SCE 594: Special Topics in Intelligent Automation & Robotics

Topic 5: Control of Fixed-Base Manipulators

Lecture 23: Motion Control

جامعة الملك فهد للبترول والمعادن
King Fahd University of Petroleum & Minerals

Ramy Rashad
Assistant Professor
CIE Department

Outline

- Open loop stability analysis
- Stabilization Control

Outline

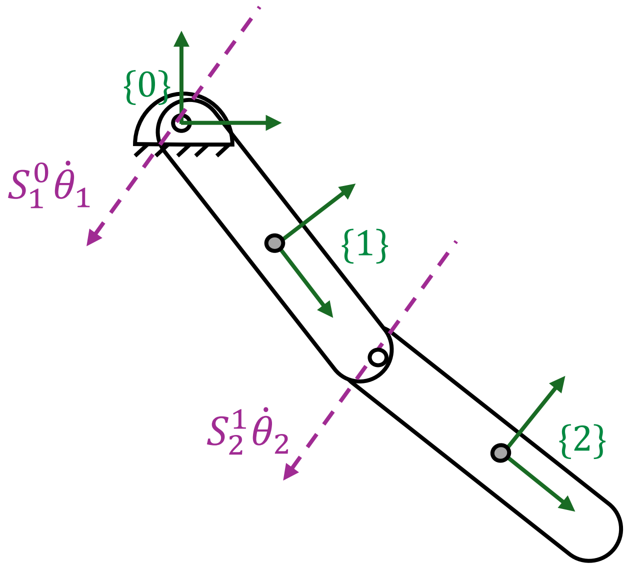
- Open loop stability analysis
- Stabilization Control

State Space Dynamics

- The governing equations of an n-link manipulator* with control torques τ are:

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + B(\theta)\dot{\theta} + g(\theta) = \tau$$

where $\theta \in Q = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \cong \mathbb{T}^n$.



*We assume for simplicity all joints are revolute.

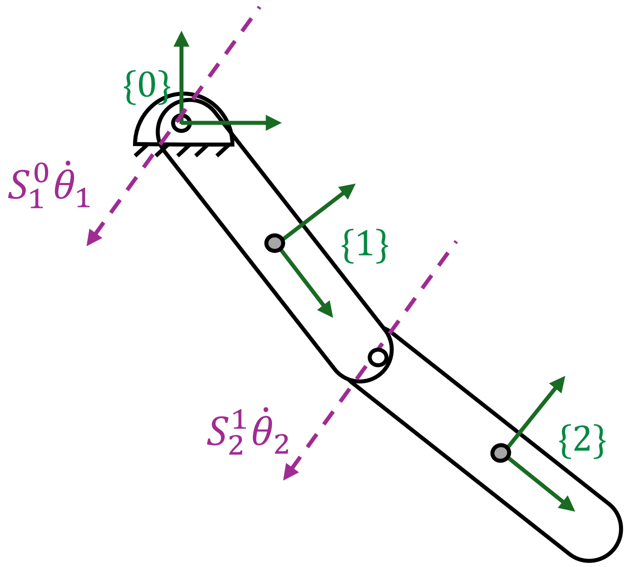
State Space Dynamics

- The governing equations of an n-link manipulator* with control torques τ are:

$$M(\theta)\ddot{\theta} + C\left(\theta, \dot{\theta}\right)\dot{\theta} + B\left(\dot{\theta}\right)\dot{\theta} + g(\theta) = \tau$$

where $\theta \in Q = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \cong \mathbb{T}^n$.

- We can cast it into state space form
  - $x = (x_1, x_2) = \left(\theta, \dot{\theta}\right) \in TQ$
  - $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)\,[c(x) + b(x_2) + g(x_1)] \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix}\tau$



$c(x) \coloneqq C(x_1, x_2)x_2 \in \mathbb{R}^n, \qquad b(x_2) \coloneqq B(x_2)x_2 \in \mathbb{R}^n, \qquad g(x_1) \in \mathbb{R}^n$

Equilibrium points of open loop system

- Proposition:
  - For $\tau = 0$, the equilibrium points of any n-link manipulator are states $x_* \coloneqq (x_{1,*}, x_{2,*})$ that satisfy

$$g(x_{1,*}) = 0_n, \qquad x_{2,*} = 0_n.$$

Equilibrium points of open loop system

- Proposition:
  - For $\tau = 0$, the equilibrium points of any n-link manipulator are states $x_* \coloneqq (x_{1,*}, x_{2,*})$ that satisfy

$$g(x_{1,*}) = 0_n, \qquad\qquad x_{2,*} = 0_n.$$

- Proof:
  - $\begin{pmatrix} 0_n \\ 0_n \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)\,[c(x) + b(x_2) + g(x_1)] \end{pmatrix} \Longrightarrow x_2 = 0_n \Longrightarrow c(x) = b(x_2) = 0_n.$
  - Thus, the equilibrium points should satisfy

$$0_n = M^{-1}(x_1)g(x_1) \quad \text{or equivalently} \quad 0 = g^\top(x_1)M^{-1}(x_1)g(x_1)$$

