SCE 594: Special Topics in Intelligent Automation & Robotics

Topic 2 – Rigid Body Modeling

Lecture 6: Configuration space of a rigid body

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Outline

- Topic 2 Intro
- Configuration Space of Rigid Bodies
- Homogeneous transformations
- Lie group structure of $SO(3)$

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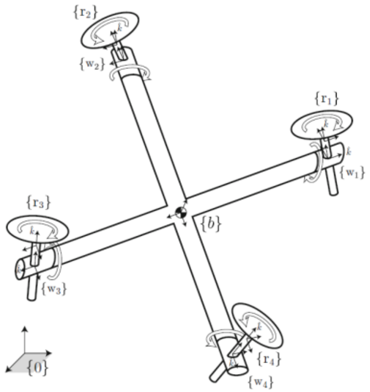
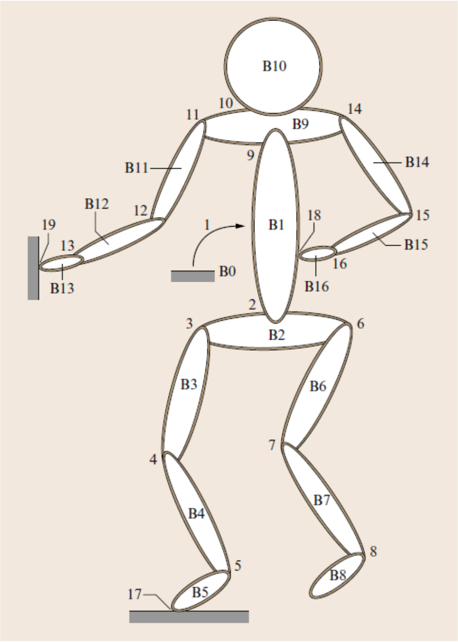
Topic 2: Rigid Body Modeling

- Most robotic mechanisms are systems of rigid bodies connected by joints.

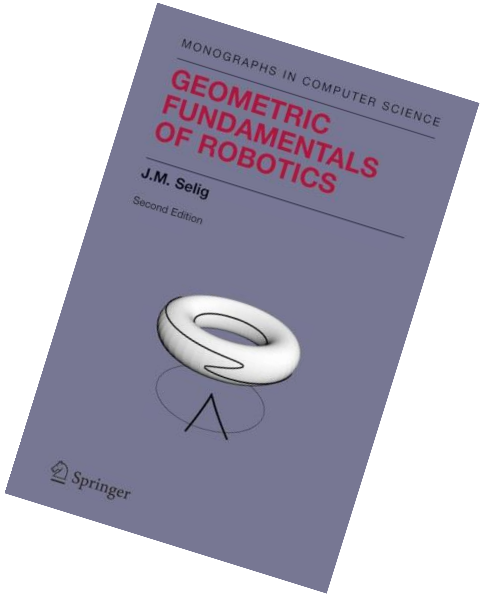
- Understanding how to model and interconnect rigid bodies is fundamental !





Configuration space of a rigid body

- We shall follow the Lie group approach for describing rigid body kinematics and dynamics.
- The configuration space of a rigid body is the group of proper isometries of Euclidean spaces, known as the special Euclidean group SE(3).

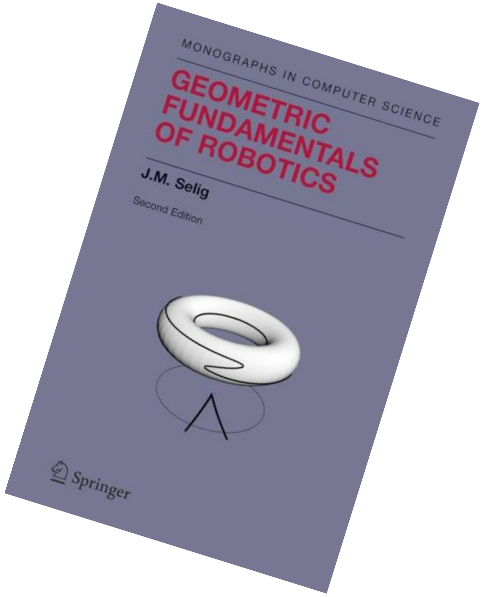


"This group is perhaps the most important one for robotics"

J. M. Selig.

Configuration space of a rigid body

- In what follows, we will start from a coordinate-free approach to:
  - Differentiate abstract coordinate-free description of rigid body motion and its coordinate representation using matrices.
  - Highlight the mathematical nature of the different maps and spaces that arise when describing rigid bodies.



