# SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 20: Geometric PD Control I

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# Outline

- Recap last lectures
- La Salle’s Invariance Principle
- PD Control of a Point Mass on $\mathbb{R}^3$
- Energy balancing formulation

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# Recap: State Space Model

- A nonlinear dynamic system can be represented by a set of nonlinear differential equations in the form

$$\dot{x} = f(x) + g(x)\, u$$
$$y = h(x)$$

which is called the state space model of the dynamic system.

- We are focusing on analyzing the stability of the equilibrium points $x_*$ systems of the form

$$\dot{x} = f(x)$$

**<u>Special case:</u>** Linear time-invariant systems

$$\dot{x} = Ax + Bu$$
$$y = C\, x$$

# Recap: Lyapunov's Direct Method

$$\dot{x} = f(x)$$

$x^*$ is an Eq. point of the system

## Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) \leq 0$

$\dot{V}(x)$ is negative semi-definite

$V(x)$ is positive definite

## Asymptotically Stable

Find a Lyapunov function $V(x)$ such that:

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$

$\dot{V}(x)$ is negative definite

## Globally Asymptotically Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$
- $V(x) \rightarrow \infty$ as $\|x - x^*\| \rightarrow \infty$

$V(x)$ is radially unbounded

# Recap: Local Definiteness

$$\boxed{V(x) \approx \frac{1}{2}(x - x_*)^\top H_V(x_*)(x - x_*)}$$

- Therefore, we have that $V(x)$ is a locally positive definite function if and only if the Hessian $H_V(x_*)$ is a positive definite matrix, which can be assessed from its eigenvalues.

- This result is a basic result from what is known as Morse theory.

# Recap: Computing $\dot{V}(x)$

- We have in general for $x \in \mathcal{X}$ that $\dot{x} \in T_x\mathcal{X}$, with $\dot{x} = f(x)$
- Therefore, we have

$$\dot{V}(x) = \mathcal{L}_{\sigma_f} V\ (x) = \langle dV(x) | f(x) \rangle = \ \langle dV(x) | \dot{x} \rangle$$

where $dV(x) \in T_x^*\mathcal{X}$ and

$$\langle \cdot \mid \cdot \rangle : T_x^*\mathcal{X} \times T_x\mathcal{X} \ \to \mathbb{R}$$

is called the duality product on tangent spaces of $\mathcal{X}$.

$dV$ is called the differential of the scalar function $V$

# Recap: Computing $\dot{V}(x)$

- For the case $\mathcal{X} = \mathbb{R}^n$ we have that

$$\dot{V}(x) = \langle dV(x) | \dot{x} \rangle = dV(x)\dot{x}$$

where $dV(x) \in (\mathbb{R}^n)^*$ is given by

$$dV(x) = \left( \frac{\partial V}{\partial x_1}(x), \cdots, \frac{\partial V}{\partial x_n}(x) \right) \in \mathbb{R}^{1\times n}$$

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- It is more common to write the above using the gradient vector

$$\nabla V(x) = \begin{pmatrix} \frac{\partial V}{\partial x_1}(x) \\ \vdots \\ \frac{\partial V}{\partial x_n}(x) \end{pmatrix} \in \mathbb{R}^n$$

The differential and gradient are dual to each other
$dV(x) = \nabla V^\top(x)$

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# Introduction

- LaSalle’s Invariance Principle is a fundamental result in the stability analysis of dynamical systems.
- It generalizes Lyapunov’s direct method to handle the case where a Lyapunov-like function’s time derivative $\dot{V}(x)$ is only negative semi-definite rather than strictly negative.

