SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 22: Geometric PD Control III

جامعة الملك فهد للبترول والمعادن
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Outline

- Recap last lectures
- PD Control of a Satellite on $SO(3)$…

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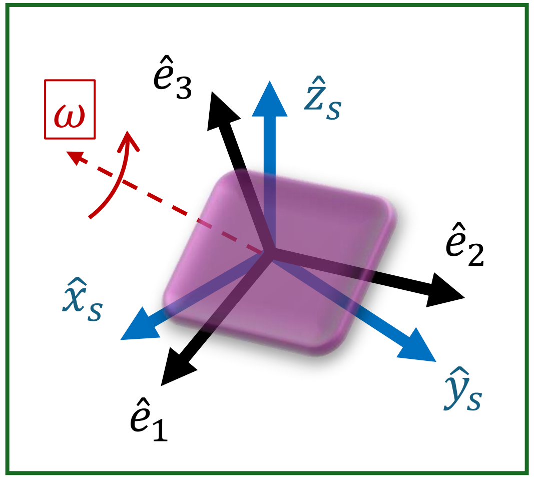
Recap: Satellite

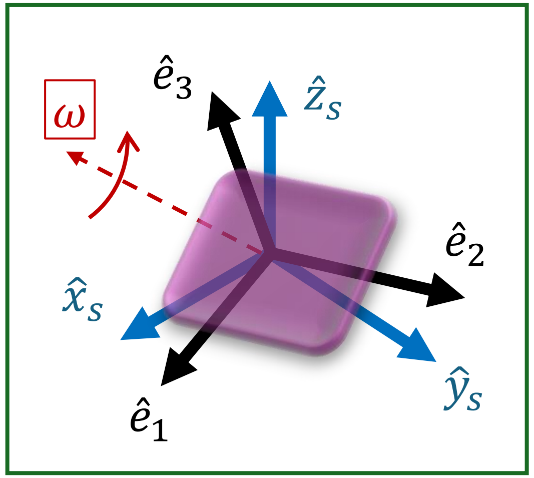
- The governing equations of a satellite with control torques τ are:
  - $\dot{R} = R\,\widetilde{\omega}$
  - $\dot{\omega} = J^{-1}(-\omega \wedge J\omega + \tau\,)$



Recap: Satellite

- The governing equations of a satellite with control torques τ are:
  - $\dot{R} = R\,\widetilde{\omega}$
  - $\dot{\omega} = J^{-1}(-\omega \wedge J\omega + \tau\,)$
- We can cast it into state space form
  - $x = (R, p) = (R, J\omega) \in SO(3) \times \mathbb{R}^3$
  - $\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega \end{pmatrix} + \begin{pmatrix} 0 \\ I_3 \end{pmatrix} \tau \qquad \Rightarrow \dot{x} = f(x) + g\,\tau$

  with
  - $\omega = J^{-1} p$
  - $\beta_R \colon \mathbb{R}^3 \to T_R SO(3), \qquad \omega \mapsto \beta_R(\omega) \coloneqq R\,\widetilde{\omega}$



Recap: PD Control of Satellite

- We aim to design a Geometric PD controller

$$\tau = \tau_p + \tau_d$$

such that $x_d = (R_d, 0)$ is an asymptotically stable equilibrium point of the closed loop system

$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega + \tau_p + \tau_d \end{pmatrix}.$$

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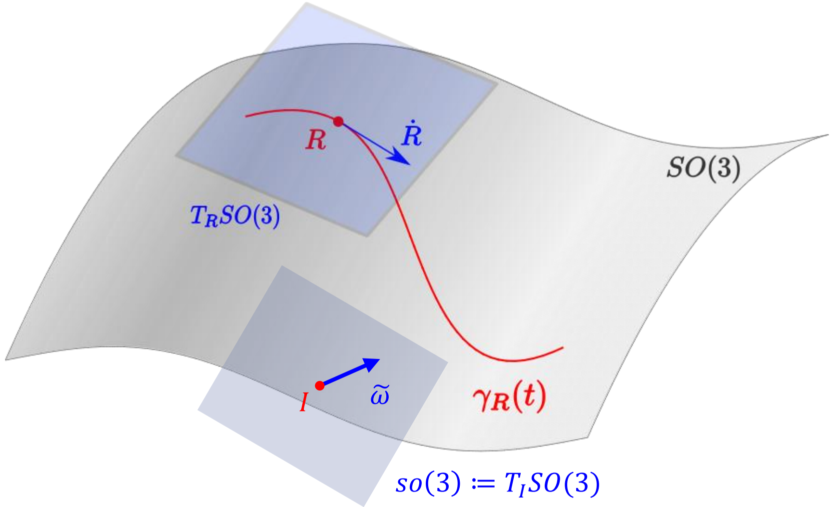
- The controller we seek is geometric in the sense that it respects the underlying non-Euclidean structure of $SO(3)$.
  - τ_p will be derived from the gradient of some positive definite potential function $\Psi(R)$ on $SO(3)$ with a minimum at $R = R_d$.
  - τ_d will be designed to inject damping.