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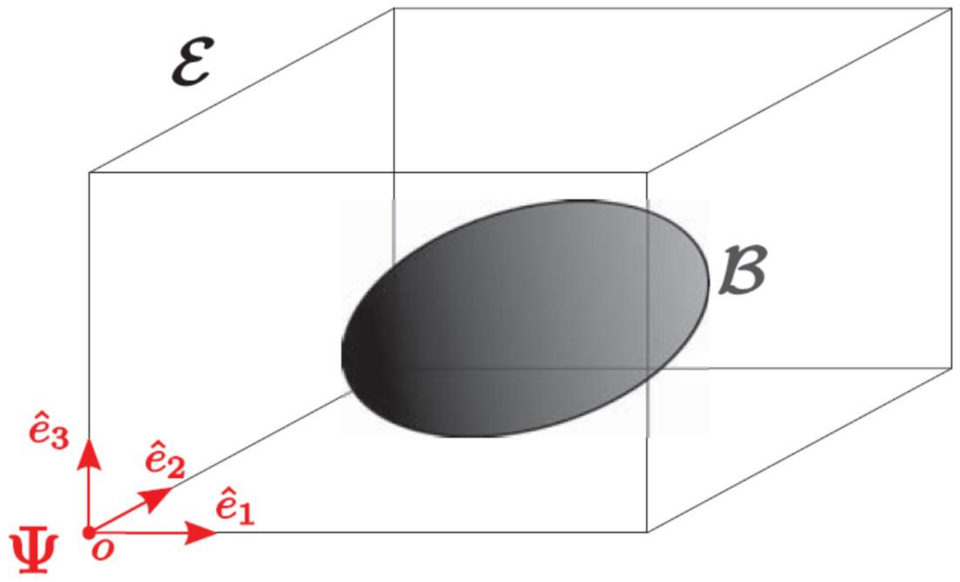
Euclidean space

- Concepts:
  1. *n*-dimensional Euclidean space $\mathcal{E}$
  2. Rigid body $\mathcal{B}$
  3. Coordinate-frame Ψ

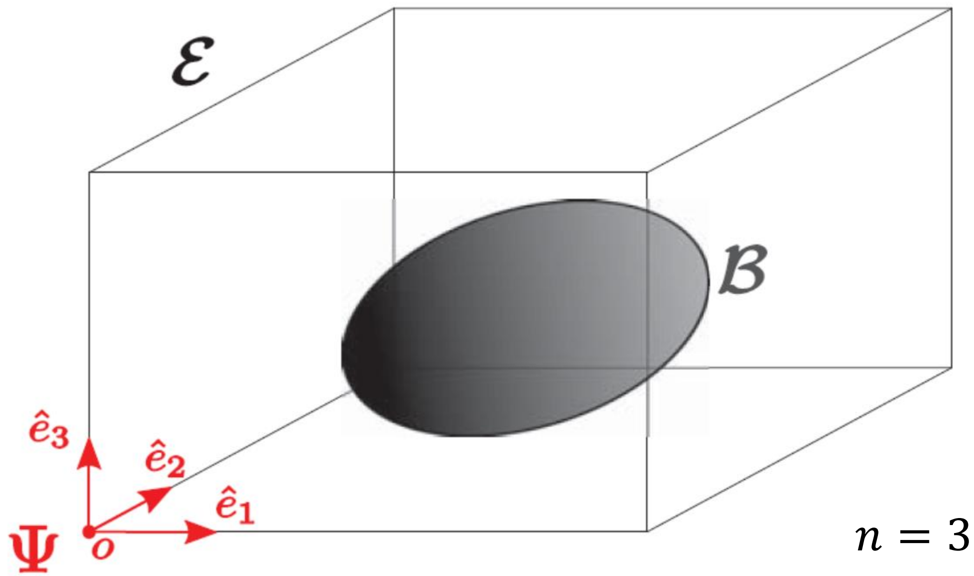


Euclidean space

1. *n*-dimensional Euclidean space $\mathcal{E}$:

- An abstraction of the physical space we are living in.
- Associated to $\mathcal{E}$ is a vector space $\mathcal{E}_*$
- Elements of $\mathcal{E}$ are points $p \in \mathcal{E}$
- Elements of $\mathcal{E}_*$ are free-vectors $v \in \mathcal{E}_*$

- $\mathcal{E}$ is equipped with a metric $d: \mathcal{E} \times \mathcal{E} \to \mathbb{R}_+$ that defines the distance between any two points in $\mathcal{E}$.

