SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 4: Vector Spaces II

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Outline

- Recap: Last Lectures
- Dual spaces
- Tensor spaces
- Bases and components

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Recap: Structure hierarchy

- A recurrent theme in mathematics is the classification of spaces by means of *structure-preserving maps* between them.
- Space = set + some structure

	Set	**Group**	**Vector space over a Field**
Mathematical object	$\mathbb{S}$	$(G, \oplus)$	$(V, \oplus, \odot)$ over $(K, +, \cdot)$
Structure-preserving map	Map $f: \mathbb{S} \to \mathbb{T}$	Group homomorphism $\rho: G \to H$	Linear map $A: V \xrightarrow{\sim} W$
Isomorphic spaces	Bijection $\mathbb{S} \cong_{\text{set}} \mathbb{T}$	Group isomorphism $G \cong_{\text{grp}} H$	Linear isomorphism $V \cong_{\text{vec}} W$

Recap: Vector space

- A vector space $(V,\oplus,\odot)$ over a field $(K,+,\cdot)$ is the set V equipped with two operations:
  - $\oplus: V \times V \to V$ called vector addition
  - $\odot : K \times V \to V$ called scalar multiplication

that should satisfy the rules:

  - $(V,\oplus)$ is an Abelian group
  - The map $\odot$ is an action of K on $(V,\oplus)$

- An element of $\mathrm{v} \in V$ is called a **vector**.

Recap: Vector space

- **Example is $(\mathbb{R}^{\boldsymbol{n}}, \oplus, \odot)$:**

- $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$

- $\lambda \odot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \lambda \cdot x_1 \\ \vdots \\ \lambda \cdot x_n \end{pmatrix}$

- Identity element of $(\mathbb{R}^{\boldsymbol{n}}, \oplus)$ is the zero **vector** $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

- **Example is $(\mathbb{R}^{\boldsymbol{m \times n}}, \oplus, \odot)$:**

- $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \oplus \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} := \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$

- $\lambda \odot \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} := \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & \ddots & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}$

- Identity element of $(\mathbb{R}^{\boldsymbol{m \times n}}, \oplus)$ is the zero **matrix** $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

Recap: Linear Maps

- A linear map $A: V \to W$ between the vector spaces $(V, \oplus, \odot)$ and $(W, \boxplus, \boxdot)$ over the same field K is defined such that:
  - $A(v_1 \oplus v_2) = A(v_1) \boxplus A(v_2), \quad \forall\, v_1, v_2 \in V$
  - $A(\lambda \odot v) = \lambda \boxdot A(v)\,, \quad \forall\, v \in V, \lambda \in K$

- If we drop the special notation for operators,

$$A(\lambda v_1 + v_2) = \lambda\, A(v_1) + A(v_2)$$

- The set of all linear maps from a vector space V to W is itself a vector-space, denoted by $\mathrm{L}(V; W)$.

Recap: Linear Maps

- Example: $V = \mathbb{R}^n$, $W = \mathbb{R}^m$
- You can prove that any linear map $A: \mathbb{R}^n \to \mathbb{R}^m$ **must** have the form:

$$A(v) = \left(\sum a_i^1 v^i, \cdots, \sum a_i^m v^i\right) ,$$

with $a_i^j \in \mathbb{R}$ for $i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}$.

In components

$$A(v) = \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \cdots & a_n^m \end{pmatrix} \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

m – rows, n - columns

- $\mathrm{L}(\mathbb{R}^n; \mathbb{R}^m) = \mathbb{R}^{m \times n}$ in this case

Example: Map that is not linear

- Example: $V = \mathbb{R}^n$, $W = \mathbb{R}^m$
- Consider the map $\Psi\colon \mathbb{R}^n \to \mathbb{R}^m$ between vector spaces defined by:

$$\Psi(v) = \left(\sum a_i^1 v^i + b^1, \cdots, \sum a_i^m v^i + b^m\right) ,$$

with $a_i^j, b^j \in \mathbb{R}$ for $i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}$.