Recap: Geometric structure of SO(3)

- The geometric nature of SO(3) will be reflected in
  1. How to compute the error between $R, R_d \in SO(3)$?
  2. How to design $\Psi(R)$ to be positive definite ?
  3. How to compute $d\Psi(R) \in T_R^* SO(3)$?
  4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
  5. How to design the derivative torque $\tau_d \in \mathbb{R}^3$?

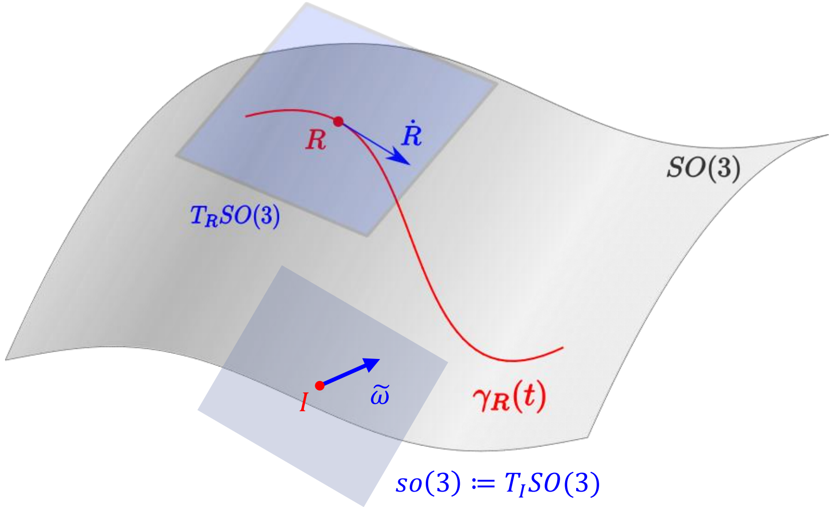


Outline

- Recap last lectures
- PD Control of a Satellite on $SO(3)$…

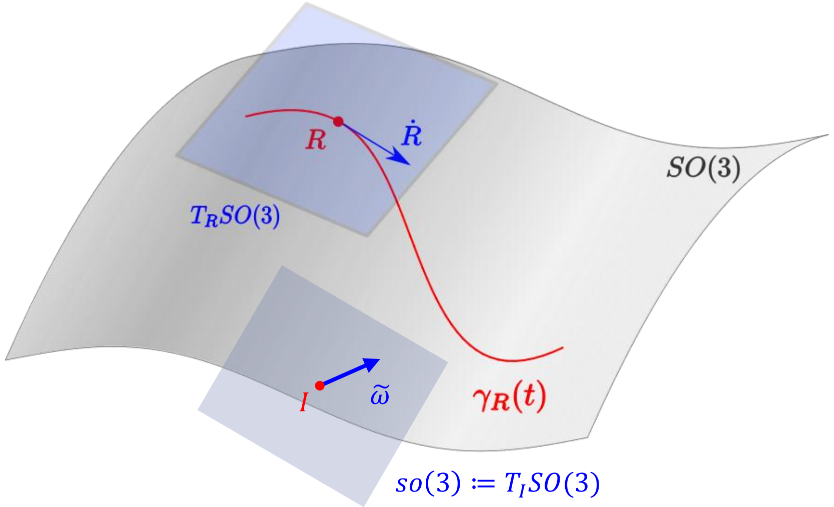
PD Control on $SO(3)$

1. How to compute the error between $R, R_d \in SO(3)$?
   - $R_e := R_d^\top R \in SO(3)$
   - $R_e \to I$ as $R \to R_d$