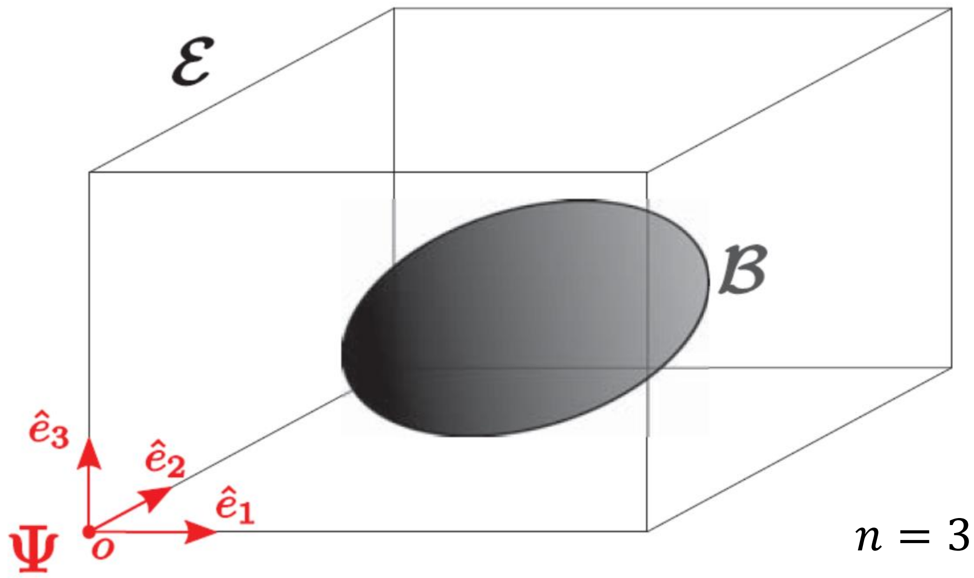


$n = 3$

Euclidean space

2. Rigid body $\mathcal{B}$

- A rigid body is mathematically the pair $(\mathcal{B}, \rho)$
- $\mathcal{B} \subset \mathcal{E}$ is the set of points where matter is present
- $\rho: \mathcal{B} \to \mathbb{R}_+$ is the mass density function



$n = 3$

Euclidean space

3. Coordinate-frame Ψ

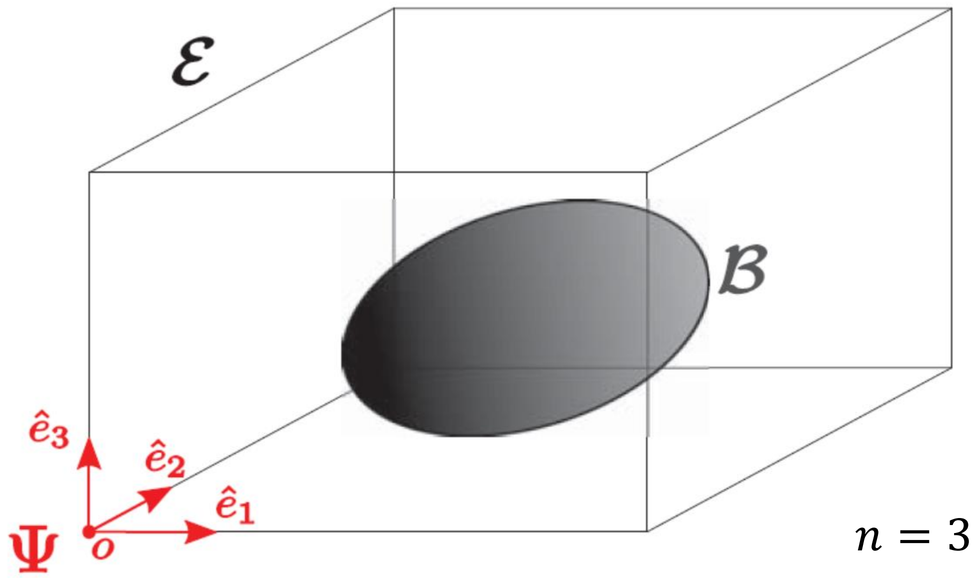
- A coordinate frame for the Euclidean space $\mathcal{E}$ is the 4-tuple

$$\Psi := \{o, \hat{e}_1, \hat{e}_2, \hat{e}_3\}$$

with $o \in \mathcal{E}$ denoting the origin of the frame and $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are linearly-independent orthonormal free vectors.