In components

$$\boxed{\Psi(v) = \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \cdots & a_n^m \end{pmatrix} \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} + \begin{pmatrix} b^1 \\ \vdots \\ b^m \end{pmatrix} = A(v) + b}$$

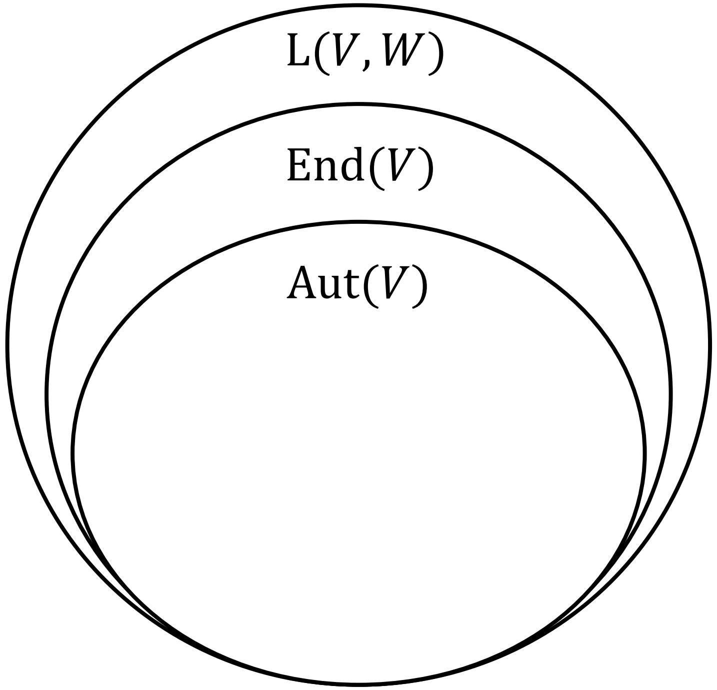
- A map Ψ of the form above is called an **affine map** and is not a linear map.

$$\boxed{\Psi(\lambda v) = A(\lambda v) + b = \lambda\, A(v) + b \neq \lambda\, \Psi(v)}$$

Recap: Subsets of $\mathrm{L}(V, W)$

- $L(V; W) \coloneqq \{A \mid A: V \xrightarrow{\sim} W\}$
- $\mathrm{End}(V) \coloneqq L(V, V) \coloneqq \{A \in L(V, W) \mid W = V\}$
- $\mathrm{Aut}(V) \coloneqq \{A \in \mathrm{End}(V) \mid A \text{ is an isomorphism}\}$

- Example $V = \mathbb{R}^n$:
  - $\mathrm{End}(\mathbb{R}^n) = \mathbb{R}^{n \times n}$
  - $\mathrm{Aut}(\mathbb{R}^n) = GL(n, \mathbb{R})$



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Dual space

- The notion of a **dual (vector) space** $\boldsymbol{V}^*$ to a vector space $\boldsymbol{V}$ is extremely important in mechanics.
- However, it is usually overlooked !!
- While an element of $\boldsymbol{V}$ is called a vector, an element of $\boldsymbol{V}^*$ is called a covector*.
  - Velocity-like variables are vectors.
  - Force-like variables are covectors.

* Also referred to as one-form.

Dual space

- Let V be a vector space over the field K
- The **dual space** to V is:

$$V^* := L(V; K),$$

where $(K, +, \cdot)$ is considered a vector space over itself.

