SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 5: Manifolds and Lie groups

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Outline

- Why differentiable structure ?
- Atlas of the world
- Manifold theory
- Maps on a manifold
- Construction of the tangent bundle

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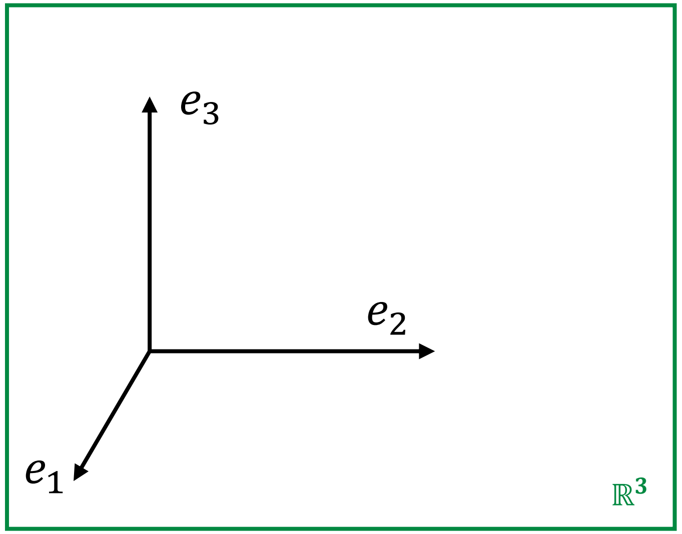
Different views of $\mathbb{R}^n$

- As a set $\qquad \mathbb{R}^n \coloneqq \mathbb{R} \times \cdots \times \mathbb{R}$
- As a vector space $\qquad (\mathbb{R}^n, \oplus, \odot)$ over $(\mathbb{R}, +, \cdot)$



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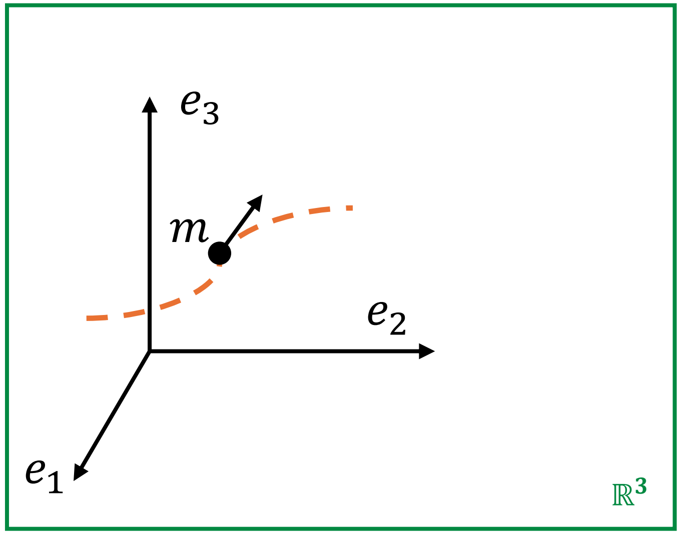
- To do calculus, analysis, and describe dynamical systems we need more structure.
- This is called a differentiable structure.

- A set M along with this differentiable structure

is called a differentiable manifold.



Differentiability Class of functions

- In mathematical analysis, the smoothness of a function is a property measured by the number of continuous derivatives (differentiability class) it has over its domain.

- Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a map from an open interval of $\mathbb{R}$ to $\mathbb{R}$. Then the function f is said to be of:
  - Class C^0: if f is continuous on I
  - Class C^1: if its derivative f' exists and both f, f' are continuous on I
  - Class C^k: if its derivatives $f', f'', \cdots, f^{(k)}$ exist and are all continuous on I
  - Class C^∞: if it has derivatives of all orders on I

C^∞ functions are called smooth / infinitely-differentiable functions

Differentiability Class of functions

- The same concept can be extended to maps on $\mathbb{R}^n$

- Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ be a map from an open interval of $\mathbb{R}^n$ to $\mathbb{R}$. Then the function f is said to be of class C^k, for some positive integer k, if all partial derivatives

$$\frac{\partial^\alpha f}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}\, \cdots\, \partial x_n^{\alpha_n}}(y_1, y_2, \ldots, y_n)$$

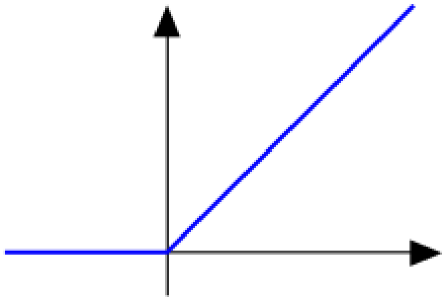
exist and are continuous on U.

