* Lyapunov Stability of origin ($\tau = 0$)

$$\dot{E}_{Pot}(x_1) = \langle d E_{Pot}(x_1) \,|\, \dot{x}_1 \rangle$$

$$y^T A y = 0 \qquad A = -A^T$$

$$V(x_1, x_2) = \frac{1}{2} x_2^T M(x_1) x_2 + E_{Pot}(x_1)$$

$$\dot{V} = \frac{1}{2} \underbrace{\dot{x}_2^T M(x_1) x_2} + \frac{1}{2} x_2^T \dot{M}(x_1) x_2 + \underbrace{\frac{1}{2} x_2^T M(x_1) \dot{x}_2} + \underbrace{\nabla_{x_1}^T E_{Pot}(x_1)}_{g^T(x_1)} \dot{x}_1$$

$$= x_2^T \underbrace{M(x_1) \dot{x}_2} + \frac{1}{2} x_2^T \dot{M}(x_1) x_2 + g^T(x_1) \dot{x}_1$$

$$= x_2^T \underbrace{\left[-C(x) x_2 - \cancel{g}(x_1) - B(x_2) x_2 \right]} + \frac{1}{2} x_2^T \dot{M}(x_1) x_2 + x_2^T \cancel{g}(x_1)$$

$$= -x_2^T B(x_2) x_2 + \cancel{\frac{1}{2} x_2^T \underbrace{\left[\dot{M}(x_1) - 2C(x) \right]} x_2} = 0$$

* PD Control:

$\theta \in Q \cong \mathbb{T}^n$ $\mathbb{R}^n$

$\Rightarrow$ Our objective is to stabilize the system at $x_d = (\theta_d, 0)$

$\Rightarrow$ Energy balancing Control : $\tau = \tau_p + \tau_d$

$\Rightarrow$ Error between θ & θ_d $\Rightarrow$ $\theta_e := \theta - \theta_d \in \mathbb{R}^n$

$\Rightarrow$ Potential function : $\Psi(\theta) = \frac{1}{2}(\theta - \theta_d)^T K_p (\theta - \theta_d)$

$\Rightarrow$ Proportional torque $\tau_p = -\nabla_\theta \Psi(\theta) = -K_p(\theta - \theta_d) \in \mathbb{R}^n$

* Lyapunov analysis for closed loop system:

$x_d = (\theta_d, 0_2)$

$$\Rightarrow V_{cl}(x) = V(x) + \Psi(x_1) = \frac{1}{2} x_2^T M(x_1) x_2 + E_{Pot}(x_1) + \Psi(x_1)$$

$$\Rightarrow V_{cl}(x_d) = 0 + E_{Pot}(\theta_d) + \underbrace{\Psi(\theta_d)}_{=0} = E_{Pot}(\theta_d) \neq 0$$

$$\Rightarrow \nabla_x V_{cl}(x) = \nabla_x V(x) + \nabla_{x_1} \Psi(x_1)$$

$$= M(x_1) x_2 + \dots + \nabla_{x_1} E_{Pot}(x_1) + \nabla_{x_1} \Psi(x_1)$$

$$\nabla_x V_{cl}(x_d) = g(\theta_d) \neq 0$$

Need to compensate gravity !!

* Now Consider

$$\Rightarrow \Psi(\theta) = \frac{1}{2}(\theta-\theta_d)^T K_p (\theta-\theta_d) - E_{Pot}(\theta)$$

$$\Rightarrow \tau_p = -\nabla_\theta \Psi(\theta) = -K_p(\theta-\theta_d) + g(\theta)$$

$$\Rightarrow V_{cl}(x) = V(x) + \Psi(x_1)$$

$$= \frac{1}{2} x_2^T M(x_1) x_2 + \frac{1}{2}(x_1-\theta_d)^T K_p (x_1-\theta_d)$$

$$\Rightarrow V_{cl}(x_d) = V_{cl}(\theta_d, 0) = 0 \qquad \Rightarrow V_{cl}(x) > 0$$

* Lyapunov analysis:

$$\Rightarrow \dot{V}_{CL} = -x_2^T B(x_2) x_2 + x_2^T \tau_d \Longrightarrow \dot{V}_{CL} = -x_2^T \underbrace{\left(B(x_2) + K_d\right)}_{>0} x_2$$

$$\Rightarrow \tau_d = -K_{\cancel{p}d}\, x_2$$

$$K_d > 0$$

$$\boxed{x = x_d \text{ is G.A.S.}}$$