- An element of $\alpha \in V^*$ is a linear map from V to K called a **covector** or **one-form**.

$$\begin{aligned} \alpha: V &\xrightarrow{\sim} K \\ v &\mapsto \alpha(v) \end{aligned}$$

Example: Dual space

- Example: $V = \mathbb{R}^n$, $K = \mathbb{R}$
- The dual space $(\mathbb{R}^n)^*$ consists of all linear maps:

$$\begin{aligned} \alpha: \mathbb{R}^n &\xrightarrow{\sim} \mathbb{R} \\ v &\mapsto \alpha(v) \end{aligned}$$

that have the form

$$\alpha(v) = \sum_{i=1}^{n} \alpha_i v^i$$

In components

$$\boxed{\alpha(v) = \begin{pmatrix} \alpha_1 & \cdots & \alpha_n \end{pmatrix} \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}}$$

Dual of a linear map

- It is conventional to introduce a **duality pairing** between vectors and covectors denoted by:

$$\langle \cdot \mid \cdot \rangle : V^* \times V \overset{\sim}{\to} K$$
$$(\alpha, v) \mapsto \langle \alpha | v \rangle := \alpha(v)$$

- Let $A: U \to V$ be a linear map between the K-vector spaces U and V.

  Then the linear map $A^*: V^* \to U^*$ defined (implicitly) by:

$$\langle A^*(\alpha) | u \rangle = \langle \alpha | A(u) \rangle, \qquad \forall u \in U, \alpha \in V^*$$

is called the **dual of the map A**.

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Tensors

- Let V be a vectorspace over K. A (p, q) tensor T on V is a multi-linear map

$$T: \underbrace{V^* \times \cdots \times V^*}_{p \text{ times}} \times \underbrace{V \times \cdots \times V}_{q \text{ times}} \overset{\sim}{\rightarrow} K$$

i.e., T is a map that *eats* p-covectors and q-vectors.

- The term multi-linear means T is a linear map in each of its entries.
- The rank of a tensor T is the sum:

$$\text{rank}(T) = p + q$$

Tensors

- Cases of interest:
  - $(0,1)$ tensor is a covector $\qquad \alpha : V \xrightarrow{\sim} K$
  - $(1,0)$ tensor is a vector $\qquad v : V^* \xrightarrow{\sim} K$
  - $(1,1)$ tensor $\qquad A : V^* \times V \xrightarrow{\sim} K$
  - $(0,2)$ tensor $\qquad B : V \times V \xrightarrow{\sim} K$
  - $(2,0)$ tensor $\qquad C : V^* \times V^* \xrightarrow{\sim} K$

- The set of all (p,q) tensor T on V is denoted by $T_q^p V$ and we can make it into a vector-space structure.

$$T_q^p V \coloneqq \{T \mid T \text{ is a } (p,q) \text{ tensor on V}\}$$

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Components of tensors

- So far, the mathematical objects on vector spaces we introduced are abstractly defined.
  - Vectors, covectors, linear maps, dual of linear maps, (p, q) tensors.
- All these objects can be written in components once a basis has been chosen.
- However, the geometric nature of these objects should be respected independent of the basis we choose.

Basis for a vector space

- Let V be a vector space. A basis S for a vector space V is a collection of vectors in V that are:
  - Linearly independent from each other
  - Generate V

Basis for a vector space

- Let V be a vector space over $\mathbb{R}$. A basis S for a vector space V is a collection of vectors in V that are:
  - **Linearly independent from each other**
    - A set $S \subset V$ of vectors is linearly independent if, for every finite subset $\{e_1, \cdots, e_k\} \subset S$, the equality

      $$\sum_{i=i}^{k} c^i e_k = c^1 e_1 + \cdots + c^k e_k = 0, \text{ for some constants } c^i \in \mathbb{R},$$

      implies that these constants should be zero, i.e., $c^i = 0\ \forall i \in \{1, \cdots, k\}$.

  - Generate V

Basis for a vector space

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      implies that these constants should be zero, i.e., $c^i = 0 \ \forall i \in \{1, \cdots, k\}$.

  - **Generate V**
    - A set $S \subset V$ of vectors generates a vector space V, if every vector $\mathrm{v} \in V$ can be written as the linear combination

      $$v = c^1 e_1 + \cdots + c^k e_k \text{ , for some constants } c^i \in \mathbb{R}$$

    - We usually write that $V = \mathrm{span}_{\mathbb{R}}(S)$