SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 21: Geometric PD Control II

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Outline

- Recap last lectures
- PD Control of a Satellite on $SO(3)$

Recap: Lyapunov’s Direct Method

$$\dot{x} = f(x)$$

x^* is an Eq. point of the system

Stable	Asymptotically Stable	Globally Asymptotically Stable

Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) \leq 0$

$\dot{V}(x)$ is negative semi-definite

$V(x)$ is positive definite

Asymptotically Stable

Find a Lyapunov function $V(x)$ such that:

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$

$\dot{V}(x)$ is negative definite

Globally Asymptotically Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$
- $V(x) \to \infty$ as $\|x - x^*\| \to \infty$

$V(x)$ is radially unbounded

Recap: La Salle's Invariance Principle

- Let $x_* \in \mathcal{X}$ be an equilibrium point of the dynamical system

$$\dot{x}(t) = f\big(x(t)\big).$$

Assume there exists a smooth Lyapunov function $V: \mathcal{X} \to \mathbb{R}$ such that in some neighborhood $\Omega \subset \mathcal{X}$ of x_*, we have that

  - V is positive definite
  - $\dot{V}$ is negative semi-definite
  - Let $R \coloneqq \{x \in \Omega \mid \dot{V}(x) = 0\} \subset \Omega$ and let $M \subset R$ be the largest invariant set in it.

- If M contains only the equilibrium point (i.e., $M = \{x_*\}$), then x_* is locally asymptotically stable with its domain of attraction defined by

$$\Omega_l \coloneqq \{x \in \Omega \mid V(x) < l\} \subset \Omega.$$

Recap: PD Control of Point Mass

- Point mass dynamics
  - $\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_3 \end{pmatrix} u,$ with $v = \frac{p}{m}$
- Control objective:
  - Stabilization at $(\xi_d, 0)$
- PD Control law:
  - $u = -K_p\,(\xi - \xi_d) - K_d v,$ with $K_p, K_d \succ 0$
- Closed loop dynamics
  - $\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -K_p\,(\xi - \xi_d) - K_d v \end{pmatrix}$

Recap: Reformulating PD Control on $\mathbb{R}^3$

- Energy balancing control law
  - $u = u_p + u_d = -\nabla\Psi(\xi) + u_d$
  - $\Psi(\xi) \coloneqq \frac{1}{2}(\xi - \xi_d)^\top K_p(\xi - \xi_d)$, with $K_p \succ 0$
  - $\Psi(\xi_d) = 0$
  - $\Psi(\xi) > 0, \quad \forall \xi \neq \xi_d$
- Closed loop dynamics
  - $\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla\Psi(\xi) + u_d \end{pmatrix}$

Recap: Reformulating PD Control on $\mathbb{R}^3$

- Geometric PD control law
  - $u = u_p + u_d = -\nabla\Psi(\xi) + u_d$
  - $\Psi(\xi) \coloneqq \frac{1}{2}(\xi - \xi_d)^\top K_p(\xi - \xi_d),$ with $K_p \succ 0$
  - $\Psi(\xi_d) = 0$
  - $\Psi(\xi) > 0, \quad \forall \xi \neq \xi_d$
- Closed loop dynamics
  - $\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla\Psi(\xi) + u_d \end{pmatrix}$
- Lyapunov analysis
  - $V(x) = \Psi(\xi) + H(p) = \frac{1}{2}(\xi - \xi_d)^\top K_p(\xi - \xi_d) + \frac{1}{2m}p^\top p$
  - $V(x)$ is globally positive definite

Recap: Reformulating PD Control on $\mathbb{R}^3$

- Lyapunov analysis
  - $\dot{V}(x) = \left\langle d\Psi(\xi) \middle| \dot{\xi} \right\rangle_{\mathbb{R}^3} + \langle d\mathrm{H}(p) | \dot{p} \rangle_{\mathbb{R}^3}$
    $= v^\top \nabla\Psi(\xi) + v^\top [-\nabla\Psi(\xi) + u_d]$
    $= v^\top u_d$
  - Choosing $u_d = -K_d v$, with $K_d \succ 0$:
  - $\dot{V}(x) = v^\top u_d = -v^\top K_d v \leq 0$

- Using La Salle's invariance principle, it follows that $x_d = (\xi_d, 0)$ is a globally asymptotically stable equilibrium point of the closed loop system.