- Using Ψ, we can represent any points $p \in \mathcal{E}$ and any free vector $v \in \mathcal{E}_*$ using coordinates in $\mathbb{R}^n$:

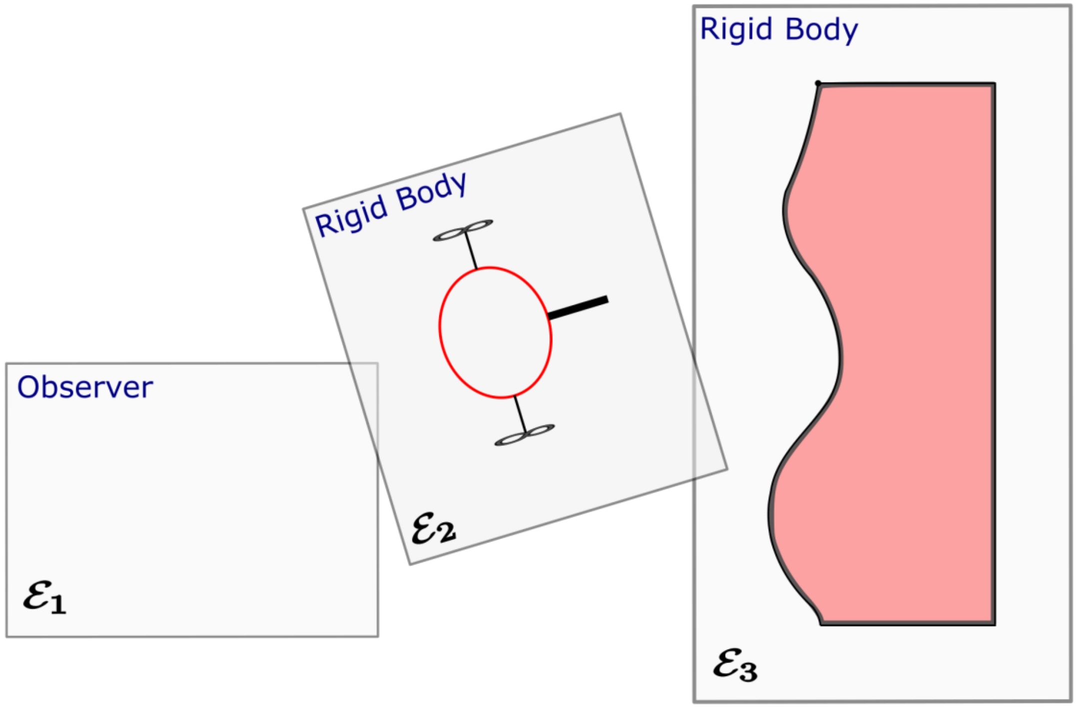
$$\boxed{p = p^i \hat{e}_i} \qquad \boxed{v = v^i \hat{e}_i}$$



