SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 19: Lyapunov’s direct method II

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Outline

- Recap last lectures
- More on Positive Definiteness
- Geometric interpretation of $\dot{V}(x)$
- Application of Lyapunov’s direct method

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Recap: State Space Model

- A nonlinear dynamic system can be represented by a set of nonlinear differential equations in the form

$$\dot{x} = f(x) + g(x)\,u$$

$$y = h(x)$$

which is called the state space model of the dynamic system.

- We are focusing on analyzing the stability of the equilibrium points x_* systems of the form

$$\dot{x} = f(x)$$

<u>Special case:</u> Linear time-invariant systems

$$\dot{x} = Ax + Bu$$

$$y = C\,x$$

Recap: Lyapunov's indirect method

- The linearization of $\dot{x} = f(x)$ around the equilibrium point x_* is given by:

$$\dot{z} = A\,z$$

where $z := x - x_* \in \mathbb{R}^n$ and $A := J_f(x_*)$ is the Jacobian of $f(x)$ evaluated at the equilibrium points.

- **Stability Conditions:**

Compute $eig(A) := (\lambda_1, \cdots \lambda_n)$, with $\lambda_i \in \mathbb{C}$

→ Is there any λ_i with $\mathrm{Re}(\lambda_i) > 0$? → x_* is **unstable** for the nonlinear system.

→ Is there any λ_i with $\mathrm{Re}(\lambda_i) = 0$? → Lyapunov's indirect method is inconclusive

→ Do all λ_i have $\mathrm{Re}(\lambda_i) < 0$? → x_* is **locally asymptotically stable** for the nonlinear system.

Recap: Lyapunov’s Direct Method

$$\dot{x} = f(x)$$

x^* is an Eq. point of the system

Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) \leq 0$

$\dot{V}(x)$ is negative semi-definite

$V(x)$ is positive definite

Asymptotically Stable

Find a Lyapunov function $V(x)$ such that:

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$

$\dot{V}(x)$ is negative definite

Globally Asymptotically Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$
- $V(x) \rightarrow \infty$ as $\|x - x^*\| \rightarrow \infty$

$V(x)$ is radially unbounded

Recap: Pendulum Case study

- **State space model:**

  - $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -c_2 x_2 - c_1 \sin x_1 \end{pmatrix}, \quad c_1, c_2 > 0$

Equilibrium Points:

$x_{*,1} \coloneqq (0,0)$, $x_{*,2} \coloneqq (\pi, 0)$

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- **Lyapunov's indirect method:**
  - $A_1 \coloneqq J_f(x_{*,1}) = \begin{pmatrix} 0 & 1 \\ -c_1 & -c_2 \end{pmatrix}$

    For $\mathrm{A}_1, \mathrm{Re}(\lambda_i) < 0, \forall c_1, c_2 > 0$
    $\therefore$ $x_{*,1} \coloneqq (0,0)$ is locally asymptotically stable.

  - $A_2 \coloneqq J_f(x_{*,2}) = \begin{pmatrix} 0 & 1 \\ c_1 & -c_2 \end{pmatrix}$

    For $\mathrm{A}_2, \mathrm{Re}(\lambda_2) > 0,, \forall c_1, c_2 > 0$
    $\therefore$ $x_{*,2} \coloneqq (\pi, 0)$ is unstable.

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  - $A_1 \coloneqq J_f(x_{*,1}) = \begin{pmatrix} 0 & 1 \\ -c_1 & -c_2 \end{pmatrix}$

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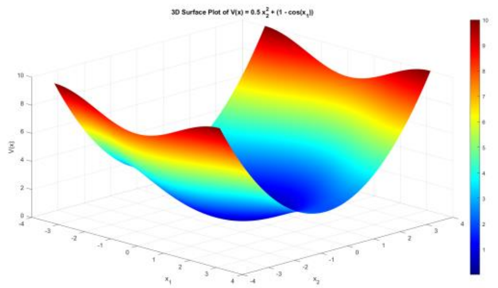
  - $A_2 \coloneqq J_f(x_{*,2}) = \begin{pmatrix} 0 & 1 \\ c_1 & -c_2 \end{pmatrix}$

    For $A_2, \mathrm{Re}(\lambda_2) > 0,, \forall c_1, c_2 > 0$
    $\therefore x_{*,2} \coloneqq (\pi, 0)$ is unstable.

- **Lyapunov's direct method:**
  - $V(x) = \frac{1}{2} x_2^2 + c_1(1 - \cos x_1)$

    $V(x_{*,1}) = 0, \quad V(x) > 0 \;\forall\, x \in \mathcal{X}/\{x_{*,1}\}$
  - $\dot{V}(x) = -c_2 x_2^2 \leq 0$

    $\dot{V}(x)$ is negative semi-definite
    $\therefore x_{*,1} \coloneqq (0,0)$ is stable.