# Invariant set

- Before discussing LaSalle’s Invariance principle, we introduce the concept of an invariant set.
- **Definition:**
  - A set $M$ is an invariant set for a dynamic system if every system trajectory which starts from a point in $M$ remains in $M$ for all future time.
- Examples of invariant sets:
  - Equilibrium points
  - Domain of attraction of an equilibrium point
  - The whole state space

# La Salle's Invariance Principle

- Let $x_* \in \mathcal{X}$ be an equilibrium point of the dynamical system

$$\dot{x}(t) = f\big(x(t)\big).$$

Assume there exists a smooth Lyapunov function $V: \mathcal{X} \to \mathbb{R}$ such that in some neighborhood $\Omega \subset \mathcal{X}$ of $x_*$, we have that

  - $V$ is positive definite
  - $\dot{V}$ is negative semi-definite
  - Let $R \coloneqq \{x \in \Omega \mid \dot{V}(x) = 0\} \subset \Omega$ and let $M \subset R$ be the largest invariant set in it.

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- If $M$ contains only the equilibrium point (i.e., $M = \{x_*\}$), then $x_*$ is locally asymptotically stable with its domain of attraction defined by

$$\Omega_l \coloneqq \{x \in \Omega \mid V(x) < l\} \subset \Omega.$$

# La Salle's Invariance Principle

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- If $M$ contains only the equilibrium point (i.e., $M = \{x_*\}$), then $x_*$ is locally asymptotically stable with its domain of attraction defined by
$$\Omega_l \coloneqq \{x \in \Omega \mid V(x) < l\} \subset \Omega.$$

This result is properly called **Barbashin-Krasovskii-LaSalle principle !**

# Recap: Pendulum Case study

- **State space model:**
  - $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -c_2 x_2 - c_1 \sin x_1 \end{pmatrix}, \quad c_1, c_2 > 0$

**Equilibrium Points:**
$x_{*,1} \coloneqq (0,0)$ , $x_{*,2} \coloneqq (\pi, 0)$

- **Lyapunov's direct method:**
  - $V(x) = \frac{1}{2}x_2^2 + c_1(1 - \cos x_1)$
  - $\dot{V}(x) = -c_2 x_2^2 \leq 0$

$V(x_{*,1}) = 0, \quad V(x) > 0 \ \forall\, x \in \mathcal{X}/\{x_{*,1}\}$

$\dot{V}(x)$ is negative semi-definite $\therefore$ $x_{*,1} \coloneqq (0,0)$ is stable according to Lyapunov's direct method.

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**Equilibrium Points:**

$x_{*,1} \coloneqq (0,0)$ , $x_{*,2} \coloneqq (\pi, 0)$

- **Further analysis:**

1. Start with $\Omega = \mathcal{X}$
2. Define the set $R \coloneqq \{x \in \Omega \mid \dot{V}(x) = 0\}$
   - $R = \{(x_1, x_2) \in \mathcal{X} \mid x_2 = 0\}$

$V(x) = \frac{1}{2} x_2^2 + c_1(1 - \cos x_1)$

$\dot{V}(x) = -c_2 x_2^2 \leq 0$

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   - $R = \{(x_1, x_2) \in \mathcal{X} \mid x_2 = 0\}$
3. Define the points that form the invariant set in $R$
   - i.e., points that have $x_2$ equal to zero
   - $\dot{x}_2 = 0 = c_2 x_2 + c_1 \sin x_1 \implies \sin x_1 = 0$
   - Thus, the invariant set $M$ is given by $M = \{x_{*,1}, x_{*,2}\}$

$$V(x) = \frac{1}{2} x_2^2 + c_1(1 - \cos x_1)$$

$$\dot{V}(x) = -c_2 x_2^2 \leq 0$$

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4. Compute level sets of $V(x)$
   - $V\left(x_{*,1}\right) = \frac{1}{2}\,(0)^2 + c_1(1 - \cos 0) = 0$
   - $V\left(x_{*,2}\right) = \frac{1}{2}\,(0)^2 + c_1(1 - \cos \pi) = 2\,c_1$

$$V(x) = \frac{1}{2}x_2^2 + c_1(1 - \cos x_1)$$

$$\dot{V}(x) = -c_2 x_2^2 \leq 0$$

# Recap: Pendulum Case study

**Equilibrium Points:**

$x_{*,1} \coloneqq (0,0)$ , $x_{*,2} \coloneqq (\pi, 0)$

- **<u>Further analysis:</u>**

5. If we consider only the region
    - $\Omega = \Omega_{2c_1} = \{x \in \mathcal{X} \,|\, V(x) < 2\, c_1\}$
    - Refine $R = \left\{(x_1, x_2) \in \Omega_{2c_1} \middle|\, x_2 = 0\right\}$
    - The largest invariant set in $R \subset \Omega_{2c_1}$ becomes $M = \{x_{*,1}\}$