PD Control on $SO(3)$

2. How to design $\Psi(R)$ to be positive definite ?
   - One choice for $\Psi\colon SO(3) \to \mathbb{R}$ is
$$\Psi(R) \coloneqq \frac{1}{2}\operatorname{tr}(I - R_d^\top R)$$
   - $\Psi(R_d) = 0$
   - We can show that it can be written as $\Psi(R) = 1 - \cos\theta$, for some $\theta \in (-\pi, \pi]$. Therefore,
$$0 \le \Psi(R) \le 2$$



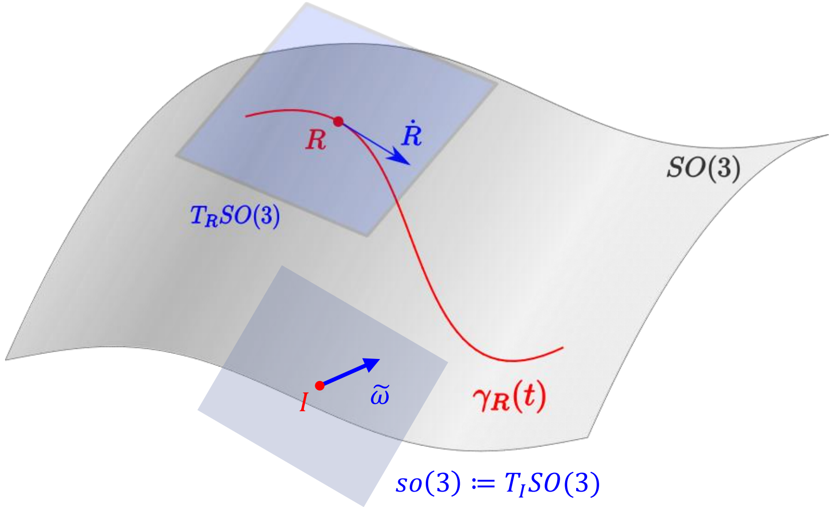
PD Control on $SO(3)$

3. How to compute $d\Psi(R) \in T_R^* SO(3)$?
   - The differential of $\Psi: \mathrm{SO}(3) \to \mathbb{R}$ is defined as the unique covector $d\Psi(R) \in T_R^* SO(3)$ that satisfies

$$\langle d\Psi(R) |\, \delta R \rangle_{T_R SO(3)} = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Psi(\mathrm{R}_\epsilon), \qquad \forall \delta R \in T_R SO(3)$$

where R_ϵ is a curve on $SO(3)$ that satisfies

$$\mathrm{R}_\epsilon|_{\epsilon=0} = R, \qquad \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathrm{R}_\epsilon = \delta R$$



The pairing between any covector in $T_R^* SO(3)$ and vector $T_R SO(3)$ is:

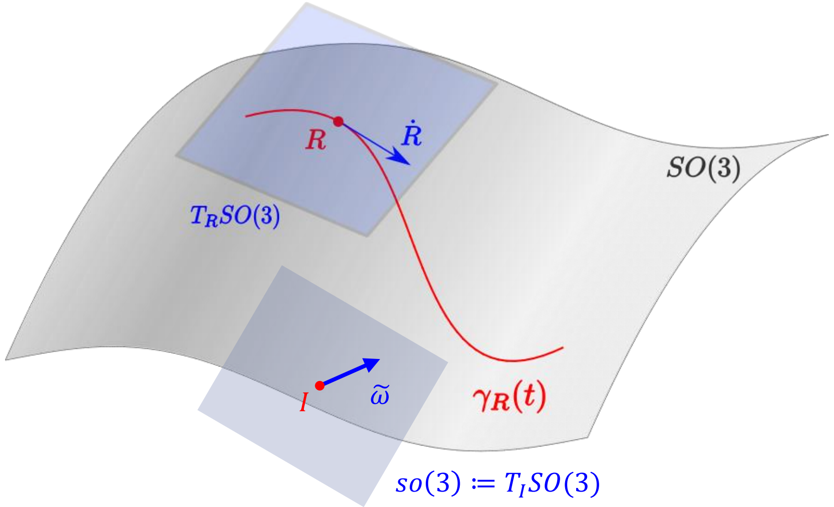
$$\langle \Gamma |\, \delta R \rangle_{T_R SO(3)} \coloneqq \mathrm{tr}\,(\Gamma \delta R)$$

