

SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 3: Vector Spaces I



Outline

- Recap: Last Lectures
- Maps between groups
- Vector Space theory I
 - Field
 - Vector space
 - Linear map



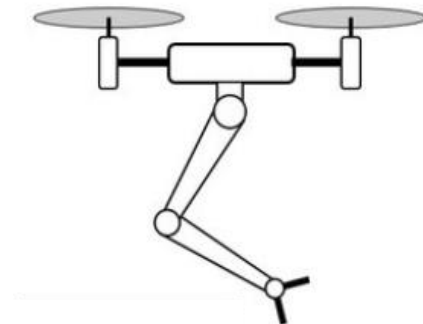
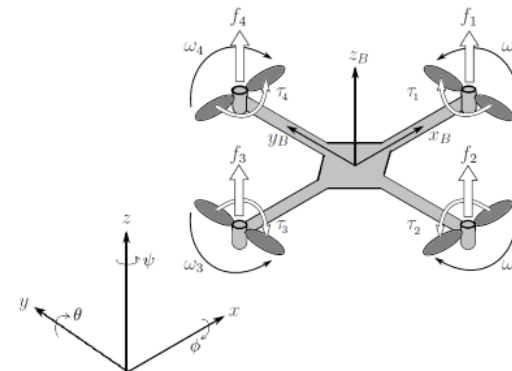
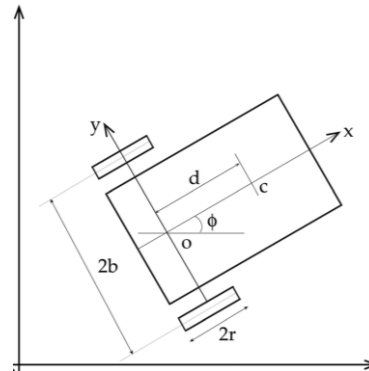
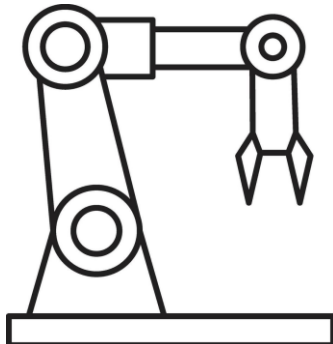
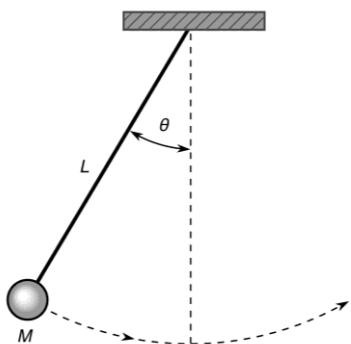
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Recap: Why Geometric approach ?

- Configuration space $\mathbb{Q}$ of (most) mechanical systems is not $\mathbb{R}^n$
 - Pendulum $\mathbb{Q} = S^1$
 - n -degree-of-freedom manipulator $\mathbb{Q} = T^n$
 - Planar mobile robot $\mathbb{Q} = SE(2)$
 - Multirotor aerial vehicle $\mathbb{Q} = SE(3)$
 - Aerial manipulator $\mathbb{Q} = SE(3) \times T^n$



Recap: Structure hierarchy

- A recurrent theme in mathematics is the classification of spaces by means of *structure-preserving maps* between them.
- Space = set + some structure