$n = 3$

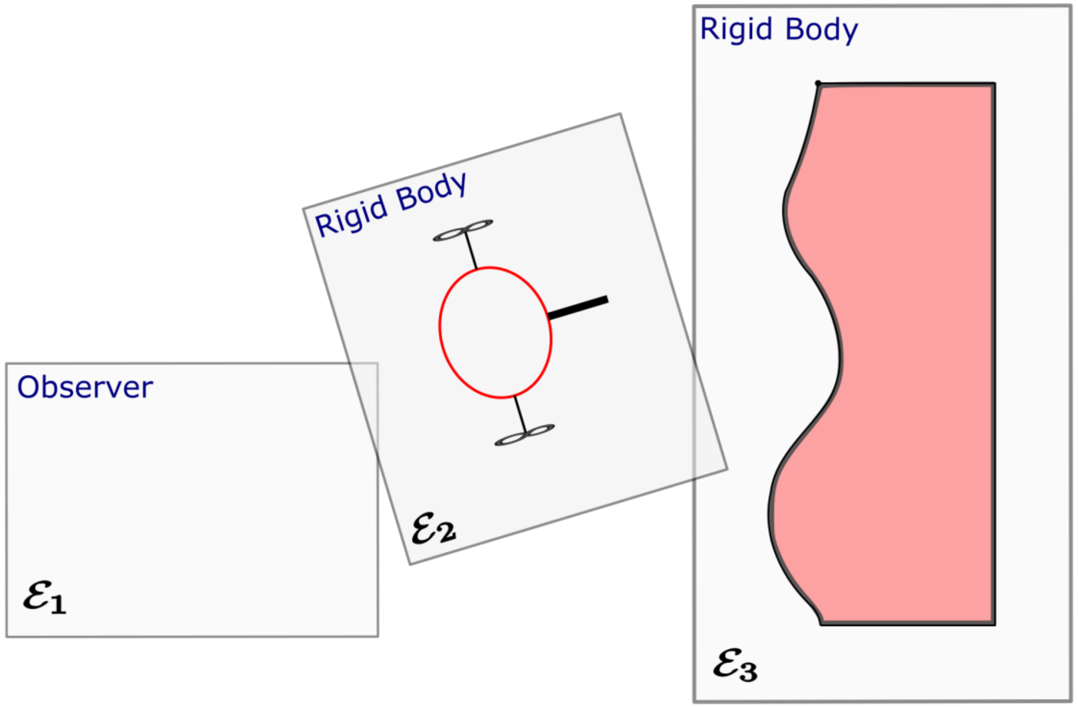
Euclidean system in 2D

- A Euclidean system consists of a collection of rigid bodies and observers.
- An observer is an example of a Euclidean space with the matter set being the empty set (e.g., a camera or external sensor).



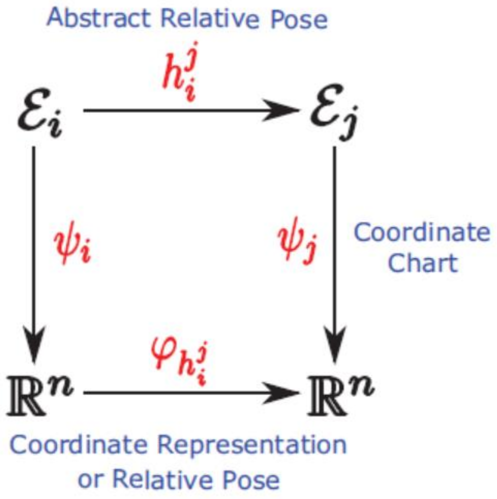
Relative Pose

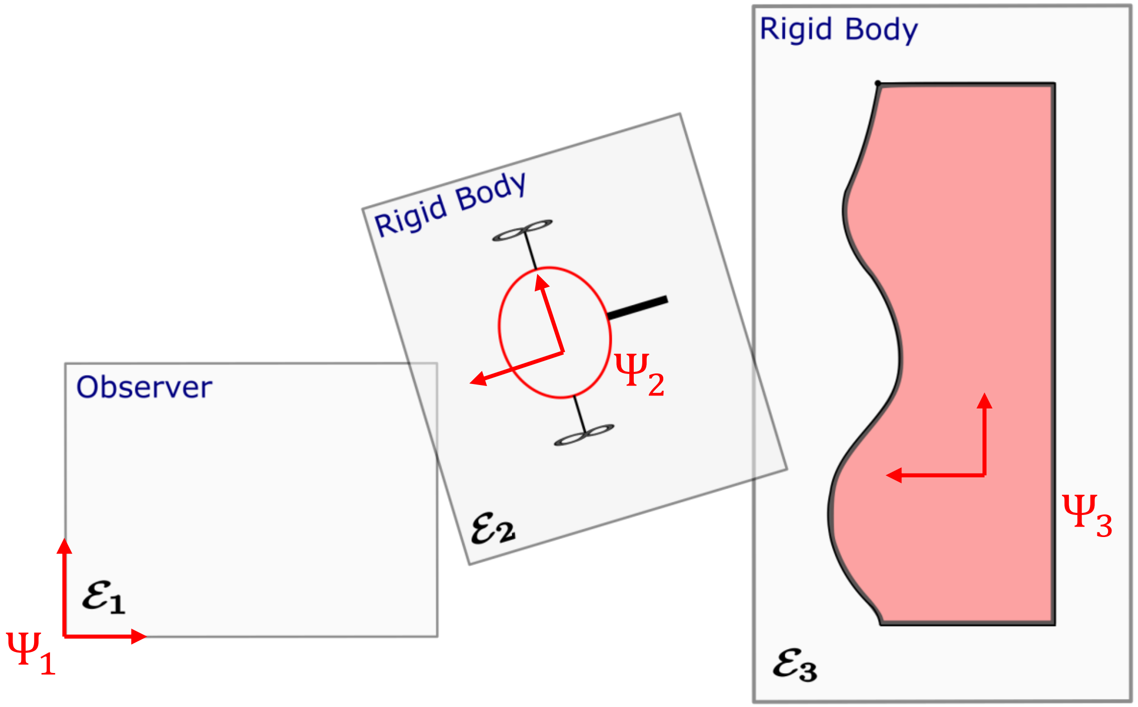
- The combined orientation and displacement of a Euclidean space is called the pose.
- The relative pose between two Euclidean spaces $\mathcal{E}_i$ and $\mathcal{E}_j$ with respect to each other will be denoted by $h_i^j: \mathcal{E}_i \to \mathcal{E}_j$.
- The set of relative poses between $\mathcal{E}_i$ and $\mathcal{E}_j$ will be denoted by $SE_i^j(n) \ni h_i^j$.