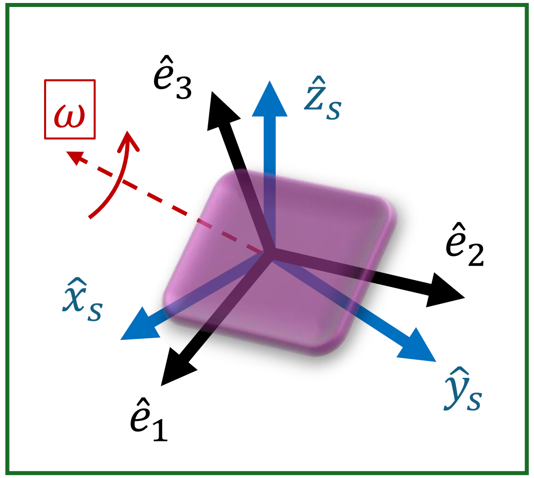
$u = u_p + u_d = -\nabla\Psi(\xi) - K_d v$
is interpreted as an energy balancing control law

Outline

- Recap last lectures
- PD Control of a Satellite on $SO(3)$

Satellite

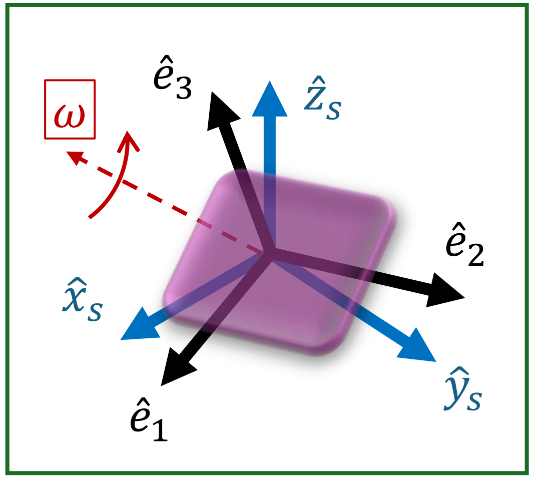
- The governing equations of a satellite with control torques τ are:
  - $\dot{R} = R\,\tilde{\omega}$
  - $\dot{\omega} = J^{-1}(-\omega \wedge J\omega + \tau\,)$



Satellite

- The governing equations of a satellite with control torques τ are:
  - $\dot{R} = R\,\tilde{\omega}$
  - $\dot{\omega} = J^{-1}(-\omega \wedge J\omega + \tau\,)$
- We can cast it into state space form
  - $x = (R, p) = (R, J\omega) \in SO(3) \times \mathbb{R}^3$
  - $\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega \end{pmatrix} + \begin{pmatrix} 0 \\ I_3 \end{pmatrix} \tau \qquad \Rightarrow \dot{x} = f(x) + g\,\tau$

  with
  - $\omega = J^{-1} p$
  - $\beta_R : \mathbb{R}^3 \to T_R SO(3), \qquad \omega \mapsto \beta_R(\omega) \coloneqq R\,\tilde{\omega}$



PD Control of Satellite

- We aim to design a Geometric PD controller

$$\tau = \tau_p + \tau_d$$

such that $x_d = (R_d, 0)$ is an asymptotically stable equilibrium point of the closed loop system

$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega + \tau_p + \tau_d \end{pmatrix}.$$

PD Control of Satellite

- We aim to design a Geometric PD controller

$$\tau = \tau_p + \tau_d$$

such that $x_d = (R_d, 0)$ is an asymptotically stable equilibrium point of the closed loop system

$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega + \tau_p + \tau_d \end{pmatrix}.$$

- The controller we seek is geometric in the sense that it respects the underlying non-Euclidean structure of $SO(3)$.
  - τ_p will be derived from the gradient of some positive definite potential function $\Psi(R)$ on $SO(3)$ with a minimum at $R = R_d$.
  - τ_d will be designed to inject damping.

Geometric structure of SO(3)

- The geometric nature of SO(3) will be reflected in
  1. How to compute the error between $R, R_d \in SO(3)$?
  2. How to design $\Psi(R)$ to be positive definite ?
  3. How to compute $d\Psi(R) \in T_R^* SO(3)$?
  4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
  5. How to design the derivative torque $\tau_d \in \mathbb{R}^3$?