Set
 S



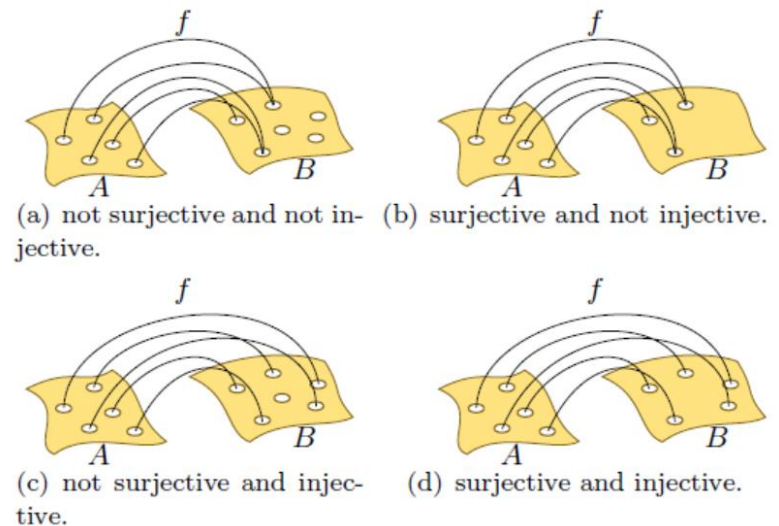
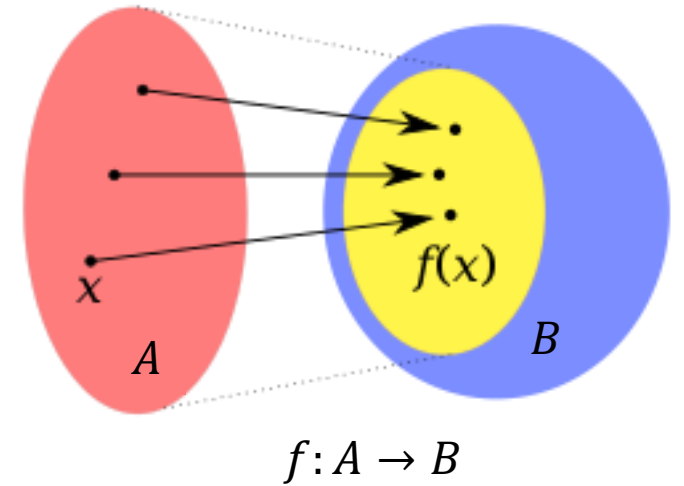
Group
 $(G, \odot)$

Vector space over a Field
 $(V, \oplus, \odot)$



Recap: Maps between sets

- The standard notation for a map is:
$$f: A \rightarrow B$$
$$x \mapsto f(x)$$
- We call:
 - A the **domain** of f .
 - B the **codomain/target** of f .
- A map can be either:
 - Surjective
 - Injective
 - Both
 - None



Recap: Inverse of a map

- Let $f: A \rightarrow B$ be a bijective map. Then the **inverse** of f , denoted by $f^{-1}: B \rightarrow A$ is defined by:
 - $f^{-1} \circ f = \text{id}_A$
 - $f \circ f^{-1} = \text{id}_B$
- Two sets A and B are called (set-theoretic) **isomorphic** if there exists a bijective map $f: A \rightarrow B$. In this case, we write that
$$A \cong_{\text{set}} B$$



Recap: Group

- A group is a pair $(G, \circ)$ where G is a set and $\circ: G \times G \rightarrow G$ is a map (called binary operation) that satisfies:
 - i. $\forall a, b, c \in G$ we have that $(a \circ b) \circ c = a \circ (b \circ c)$
 - ii. $\exists e \in G$ such that $\forall g \in G$ we have that $e \circ g = g \circ e = g$
 - iii. $\forall g \in G, \exists g^{-1} \in G$ such that $g^{-1} \circ g = g \circ g^{-1} = e$
- A group $(G, \circ)$ is also called **Abelian** if it satisfies:
 - iv. $a \circ b = b \circ a, \quad \forall a, b \in G$



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Maps between groups

- Let $(G, \oplus)$ and $(H, \odot)$ be two groups.
- If there exists a map $\rho: G \rightarrow H$ that satisfies:
 - $\rho(a \oplus b) = \rho(a) \odot \rho(b) \quad \forall a, b \in G$

Then we call the map $\rho: G \rightarrow H$ a **group homomorphism**.

- If ρ is also a bijective map, then we call ρ a (group) **isomorphism**
- If there exists an isomorphism between $(G, \oplus)$ & $(H, \odot)$, then we say that G and H are (group-theoretic) isomorphic to each other.

$$G \cong_{\text{grp}} H$$



Examples

1. The map

$$\begin{aligned}\exp: \mathbb{R} &\rightarrow \mathbb{R}_+ \\ t &\mapsto e^t\end{aligned}$$

is a group *isomorphism* between $(\mathbb{R}, +)$ and $(\mathbb{R}_+, \cdot)$ because of the property $e^{t_1+t_2} = e^{t_1} \cdot e^{t_2}$.