Equilibrium points of open loop system

- Proposition:
  - For $\tau = 0$, the equilibrium points of any n-link manipulator are states $x_* := \left(x_{1,*}, x_{2,*}\right)$ that satisfy

$$g\left(x_{1,*}\right) = 0_n, \qquad\qquad x_{2,*} = 0_n.$$

- Proof:
  - $\begin{pmatrix} 0_n \\ 0_n \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)\left[c(x) + b(x_2) + g(x_1)\right] \end{pmatrix} \Longrightarrow x_2 = 0_n \Longrightarrow c(x) = b(x_2) = 0_n.$
  - Thus, the equilibrium points should satisfy

$$0_n = M^{-1}(x_1)g(x_1) \quad \text{or equivalently} \quad 0 = g^\top(x_1)M^{-1}(x_1)g(x_1)$$

  - Since $M(x_1) \succ 0$, we have that $M^{-1}(x_1) \succ 0$
  - Consequently, $g^\top(x_1)M^{-1}(x_1)g(x_1) = 0$ if an only if $g(x_1) = 0_n$.

Example: 2-link manipulator

- Recall that the gravity torques for a 2-link manipulator are given by:

$$g(\theta) = \begin{pmatrix} \gamma_1 \sin\theta_1 + \gamma_2 \sin(\theta_1 + \theta_2) \\ \gamma_2 \sin(\theta_1 + \theta_2) \end{pmatrix}$$

where $\theta \coloneqq (\theta_1, \theta_2) \in (-\pi, \pi] \times (-\pi, \pi]$ and $\gamma_1, \gamma_2 \in \mathbb{R}_+$ are constants.

Example: 2-link manipulator

- Recall that the gravity torques for a 2-link manipulator are given by:

$$g(\theta) = \begin{pmatrix} \gamma_1 \sin\theta_1 + \gamma_2 \sin(\theta_1 + \theta_2) \\ \gamma_2 \sin(\theta_1 + \theta_2) \end{pmatrix}$$

where $\theta \coloneqq (\theta_1, \theta_2) \in (-\pi, \pi] \times (-\pi, \pi]$ and $\gamma_1, \gamma_2 \in \mathbb{R}_+$ are constants.

- The equilibrium points $x_* = (\theta_*, 0_2)$ of this manipulator should satisfy $g(\theta_*) = 0_n$:

$$\sin(\theta_1 + \theta_2) = 0 \qquad \text{and} \qquad \sin\theta_1 = 0$$

  - Thus, we have that

$$\theta_* = (0,0), \qquad \theta_* = (0,\pi), \qquad \theta_* = (\pi,0), \qquad \theta_* = (\pi,\pi),$$

  - Consequently:

$$x_* = (0,0,0,0), \qquad x_* = (0,\pi,0,0), \qquad x_* = (\pi,0,0,0), \qquad x_* = (\pi,\pi,0,0),$$

Stability properties of the origin

- Total energy of the system

$$E_{\mathrm{tot}}(\theta, \dot{\theta}) = E_{\mathrm{kin}}(\theta, \dot{\theta}) + E_{\mathrm{pot}}(\theta) = \frac{1}{2}\, \dot{\theta}^{\top} M(\theta)\dot{\theta} \; + E_{\mathrm{pot}}(\theta)$$

where $E_{\mathrm{pot}}: Q \to \mathbb{R}$ is defined such that its gradient:

$$\nabla_{\theta} E_{\mathrm{pot}}(\theta) = g(\theta) \in \mathbb{R}^{n}.$$

In general, we have that $E_{\mathrm{kin}}: TQ \to \mathbb{R}$ and $E_{\mathrm{pot}}: Q \to \mathbb{R}$

Stability properties of the origin

- Consider the Lyapunov function given by the total energy of the mechanical system

$$V(x_1, x_2) = \frac{1}{2}\, x_2^\top M(x_1) x_2 \; + E_{\text{pot}}(x_1)$$

- Since it represents energy, $V(x_1, x_2)$ is a positive definite Lyapunov function with a minimum at $V(0_2, 0_2) = 0$.

Stability properties of the origin

- Consider the Lyapunov function given by the total energy of the mechanical system

$$V(x_1, x_2) = \frac{1}{2}\, x_2^\top M(x_1) x_2 \; + E_{\mathrm{pot}}(x_1)$$

- Since it represents energy, $V(x_1, x_2)$ is a positive definite Lyapunov function with a minimum at $V(0_2, 0_2) = 0$.
- One can show* that

$$\dot{V}(x_1, x_2) = -x_2^\top B(x_2) x_2 \leq 0$$

which along with LaSalle's invariance principle shows that $x_* = (0_2, 0_2)$ is locally asymptotically stable.

*Done on board !

Outline

- Open loop stability analysis
- Stabilization Control

Done on board !

Summary

- The control law

$$\tau = \tau_p + \tau_d = -K_p\,(x_1 - \theta_d) - K_d x_2 + g(x_1)$$

makes the closed loop system

$$\begin{pmatrix} \dot{x}_1 \\ M(x_1)\dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -c(x) - b(x_2) - K_p\,(x_1 - \theta_d) - K_d x_2 \end{pmatrix}$$

have a globally asymptotically equilibrium point at $x_d = (\theta_d, 0_2)$.