Recap: Satellite Case study

- **State space model:**
  - $\dot{\omega} = -J^{-1}(\omega \wedge J\omega)$
  - $\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} c_1\omega_2\omega_3 \\ c_2\omega_3\omega_1 \\ c_3\omega_1\omega_2 \end{pmatrix}$, $c_1, c_3 > 0$, and $c_2 < 0$

- **Equilibrium Points:**
- $\omega_{*,a} = (0,0,0)$
- $\omega_{*,b} = (\Omega, 0, 0)$
- $\omega_{*,c} = (0, \Omega, 0)$
- $\omega_{*,d} = (0, 0, \Omega)$

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- **Equilibrium Points:**
- $\omega_{*,a} = (0,0,0)$
- $\omega_{*,b} = (\Omega, 0, 0)$
- $\omega_{*,c} = (0, \Omega, 0)$
- $\omega_{*,d} = (0, 0, \Omega)$

- **Lyapunov's indirect method:**

$$A_a = J_f(\omega_{*,a}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_c = J_f(\omega_{*,c}) = \begin{pmatrix} 0 & 0 & c_1\Omega \\ 0 & 0 & 0 \\ c_3\Omega & 0 & 0 \end{pmatrix}$$

$$A_b = J_f(\omega_{*,b}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_2\Omega \\ 0 & c_3\Omega & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= \Omega\sqrt{c_1 c_3} > 0 \\ \lambda_3 &= -\Omega\sqrt{c_1 c_3} < 0 \end{aligned}}$$

$$A_d = J_f(\omega_{*,d}) = \begin{pmatrix} 0 & c_1\Omega & 0 \\ c_2\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore \omega_{*,b} = (\Omega, 0, 0)$ is unstable.

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Positive definite functions

- A function $\psi: \mathcal{X} \to \mathbb{R}$ is locally positive definite about $x_* \in \mathcal{X}$ if:
  - $\psi\,(x_*) = 0$
  - There exists a neighborhood $\mathcal{U} \subset \mathcal{X}$ that includes x_* (i.e., $x_* \in \mathcal{U}$) such that $\psi(x) > 0\ \forall\ x \in \mathcal{U} \backslash \{x_*\}$.

- $\psi : \mathcal{X} \to \mathbb{R}$ is locally positive semi-definite if $\psi(x) \geq 0\ \forall\ x \in \mathcal{U} \backslash \{x_*\}$.
- $\psi : \mathcal{X} \to \mathbb{R}$ is globally positive definite if $\psi(x) > 0\ \forall\ x \in \mathcal{X} \backslash \{x_*\}$.
- $\psi : \mathcal{X} \to \mathbb{R}$ is locally negative (semi-)definite if $-\psi(x)$ is locally positive (semi-)definite.

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- In other words, x_* is a strict local minimum of $\psi(x)$, and the function increases in every direction away from x_* within $\mathcal{U}$.
- Recall the classic result of calculus:
  - If ψ is differentiable, and x_* is a local minimum, then the gradient must vanish there i.e., $J_\psi(x_*) = 0$.
- We call points $x_* \in \mathcal{X}$ a critical zero for the smooth function $\psi: \mathcal{X} \rightarrow \mathbb{R}$ if $\psi\,(x_*) = 0$ and $J_\psi(x_*) = 0$.

Positive Definite Matrices

- A real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called a positive definite matrix if we have that

$$x^\top A\, x > 0\,, \quad \forall x \in \mathbb{R}^n \backslash \{0\}.$$

- $A \in \mathbb{R}^{n\times n}$ is called a positive semi-definite matrix if $x^\top A\, x \geq 0$.
- $A \in \mathbb{R}^{n\times n}$ is called a negative definite matrix if $x^\top A\, x < 0$.
- $A \in \mathbb{R}^{n\times n}$ is called a negative semi-definite matrix if $x^\top A\, x \leq 0$.

- Key property:
  - $A \succ 0$ is positive definite if and only if all its eigenvalues are positive.
  - $A \succcurlyeq 0$ is positive semi-definite if and only if all its eigenvalues are non-negative.
  - $A \prec 0$ is negative definite if and only if all its eigenvalues are negative.
  - $A \preccurlyeq 0$ is negative definite if and only if all its eigenvalues are non-positive.

Using Taylor Expansion for Positive Definiteness

- Suppose you have a smooth scalar function $V\colon \mathbb{R}^n \to \mathbb{R}$, and you want to check whether it's positive definite around $x_* \in \mathbb{R}^n$.
- You can expand $\mathrm{V}(x)$ as a Taylor series of order 2:

$$V(x) = V(x_*) + J_V(x_*)\,(x - x_*) + \frac{1}{2}(x - x_*)^\top H_V(x_*)(x - x_*) + \ \ldots$$

where $H_V(x_*) \in \mathbb{R}^{n\times n}$ is called the Hessian matrix with the entry of the ith row and the jth column is $(H_V)_{ij} \coloneqq \frac{\partial^2 V}{\partial x_i \partial x_j}\colon \mathbb{R}^n \to \mathbb{R}$.

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where $H_V(x_*) \in \mathbb{R}^{n\times n}$ is called the Hessian matrix with the entry of the ith row and the jth column is $(H_V)_{ij} \coloneqq \frac{\partial^2 V}{\partial x_i \partial x_j}: \mathbb{R}^n \to \mathbb{R}$.

- However, since x_* is a critical zero of V, the function is locally approximated by

$$\boxed{V(x) \approx \frac{1}{2}(x - x_*)^\top H_V(x_*)(x - x_*)}$$

Using Taylor Expansion for Positive Definiteness

$$\boxed{V(x) \approx \frac{1}{2}(x - x_*)^\top H_V(x_*)(x - x_*)}$$

- Therefore, we have that $V(x)$ is a locally positive definite function if and only if the Hessian $H_V(x_*)$ is a positive definite matrix, which can be assessed from its eigenvalues.

- This result is a basic result from what is known as Morse theory.

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