PD Control on $SO(3)$

3. How to compute $d\Psi(R) \in T_R^* SO(3)$?
   - $\Psi(\mathrm{R}_\epsilon) = \frac{1}{2}\operatorname{tr}\left(I - R_d^\top \mathrm{R}_\epsilon\right)$
   - $\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Psi(\mathrm{R}_\epsilon) = -\frac{1}{2}\operatorname{tr}\left(R_d^\top \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathrm{R}_\epsilon\right) = -\frac{1}{2}\operatorname{tr}\left(R_d^\top \delta R\right)$
   - $\langle d\Psi(R) |\, \delta R\rangle_{T_R SO(3)} = \operatorname{tr}\left(d\Psi(R)\delta R\right)$

   - By comparing both sides, we have that

$$d\Psi(R) = -\frac{1}{2} R_d^\top$$

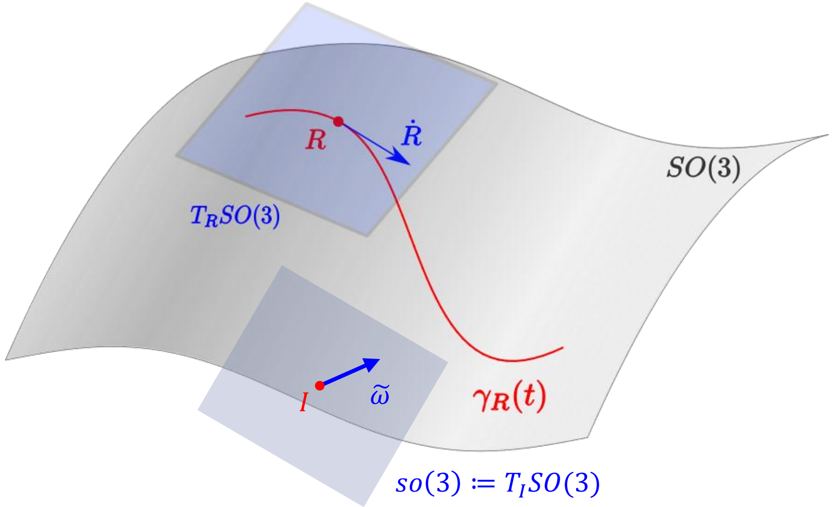


PD Control on $SO(3)$

4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
   - We use the duality of linear maps !
   - Recall the definition of the dual of a linear map:

- Let $A: U \to V$ be a linear map between the vector spaces U and V. Then its dual map $A^*: V^* \to U^*$ is defined (implicitly) by:

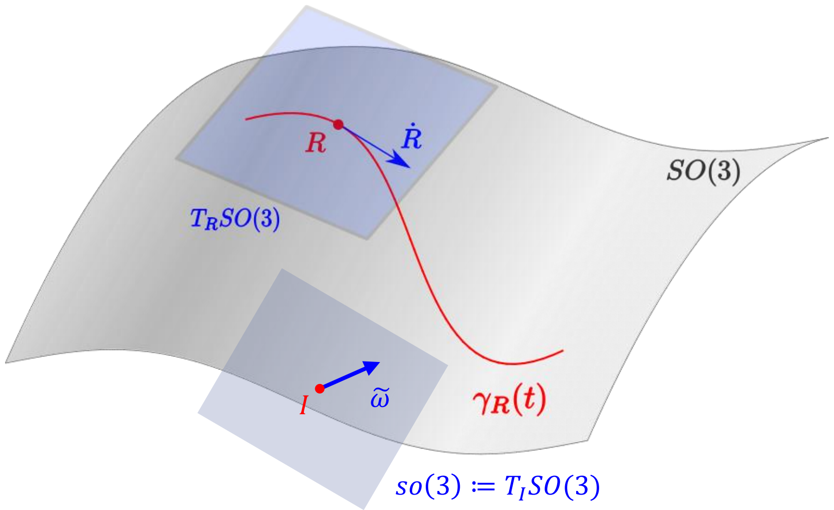
$$\langle A^*(\alpha) | u \rangle_U = \langle \alpha | A(u) \rangle_V, \qquad \forall u \in U, \alpha \in V^*$$



PD Control on $SO(3)$

4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?