- Therefore, using La Salle's Invariance Principle, every solution originating in the region $\Omega_{2c_1}$ will tend to $\{x_{*,1}\}$ as $t \to \infty$.
- Therefore, $x_{*,1} \coloneqq (0,0)$ is locally asymptotically stable.

$$V(x) = \frac{1}{2}x_2^2 + c_1(1 - \cos x_1)$$

$$\dot{V}(x) = -c_2 x_2^2 \leq 0$$

# Summary

- Lyapunov's direct method:
  - If you have $\dot{V}(x) < 0$ (negative definite), you can conclude asymptotic stability of that equilibrium.

- LaSalle's Invariance Principle:
  - Works with $\dot{V}(x) \leq 0$ (negative semi-definite).
  - This weaker condition only implies that the trajectory stays in or approaches the set $R$ where $\dot{V}(x)$.
  - You then must check what the motion does within $R$ to conclude asymptotic convergence.

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# PD Control of Point Mass

- To provide some intuition, let's start in a simple Euclidean space $\mathbb{R}^n$.
- The governing equations of a point mass (with no gravity) are:
  - $\dot{\xi} = v, \qquad \dot{v} = \frac{1}{m} u$

where $\xi, v, u \in \mathbb{R}^3$ denote the position, velocity and control forces.

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where $\xi, v, u \ \in \mathbb{R}^3$ denote the position, velocity and control forces.

- We can rewrite it as the state space model
  - $x = (\xi, p) = (\xi, mv) \in \mathbb{R}^3 \times \mathbb{R}^3$
  - $\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_3 \end{pmatrix} u \qquad\qquad \Rightarrow \dot{x} = f(x) + g\, u$

  with $v = \frac{p}{m}$.

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  with $v = \frac{p}{m}$.

Note that the total energy of the system is given by
$H(p) = \frac{1}{2m} p^\top p = \frac{1}{2}\, m v^\top v$

# PD Control of Point Mass

- Our control objective is to stabilize the system at the desired state
$$x_d = (\xi_d, 0)$$
- A classical proportional-derivative (PD) controller can often be written as
  - $e_\xi \coloneqq \xi - \xi_d \in \mathbb{R}^3$
  - $u = -K_p\, e_\xi - K_d \dot{e}_\xi$
  $= -K_p\,(\xi - \xi_d) - K_d v \qquad \Rightarrow u = \gamma(x)$

where $K_p$ and $K_d$ are positive-definite gain matrices

# PD Control of Point Mass

- Closed loop system can be written as

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -K_p\,(\xi - \xi_d) - K_d v \end{pmatrix} \qquad \Rightarrow \dot{x} = f(x) + g\,\gamma(x)$$

We will show later the stability of this system using Lyapunov's direct method

# Moving to a "Geometric" Setting

- It is not straightforward to extend such PD controller to a non-Euclidean space.
- For example, for the satellite problem we have that

$$x = (R, \omega) \in SO(3) \times \mathbb{R}^3$$

- If we wish to stabilize the system at the desired state $x_d = (R_d, 0)$, one cannot simply compute

$$u = -K_p\, e_R \; - K_d \dot{e}_R$$

with $e_R = R \; - R_d \notin SO(3)$ and $\dot{e}_R = \dot{R} \in T_R\, SO(3)$ to compute the control torques $u \in (\mathbb{R}^3)^*$.