2. The map

$$\begin{aligned}\det: GL(n, \mathbb{R}) &\rightarrow \mathbb{R} \setminus \{0\} \\ A &\mapsto \det(A)\end{aligned}$$

is a group *homomorphism* between $(GL(n, \mathbb{R}), \odot)$ and $(\mathbb{R} \setminus \{0\}, \cdot)$ because of the property $\det(A \odot B) = \det(A) \cdot \det(B)$



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Vector Spaces

- Now we will turn attention to vector spaces (*aka linear spaces*)
- It is convenient to consider them in more abstract terms than to simply think of $\mathbb{R}^n$.
- A vector space $(V, \oplus, \odot)$ is a set that is equipped with two operations satisfying certain properties, not just a set of *n-tuples*.
- To define a vector space, we need to define first what is a field
 $(K, +, \cdot)$



Field

- An (algebraic) field is a triple $(K, +, \cdot)$ where K is a set equipped with the maps $+, \cdot : K \times K \rightarrow K$ satisfying:
 - $(K, +)$ is an Abelian group
 - $(K \setminus \{0\}, \cdot)$ is an Abelian group
 - The maps $+$ and $\cdot$ satisfy the distributive property i.e.
 $\forall a, b, c \in K$ we have that $(a + b) \cdot c = a \cdot c + b \cdot c$

Recall $(K, +)$ is an Abelian group

- i. $\forall a, b, c \in K$ we have that $(a + b) + c = a + (b + c)$
- ii. $\exists 0 \in K$ such that $\forall a \in K$ we have that $0 + a = a + 0 = a$
- iii. $\forall a \in K, \exists -a \in K$ such that $a + (-a) = (-a) + a = 0$
- iv. $a + b = b + a, \quad \forall a, b \in K$



Field

- An (algebraic) field is a triple $(K, +, \cdot)$ where K is a set equipped with the maps $+, \cdot : K \times K \rightarrow K$ satisfying:
 - $(K, +)$ is an Abelian group
 - $(K \setminus \{0\}, \cdot)$ is an Abelian group
 - The maps $+$ and $\cdot$ satisfy the distributive property i.e.
$$\forall a, b, c \in K \text{ we have that } (a + b) \cdot c = a \cdot c + b \cdot c$$

Example:

- The sets $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are all fields under the usual addition and multiplication operations
- The triple $(\mathbb{Z}, +, \cdot)$ is not a field

Recall $(K, +)$ is an Abelian group

- i. $\forall a, b, c \in K$ we have that $(a + b) + c = a + (b + c)$
- ii. $\exists 0 \in K$ such that $\forall a \in K$ we have that $0 + a = a + 0 = a$
- iii. $\forall a \in K, \exists -a \in K$ such that $a + (-a) = (-a) + a = 0$
- iv. $a + b = b + a, \forall a, b \in K$



Vector space

- A vector space $(V, \oplus, \odot)$ over a field $(K, +, \cdot)$ is the set V equipped with two operations:
 - $\oplus: V \times V \rightarrow V$ called vector addition
 - $\odot: K \times V \rightarrow V$ called scalar multiplication

that should satisfy the rules:

- $(V, \oplus)$ is an Abelian group
- The map $\odot$ is an action of K on $(V, \oplus)$:

$$\text{i) } \forall \lambda \in K : \forall v, w \in V : \lambda \odot (v \oplus w) = (\lambda \odot v) \oplus (\lambda \odot w);$$

$$\text{ii) } \forall \lambda, \mu \in K : \forall v \in V : (\lambda + \mu) \odot v = (\lambda \odot v) \oplus (\mu \odot v);$$

$$\text{iii) } \forall \lambda, \mu \in K : \forall v \in V : (\lambda \cdot \mu) \odot v = \lambda \odot (\mu \odot v);$$

$$\text{iv) } \forall v \in V : 1 \odot v = v.$$

- An element of $v \in V$ is called a **vector**.



Prototypical Example

- $(\mathbb{R}^n, \oplus, \odot)$ is a vector space over the field $(\mathbb{R}, +, \cdot)$.
- The vector space $\mathbb{R}^n$ (not the set !!) is frequently called the **n-dimensional Euclidean space**.



Maps between vector spaces

Set
 $\mathbb{S}$

Maps

Bijections

Group
 $(G, \odot)$

**Group
homomorphisms**

**Group
isomorphisms**

Vector space over a Field
 $(V, \oplus, \odot)$

**Linear
maps**

**Linear
isomorphisms**

