SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 2: Maps, Groups & Fields

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Outline

- Recap: Set theory basics
- Maps between sets
- Adding structure to sets
  - Groups
  - Fields

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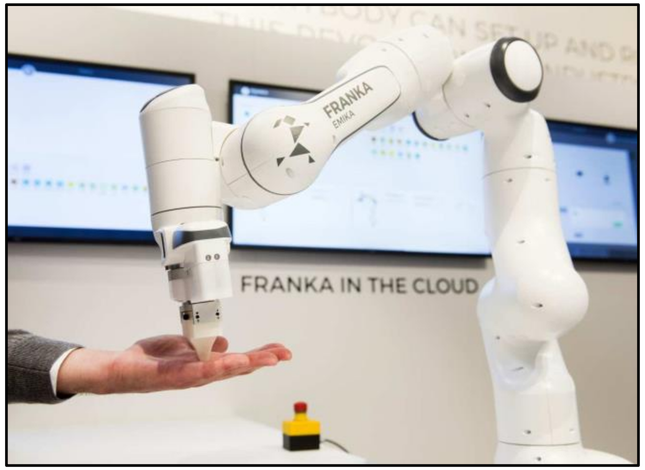
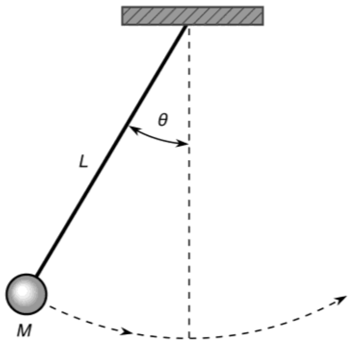
Recap: Robotic Systems

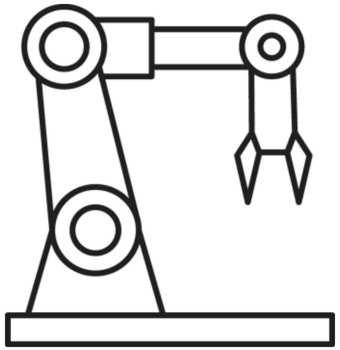
- Multi-rotor aerial vehicles
- Fixed-based manipulators
- Floating-base manipulators
  - Ground, Aerial, Underwater

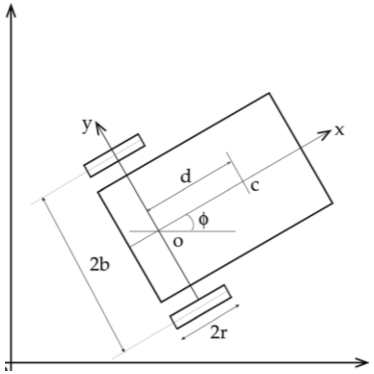


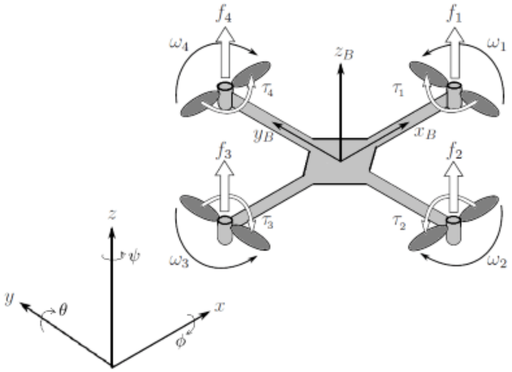
Recap: Why Geometric approach ?

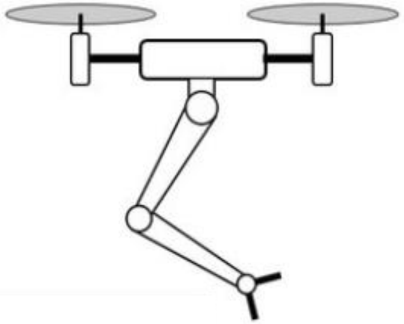
- Configuration space $\mathbb{Q}$ of (most) mechanical systems is not $\mathbb{R}^n$
  - Pendulum $\quad \mathbb{Q} = S^1$
  - n-degree-of-freedom manipulator $\quad \mathbb{Q} = T^n$
  - Planar mobile robot $\quad \mathbb{Q} = SE(2)$
  - Multirotor aerial vehicle $\quad \mathbb{Q} = SE(3)$
  - Aerial manipulator $\quad \mathbb{Q} = SE(3) \times T^n$











