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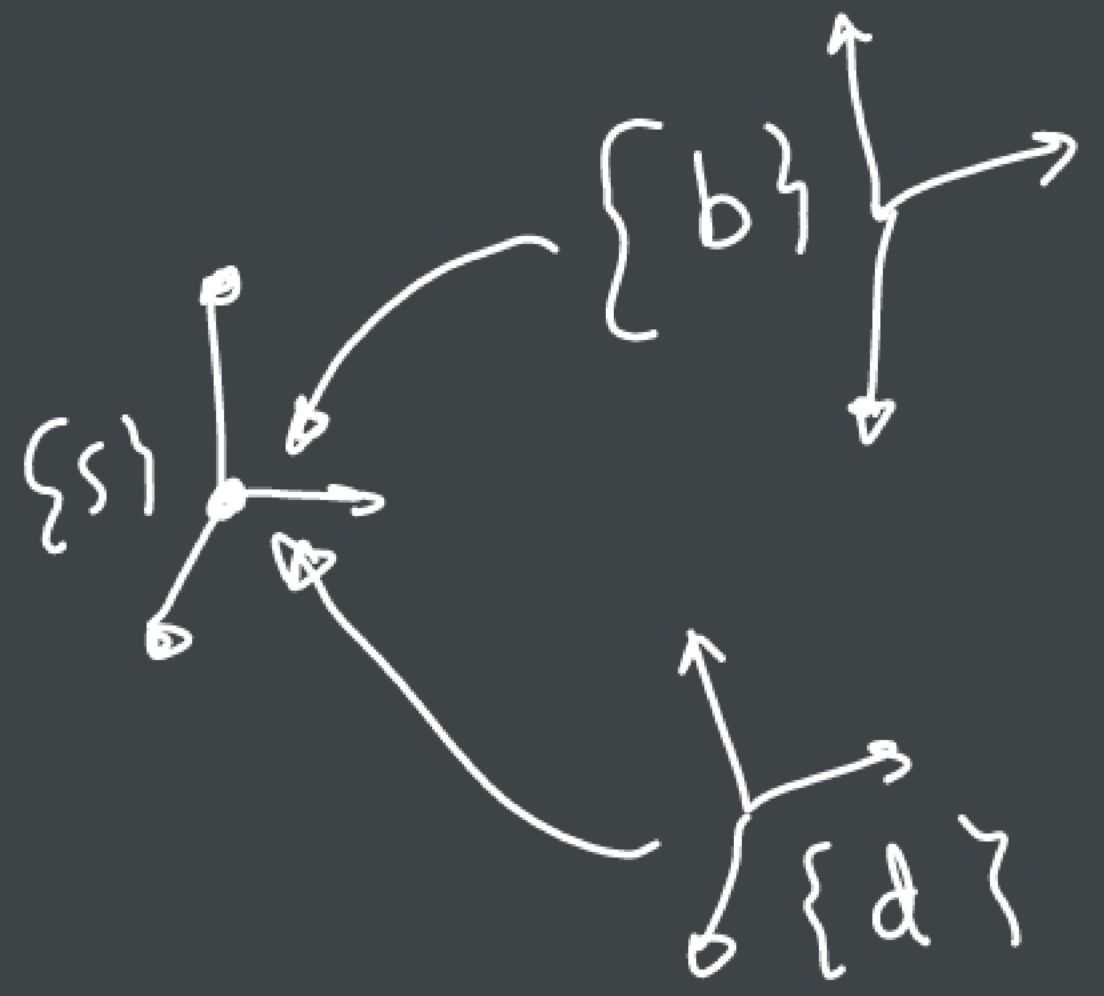
1) Error between $R, R_d \in SO(3)$:

$$R \equiv R^s_b \quad , \quad R_d \equiv R^s_d$$



Error

* $R^d_b = R^d_s R^s_b = (R^s_d)^T R^s_b$

$$\boxed{R_e := R_d^T R} \qquad R_e \to I \quad \text{as } t \to \infty$$

2) Potential function: $\Psi : SO(3) \to \mathbb{R}$

$$\boxed{\Psi(R_e) = \frac{1}{2}\, tr(I - R_e)}$$

or

$$\boxed{\Psi(R) = \frac{1}{2}\, tr(I - R_d^T R)}$$

$\boxed{\begin{matrix} R_e = I \\ \theta = 0 \end{matrix}}$

$$\Psi(R_e = I) = \frac{1}{2}\, tr(I - I) = 0$$

$$\Psi(R_e) = \frac{1}{2} \underbrace{tr(I)}_{=3} - \frac{1}{2} tr(R_e)$$

$$= 1 - \cos\theta \implies \boxed{0 < \Psi(R_e) < 2}$$

$tr(\lambda\Sigma)$

$\lambda\, tr(\Sigma)$

Recall $\mathbb{R}^3$

$$\Psi(e_\xi) = \frac{1}{2} e_\xi^T K_p e_\xi$$

$$\Psi(\xi) = \frac{1}{2}(\xi - \xi_d)^T K_p (\xi - \xi_d)$$

$(\theta, \hat{n})$

$$R_e = I + \sin\theta\, \tilde{n} + (1-\cos\theta)\, \tilde{n}^2$$

$$tr(R_e) = 3 + 0 + (1-\cos\theta)(-2)$$

$$= 1 + 2\cos\theta$$

$$\Psi(R_e) = \frac{1}{2}(3 - 1 - 2\cos\theta)$$

$$= 1 - \cos\theta \quad \#$$

3) Differential of $\Psi(R)$: $\Psi: SO(3) \to \mathbb{R}$

a) Diff. of a function on a vector space W:

let $\Psi: W \longrightarrow \mathbb{R}$
$x \longmapsto \Psi(x)$

The diff. of Ψ is the unique covector $d\Psi(x) \in W^*$ that satisfies:

$$\langle d\Psi(x) \mid \delta x \rangle_W = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Psi(x + \epsilon\,\delta x) \quad , \forall\ \delta x \in W$$

b) Example : $x \in \mathbb{R}^n$ $\quad K_p = K_p^T$

$$\langle d\psi(x) \mid \delta x \rangle_{\mathbb{R}^n} = d\psi(x) . \delta x$$

For $\psi(x) = \frac{1}{2} x^T K_p x$

$$d\psi(x) \in (\mathbb{R}^n)^*$$

$$\psi(x + \epsilon \delta x) = \frac{1}{2} (x + \epsilon \delta x)^T K_p (x + \epsilon \delta x) = \underbrace{\frac{1}{2} x^T K_p x}_{=0} + \epsilon \underbrace{\delta x^T K_p x}_{} + \frac{1}{2} \epsilon^2 \underbrace{\delta x^T K_p \delta x}_{=0}$$

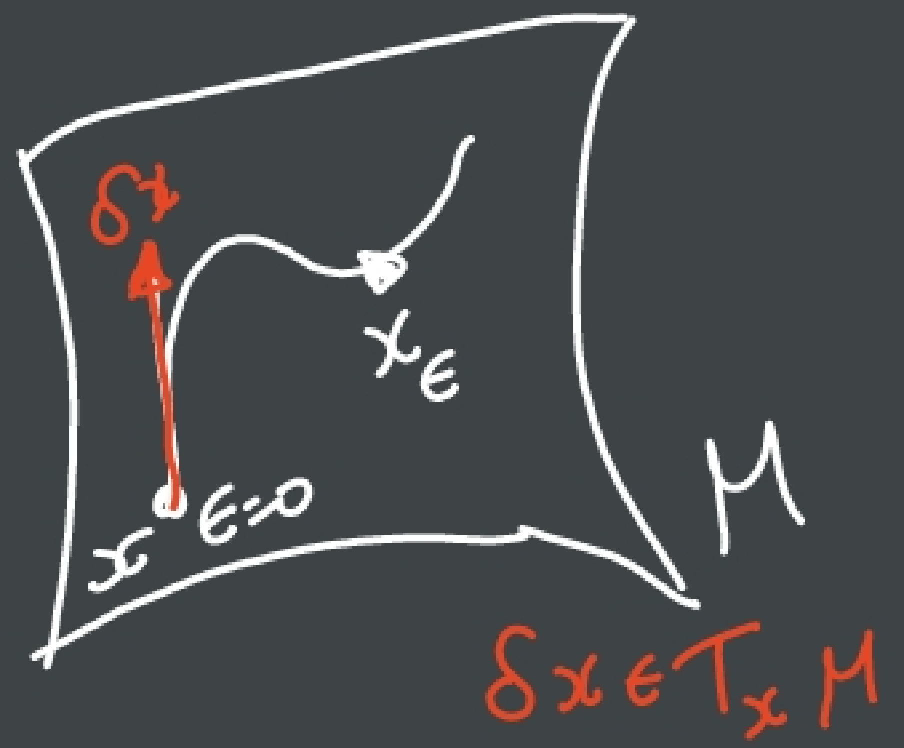
$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (\;) = \delta x^T K_p x = (K_p x)^T \delta x$$

By comparison of RHS & LHS $\quad \boxed{d\psi(x) = (K_p x)^T}$

c) Compute diff. of a function on a manifold: M

Let $\Psi : M \to \mathbb{R}$

$\quad\quad x \mapsto \Psi(x)$



The diff. of Ψ is the unique covector $d\Psi(x) \in T_x^* M$ that satisfies for all $\delta x \in T_x M$

$$\langle d\Psi(x) \,|\, \delta x \rangle_{T_x M} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Psi(x_\epsilon)$$

where x_ϵ is a curve on M defined as

$$x_\epsilon\Big|_{\epsilon=0} = x \in M \quad , \quad \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} x_\epsilon = \delta x \in T_x M$$

d) Example: $R \in SO(3)$

$$\Psi(R) = \frac{1}{2} tr(I - R_d^T R)$$, how compute $d\Psi(R) \in T_R^* SO(3)$

$$\Psi(R_\epsilon) = \frac{1}{2} tr(I - R_d^T R_\epsilon)$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Psi(R_\epsilon) = \left.\frac{d}{d\epsilon}\right|_{\varepsilon=0} \left[\frac{1}{2} tr(I) - \frac{1}{2} tr(R_d^T R_\epsilon) \right] = -\frac{1}{2} tr\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (R_d^T R_\epsilon) \right)$$

$$= -\frac{1}{2} tr\left(R_d^T \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (R_\epsilon) \right) = -\frac{1}{2} tr(R_d^T \delta R)$$

$$\langle d\Psi(R) \mid \delta R \rangle_{T_R SO(3)} = \operatorname{tr}\left(d\Psi(R)\, \delta R \right)$$

$$\Longrightarrow \boxed{d\Psi(R) = -\frac{1}{2} R_d^T}$$