy}} + \underbrace{(1 - \cos\theta)}_{\text{Potential Energy}}$
  - $\dot{V}(x) = \dot{\theta}\ddot{\theta} + \dot{\theta}\sin\theta = \dot{\theta}\left(-\dot{\theta} - \sin\theta\right) + \dot{\theta}\sin\theta = -\dot{\theta}^2$

$V(x) > 0$, but **not** radially unbounded

$\dot{V}(x) \leq 0$, therefore we can only conclude that the origin is stable but not asymptotically stable

Motivation

- In many physical or engineering systems (especially mechanical systems with dissipation), we might only show $\dot{V}(x) \leq 0$ (i.e., negative semidefinite).
- That tells us the system's "energy-like" quantity never increases, but it might stay constant for certain motions.
- Simply applying Lyapunov's theorem with negative semidefinite $\dot{V}(x)$ gives us stability but not necessarily asymptotic stability.

- One solution is to use the invariant set theorem attributed to J.P. LaSalle