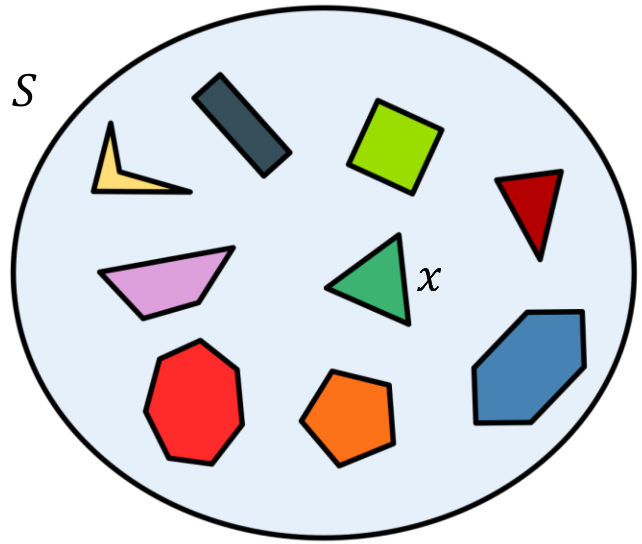
Recap: Structure hierarchy

- A recurrent theme in mathematics is the classification of spaces by means of *structure-preserving maps* between them.
- Space = set + some structure

Lie Group	**Lie Algebra**
Symplectic manifold	**Algebra**
Riemannian manifold	**Vector space**
Smooth manifold	**Field**
Topological manifold	**Group**
Set	

Recap: What is a set ?

- A set is intuitively a collection of elements.
  - $x \in S$
  - $A \coloneqq \{x \in S \mid \text{conditions on } x\} \subset S$
  - $A \coloneqq \{\text{square}, \text{rectangle}, \text{triangle}, \dots\}$
- We can define a set from two other ones by:
  - Cartesian product $A \times B$
  - Union $A \cup B$
  - Intersection $A \cap B$



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Maps between sets

- Let A and B denote two sets, a **map** f from A to B assigns to each element $x \in A$ an element $f(x) \in B$.
- The standard notation for a map is:

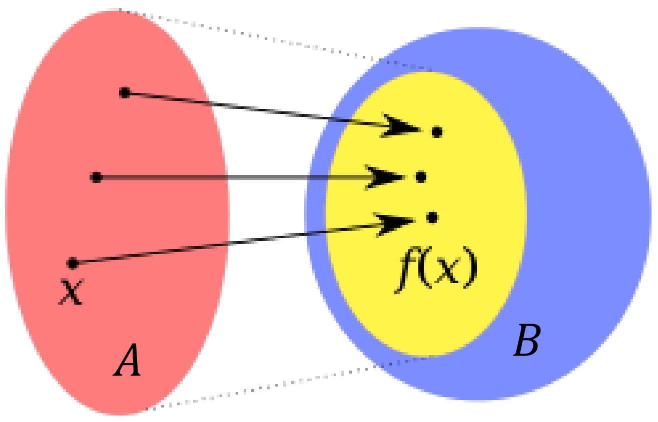
$$\begin{aligned} f: A &\to B \\ x &\mapsto f(x) \end{aligned}$$

Maps between sets

- Let A and B denote two sets, a **map** f from A to B assigns to each element $x \in X$ an element $f(x) \in Y$.
- The standard notation for a map is:

$$\begin{aligned} f: A &\to B \\ x &\mapsto f(x) \end{aligned}$$

- We call:
  - A the **domain** of f.
  - B the **codomain/target** of f.
  - $im_f(A) \coloneqq \{f(x) \in B \mid x \in A\}$ the **image** of A under f



$f: A \to B$

*A **map** is also known as a **function** or **mapping.**

Classification of Maps

- A map $f: A \to B$ is said to be:
  - **Surjective**: if for each $y \in B$, there exists at least one $x \in A$ such that $f(x) = y$
  - **Injective**: if the equality $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$ implies that $x_1 = x_2$
  - **Bijective:** if it is both surjective and injective

*A **surjective** map is also called **onto,** while an **injective** map is also called **one-to-one**