What does a continuous function mean ?

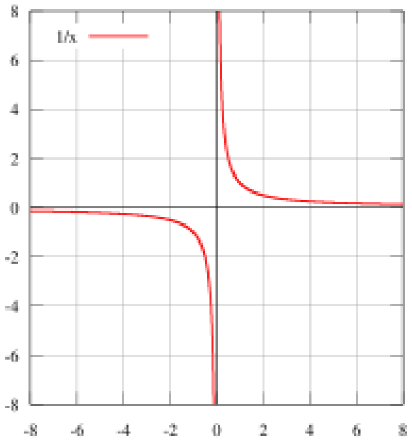
- The formal way of analyzing continuity of functions is by making a set M into a topological space by equipping it with a topology $\mathcal{O}$.

- If the topological space $(M, \mathcal{O})$ satisfies some conditions (it is paracompact and Hausdorff) it is called a topological manifold.

- When you do analysis on Euclidean space $\mathbb{R}^n$, you are using its standard topology $\mathcal{O}_{\text{std}}$.
- How do we do analysis on general topological spaces that are not Euclidean?



C^0 function on $\mathbb{R}$



Not a C^0 function on $\mathbb{R}$

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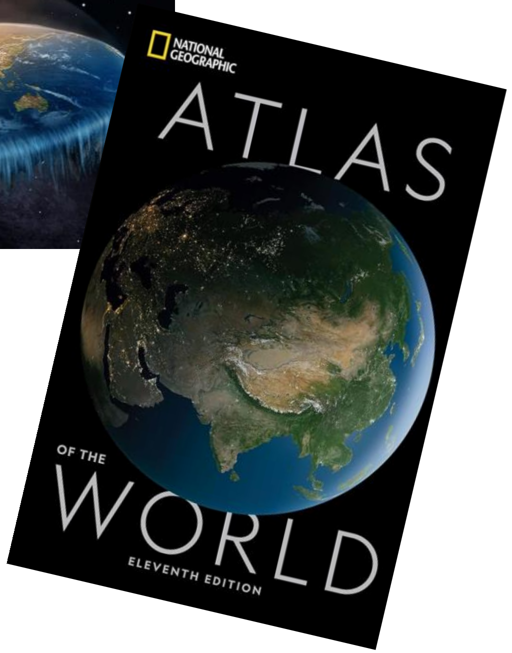
Atlas of the world

- The surface of the earth is an example of a two-dimensional non-Euclidean space.
- Locally, S^2 “looks-like” $\mathbb{R}^2$ but globally it is not.

$$S^2 \coloneqq \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = c^2\}$$

- An atlas of Earth is a collection of charts.
- Each chart maps a “local” region of Earth

into a piece of paper $\mathbb{R}^2$.



Charts

- Consider two charts over Egypt and Saudi Arabia.