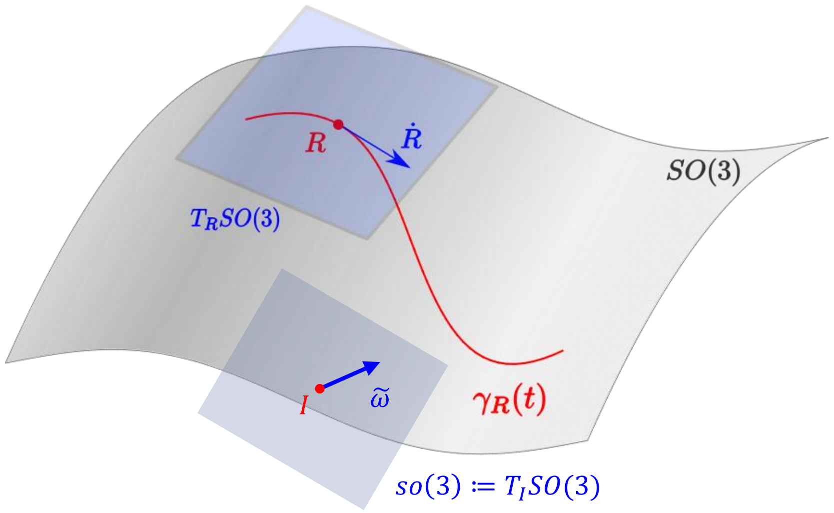
$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "S"] \arrow[rr, bend left, "\beta_R"] & so(3) \arrow[r, "L_{R,*}"] & T_R SO(3) \\
(\mathbb{R}^3)^* & so^*(3) \arrow[l, "S"'] & T_R^* SO(3) \arrow[l, "L_R^*"'] \arrow[ll, bend left, "\beta_R^*"]
\end{tikzcd}$$



PD Control on $SO(3)$

4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
    - $d\Psi(R) = -\frac{1}{2} R_d^\top \in T_R^* SO(3)$
    - $\tilde{e}_R \coloneqq L_R^*\big(d\Psi(R)\big) = \mathrm{sk}\big(R_d^\top R\big) = \frac{1}{2}(R_d^\top R - R^\top R_d) \in so^*(3)$
    - $e_R \coloneqq \beta_R^*\big(d\Psi(R)\big) = S^{-1}(\tilde{e}_R) = \frac{1}{2} S^{-1}\big(R_d^\top R - R^\top R_d\big) \in (\mathbb{R}^3)^*$
    - The proportional torque is then designed as

$$\tau_p \coloneqq -K_p e_R$$



PD Control on $SO(3)$

5. How to design the derivative torque $\tau_d \in \mathbb{R}^3$?
   - Let's conduct a Lyapunov analysis for the closed loop system

$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega + \tau_p + \tau_d \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega - K_p\,\beta_R^*\big(d\Psi(R)\big) + \tau_d \end{pmatrix}$$

PD Control on $SO(3)$

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   - Lyapunov function:

$$V(R, p) = H(p) + \Psi(R)$$

where

$$H(p) \coloneqq \frac{1}{2}\,p^\top J^{-1} p, \qquad \text{and} \qquad \Psi(R) \coloneqq \frac{1}{2}\,\mathrm{tr}(I - R_d^\top R)$$

with

$$dH(p) = (J^{-1}p)^\top = \omega^\top, \qquad \text{and} \qquad d\Psi(R) = -\frac{1}{2} R_d^\top$$

Stability Analysis

Compute the time derivative of V

- $V(R,p) = H(p) + \Psi(R)$
- $\dot{V}(R,p) = \dot{H}(p) + \dot{\Psi}(R) = \langle dH(p)|\dot{p}\rangle_{\mathbb{R}^3} + \left\langle d\Psi(R)|\, \dot{R}\right\rangle_{T_R SO(3)}$

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$$= \langle dH(p)|\dot{p}\rangle_{\mathbb{R}^3} + \langle d\Psi(R)|\beta_R(\omega)\rangle_{T_R SO(3)}$$