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# Reformulating PD Control on $\mathbb{R}^3$

- Our starting point to develop a geometric PD controller is to express the proportional term as the gradient of some potential function.
- For the PD controller

$$u = u_p + u_d = -K_p\, e_\xi \, - K_d \dot{e}_\xi$$

we can view the proportional part as

$$u_p = -\nabla\Psi(\xi) = -K_p\, e_\xi$$

if we pick

$$\Psi(\xi) \coloneqq \frac{1}{2}(\xi \, - \xi_d)^\top K_p(\xi \, - \xi_d)$$

which is a (global) positive definite function of $\xi \in \mathbb{R}^3$.

Recall:
$\nabla\Psi^\top(e_\xi) = d\Psi(e_\xi)$

# Reformulating PD Control on $\mathbb{R}^3$

- Such interpretation allows us to perform Lyapunov analysis of the closed loop system easily.
- We can choose

$$\begin{aligned} V(x) &= \Psi(\xi) + H(p) \\ &= \frac{1}{2}(\xi - \xi_d)^\top K_p (\xi - \xi_d) + \frac{1}{2m} p^\top p \end{aligned}$$

Closed loop system

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla\Psi(\xi) + u_d \end{pmatrix}$$

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- We have that $V(x_d) = V(\xi_d, 0) = 0$, and its Hessian given by

$$H_V(x_d) = \begin{pmatrix} K_p & 0_{3\times3} \\ 0_{3\times3} & \frac{1}{m} I_3 \end{pmatrix} \succ 0 \,,$$

- Therefore, $V(x)$ is globally positive definite.

Closed loop system

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla\Psi(\xi) + u_d \end{pmatrix}$$

# Reformulating PD Control on $\mathbb{R}^3$

- Lyapunov function

$$V(x) = \Psi(\xi) + H(p)$$

- The time derivative $\dot{V}(x)$ along trajectories of the closed loop system can be written as

$$\begin{aligned}
\dot{V}(x) &= \langle dV(x) \mid \dot{x} \rangle_{\mathbb{R}^6} \\
&= \left\langle d\Psi(\xi) \middle| \, \dot{\xi} \right\rangle_{\mathbb{R}^3} + \langle d\mathrm{H}(p) | \, \dot{p} \rangle_{\mathbb{R}^3} \\
&= \dot{\xi}^{\top} \nabla \Psi(\xi) + \nabla \mathrm{H}^{\top}(p) \, \dot{p} \\
&= v^{\top} \, \nabla \Psi(\xi) + v^{\top} \, [-\nabla \Psi(\xi) + u_d \,] \\
&= v^{\top} u_d
\end{aligned}$$

Closed loop system

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# Reformulating PD Control on $\mathbb{R}^3$

- Lyapunov function

$$V(x) = \Psi(\xi) + H(p)$$

- If we choose $u_d = -K_d v$, we have hat

$$\dot{V}(x) = v^\top u_d = -v^\top K_d v \leq 0$$

- Using La Salle's invariance principle, it follows that $x_d = (\xi_d, 0)$ is a globally asymptotically stable equilibrium point of the closed loop system.

Closed loop system

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla\Psi(\xi) + u_d \end{pmatrix}$$

# Energy-balancing interpretation of PD Control on $\mathbb{R}^3$

- The PD controller

$$u = u_p + u_d$$

can be interpreted as a sum of an energy-shaping term $u_p$ and a damping injection term $u_d$.

- For a chosen locally positive definite function $\Psi(\xi)$ designed such that $\xi_d$ is a minimum, one has that $u_p = -\nabla\Psi(\xi)$ which yields $\dot{V}(x) = v^\top u_d$ .

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- Choosing $u_d = \gamma(v)$ to inject damping such that $\dot{V}(x) \leq 0,$ one has with La Salle's invariance principle that $x_d = (\xi_d, 0)$ is locally asymptotically stable.
- If $\Psi(\xi)$ has $\xi_d$ to be a global minimum, then $x_d = (\xi_d, 0)$ is globally asymptotically stable.