S^2 Earth

$\mathbb{R}^2$ charts

Charts

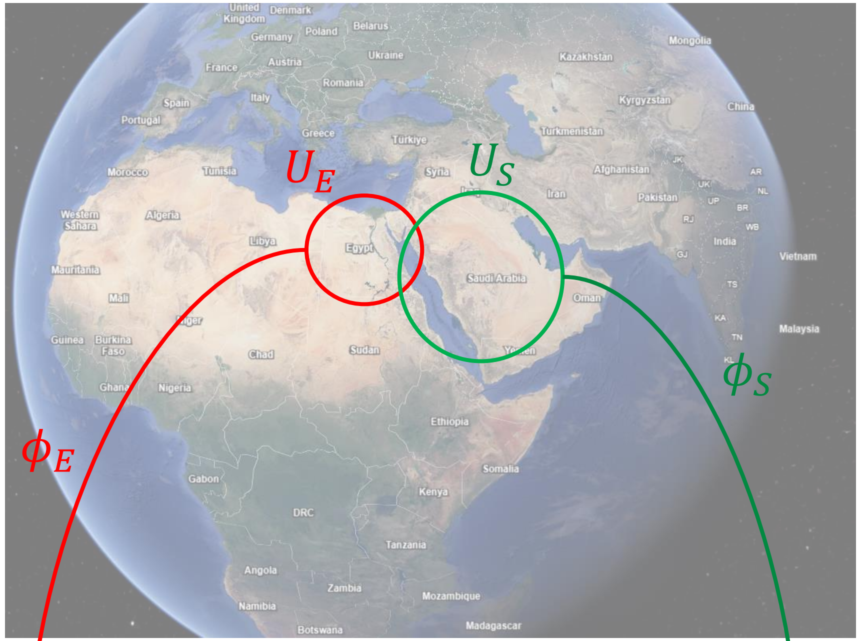
- A chart of Earth consists of the pair (U, ϕ) where $U \subset S^2$ and $\phi: U \to \mathbb{R}^2$ is a continuous map.

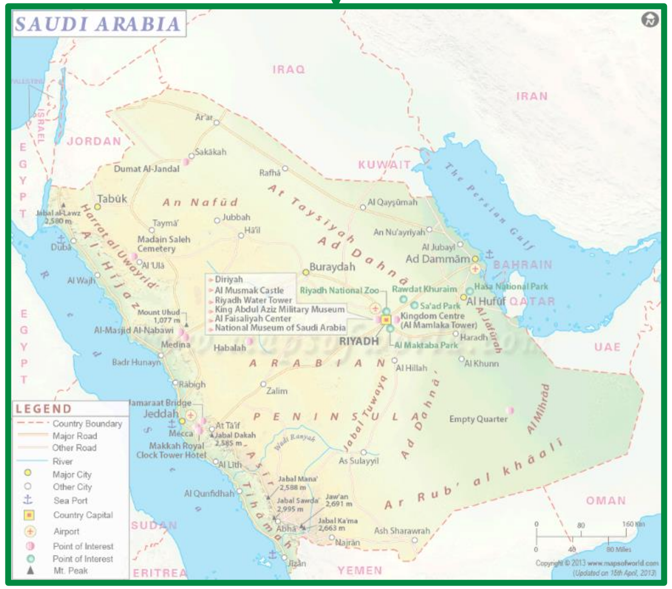
- A collection of charts that cover all of Earth S^2 is called an Atlas $\mathcal{A}$:

$$\mathcal{A} \coloneqq \{(U_i, \phi_i)\}_{i \in A}$$

with the property that

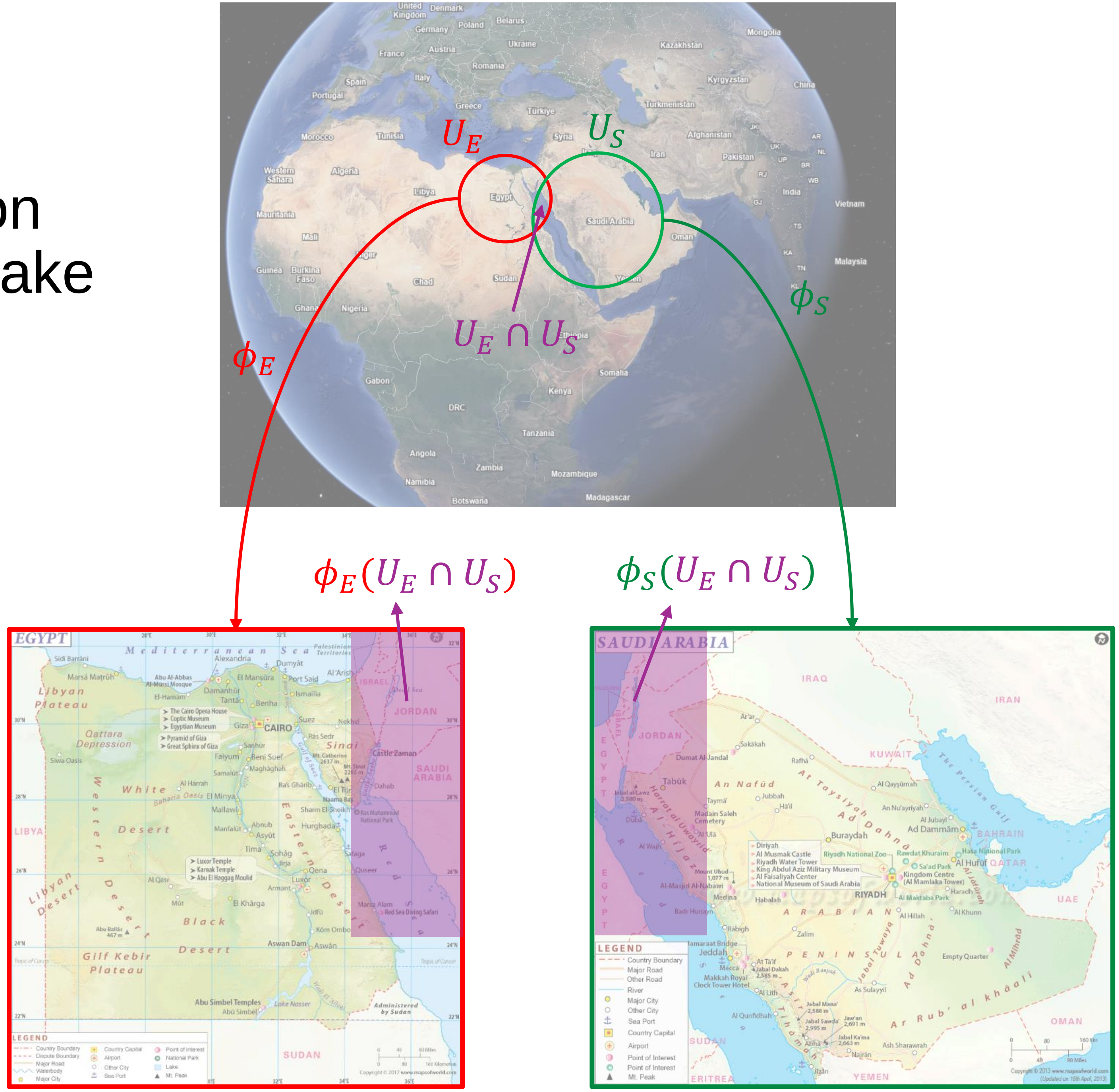
$$S^2 = \bigcup_{i \in A} U_i$$





Charts

- However, in an overlap region (e.g. $U_E \cap U_S$) we need to make sure that the charts are compatible.



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Differentiable Manifold

- The fundamental object in differential geometry is a differentiable manifold.
- Intuitively, an n-dimensional manifold is a set that locally "looks like" an open subset of Euclidean space $\mathbb{R}^n$.

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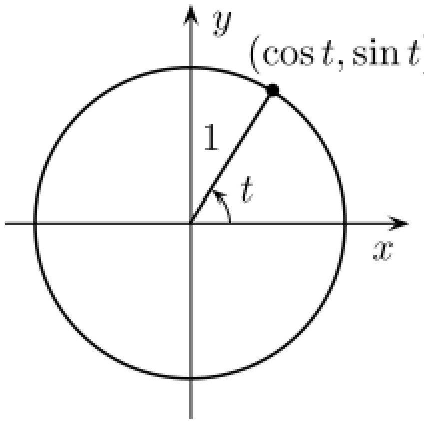
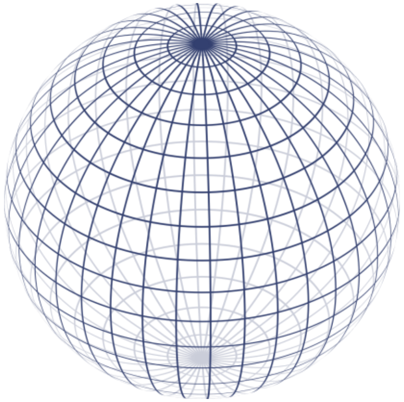
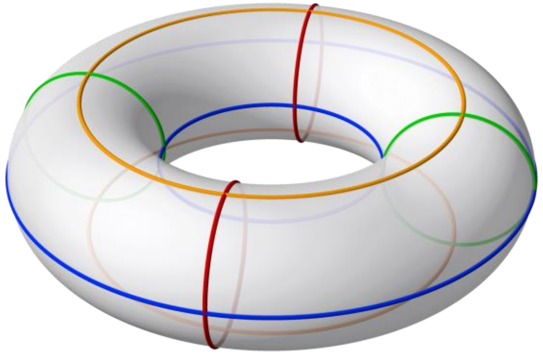
- Examples:



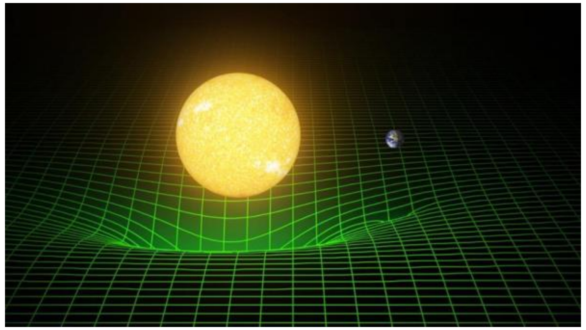
Circle S^1



Sphere S^2



Torus T^2



Spacetime

Differentiable Manifold

- Intuitively, an n-dimensional manifold is a set that locally "looks like" an open subset of Euclidean space $\mathbb{R}^n$.
- Formally, a C^k differentiable manifold is the triple $(M, \mathcal{O}, \mathcal{A})$ where $(M, \mathcal{O})$ is a topological manifold* and $\mathcal{A}$ is a C^k-atlas for M.

*$(M, \mathcal{O})$ is a topological manifold is the formal way of saying it locally looks like $\mathbb{R}^n$.

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- An C^k-atlas for M is a collection of charts that cover the entire manifold while satisfying certain overlap conditions.
- Given this C^k-differentiable structure on M, we can then talk about curves on manifolds, maps between manifolds, differentiability of maps,… etc.

*$(M, \mathscr{O})$ is a topological manifold is the formal way of saying it locally looks like $\mathbb{R}^n$.

Charts and Atlas of a manifold

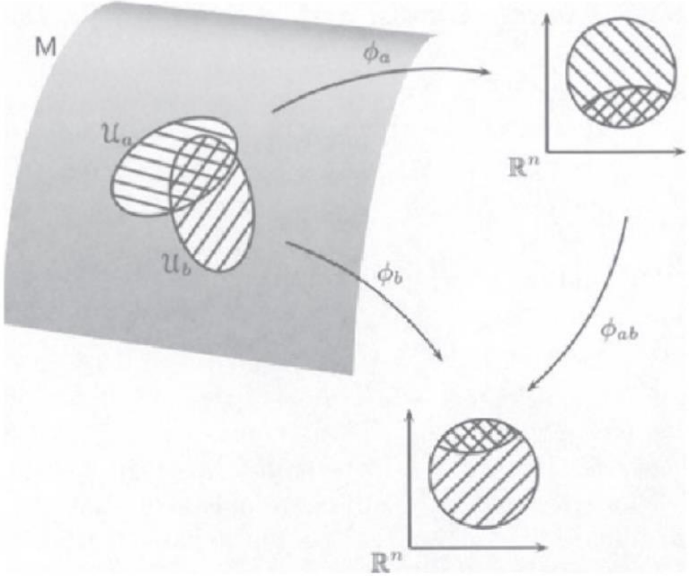
- Let $(M, \mathcal{O})$ be a set equipped with a topology.
- A chart for M is the pair (U, ϕ) with $U \subset M$ an open subset* of M and $\phi: U \to \mathbb{R}^n$ with $\phi(U) \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$.
- A C^k-atlas for M is the collection $\mathcal{A} := \{(U_i, \phi_i)\}_{i \in A}$

with the properties that $M = \bigcup_{i \in A} U_i$ and whenever

$U_a \cap U_b \neq \emptyset$ we have that the overlap/transition map

$$\phi_{ab} := \phi_b \circ \phi_a^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$

is of class C^k.



*open subset $U \subset M$ is equivalent to $U \in \mathcal{O}$

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