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$$= \langle dH(p)|\dot{p}\rangle_{\mathbb{R}^3} + \langle d\Psi(R)|\beta_R(\omega)\rangle_{T_R SO(3)}$$

$$= \langle dH(p)|\dot{p}\rangle_{\mathbb{R}^3} + \left\langle \beta_R^*\big(d\Psi(R)\big)|\omega\right\rangle_{\mathbb{R}^3}$$

$$= \langle dH(p)|\dot{p}\rangle_{\mathbb{R}^3} + \langle e_R|\omega\rangle_{\mathbb{R}^3}$$

$$e_R \coloneqq \beta_R^*\big(d\Psi(R)\big)$$

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$$
\begin{aligned}
&= \langle dH(p) | \dot{p} \rangle_{\mathbb{R}^3} + \langle d\Psi(R) | \beta_R(\omega) \rangle_{T_R SO(3)} \\
&= \langle dH(p) | \dot{p} \rangle_{\mathbb{R}^3} + \left\langle \beta_R^* \big(d\Psi(R) \big) | \omega \right\rangle_{\mathbb{R}^3} \\
&= \langle dH(p) | \dot{p} \rangle_{\mathbb{R}^3} + \langle e_R | \omega \rangle_{\mathbb{R}^3} \\
&= \omega^\top \big(\tilde{p} \, \omega - K_p e_R + \tau_d \big) + \omega^\top e_R \\
&= \omega^\top \tilde{p} \, \omega \, - \omega^\top \big(K_p - I \big) e_R + \omega^\top \tau_d
\end{aligned}
$$

$e_R \coloneqq \beta_R^* \big(d\Psi(R) \big)$

$\tau_p = -K_p \, e_R$

Stability Analysis

Consider the modified Lyapunov function:

- $V(R,p) = H(p) + \bar{\Psi}(R)$
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$$= \langle dH(p)|\dot{p}\rangle_{\mathbb{R}^3} + \left\langle K_p e_R|\omega\right\rangle_{\mathbb{R}^3}$$

$$= \omega^\top\left(\tilde{p}\,\omega - K_p e_R + \tau_d\right) + \omega^\top K_p e_R$$

$$= \omega^\top \tilde{p}\,\omega - \omega^\top\left(K_p - K_p\right)e_R + \omega^\top \tau_d$$

$$K_p e_R \coloneqq \beta_R^*\left(d\bar{\Psi}(R)\right)$$

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$$= \omega^\top\left(\tilde{p}\,\omega - K_p e_R + \tau_d\right) + \omega^\top K_p e_R$$

$$= \underbrace{\omega^\top \tilde{p}\,\omega}_{=0} - \underbrace{\omega^\top\left(K_p - K_p\right)e_R}_{=0} + \omega^\top \tau_d$$

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$$= \omega^\top \tau_d$$

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&= \omega^\top \tau_d = -\omega^\top K_d \omega \leq 0
\end{aligned}
$$

Choose $\tau_d = -K_d \omega$ with $K_d \succ 0$

La Salle's Invariance Principle

- Let $\Omega = \mathcal{X} = SO(3) \times \mathbb{R}^3$
- $\mathcal{R} \coloneqq \{(R, p) \in \Omega \,|\dot{V}(R, p)\}$
  - $\dot{V}(R, p) = \omega^\top K_d \omega$ $\qquad\qquad \Longrightarrow \omega = p = 0$

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  - $\dot{V}(R,p) = \omega^\top K_d \omega$ $\qquad \Longrightarrow \omega = p = 0$
- $\mathcal{M} \coloneqq \{(R,p) \in \mathcal{R} \,|\dot{p} = 0\}$
  - $\dot{p} \ = \ \tilde{p}\,\omega - K_p\,\beta_R^*\big(d\Psi(R)\big) - K_d\omega$ $\qquad \Longrightarrow \beta_R^*\big(d\Psi(R)\big) = 0$

$$\Longrightarrow L_R^*\big(d\Psi(R)\big) = \ \mathrm{sk}\big(R_d^\top R\big) = 0$$

$$\Longrightarrow R_e \ - R_e^\top = 0 \qquad \Longrightarrow R_e = R_e^\top$$