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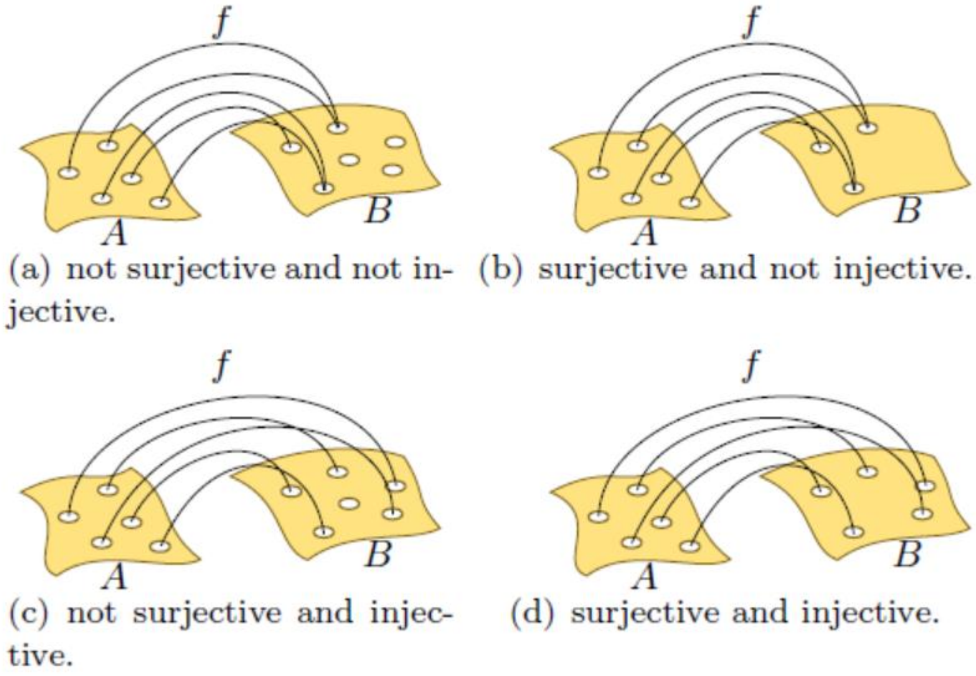

(a) not surjective and not injective. (b) surjective and not injective. (c) not surjective and injective. (d) surjective and injective.

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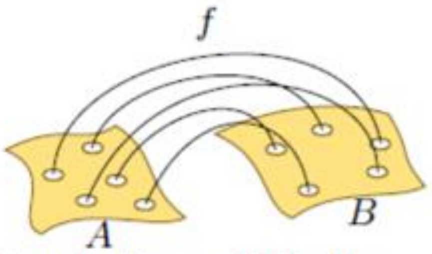
- Examples:
  - $\mathrm{id}_A: A \to A, \quad x \mapsto x$
  - $f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3$



(d) surjective and injective.

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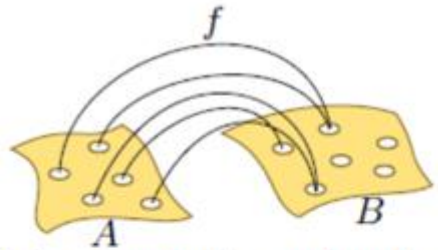
- Examples:
  - $f: \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^2$
  - $f: \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \sin(x)$



(a) not surjective and not injective.

Classification of Maps

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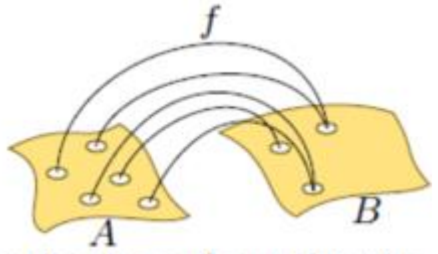
- Examples:
  - $f: \mathbb{R} \to \mathbb{R}$, $\quad x \mapsto \tan(x)$
  - $f: \mathbb{R} \to [-1,1]$, $\quad x \mapsto \sin(x)$



(b) surjective and not injective.

Classification of Maps

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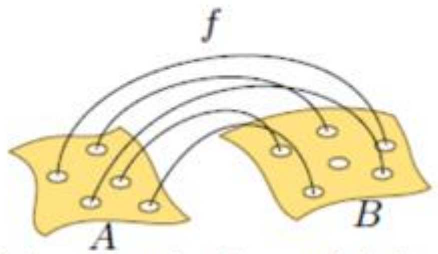
  - **Bijective:** if it is both surjective and injective

- Examples:
  - $\exp: \mathbb{R} \to \mathbb{R}$, $\quad x \mapsto e^x$



(c) not surjective and injective.

* But $\exp: \mathbb{R} \to \mathbb{R}_+$ is bijective
In general, if you restrict the codomain of an injective map to its image, the resulting function becomes bijective.

Composition of Maps

- Given two maps $f: A \to B$ and $g: B \to C$, we can construct a third map, called the **composition** of f and g, denoted by $g \circ f$ and defined as:

$$g \circ f: A \to C$$
$$x \mapsto g\big(f(x)\big).$$

- This is often represented by drawing the following **commutative diagram:**

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd}$$

Inverse of a map

- Let $f: A \to B$ be a bijective map. Then the **inverse** of f, denoted by $f^{-1}: B \to A$ is defined by:
  - $f^{-1} \circ f = \mathrm{id}_A$
  - $f \circ f^{-1} = \mathrm{id}_B$

- Two sets A and B are called (set-theoretic) **isomorphic** if there exists a bijective map $f: A \to B$. In this case, we write that

$$A \cong_{\text{set}} B$$

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Adding structure to sets

- So far, we've considered sets with no additional structure
- Now we'll add more structure to define:
  - Groups
  - Fields