Relative Pose

- By associating to every Euclidean space $\mathcal{E}_k$ a coordinate frame Ψ_k , we can assign to every $h_i^j \in SE_i^j(n)$ a map $\varphi_{h_i^j}: \mathbb{R}^n \to \mathbb{R}^n$ which corresponds to the numerical representation of the relative pose of frames Ψ_i and Ψ_j.

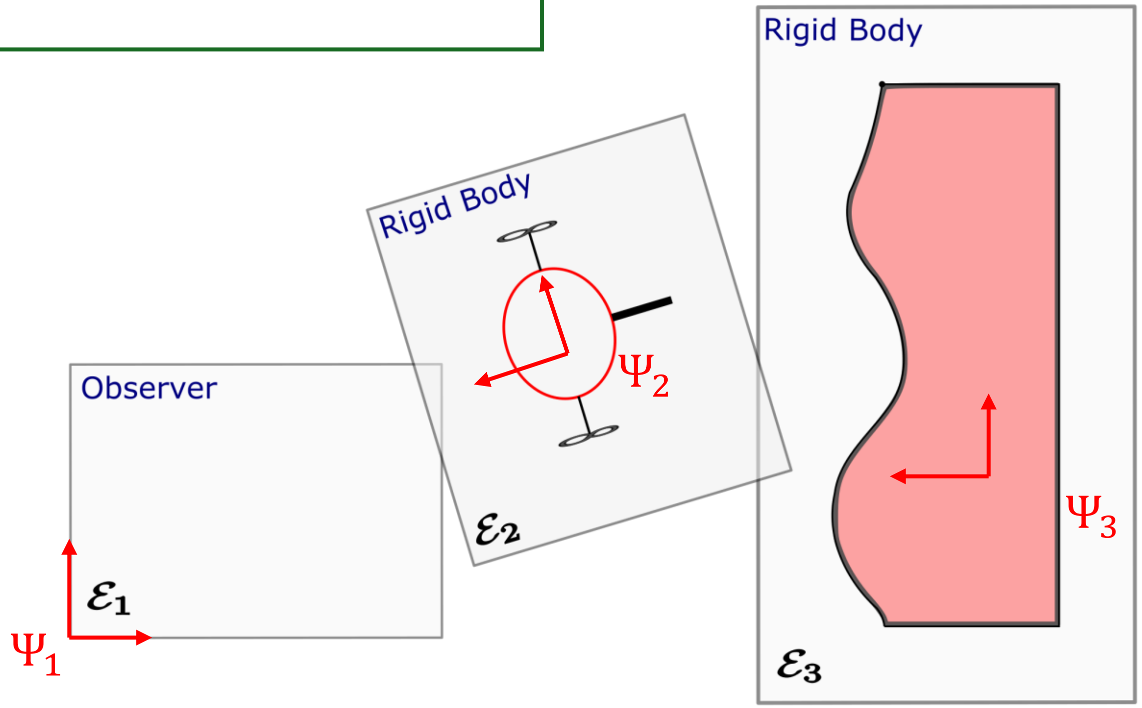




Isometries on $\mathbb{R}^n$

- For $\varphi_{h_i^j} : \mathbb{R}^n \to \mathbb{R}^n$ to represent a rigid body transformation, it needs to preserve distances between points.
- Such a map is called an isometry on $\mathbb{R}^n$:

$$\boxed{\left\| \varphi_{h_i^j}(v) - \varphi_{h_i^j}(w) \right\|_{\mathbb{R}^n} = \|v - w\|_{\mathbb{R}^n}, \qquad \forall v, w \in \mathbb{R}^n}$$

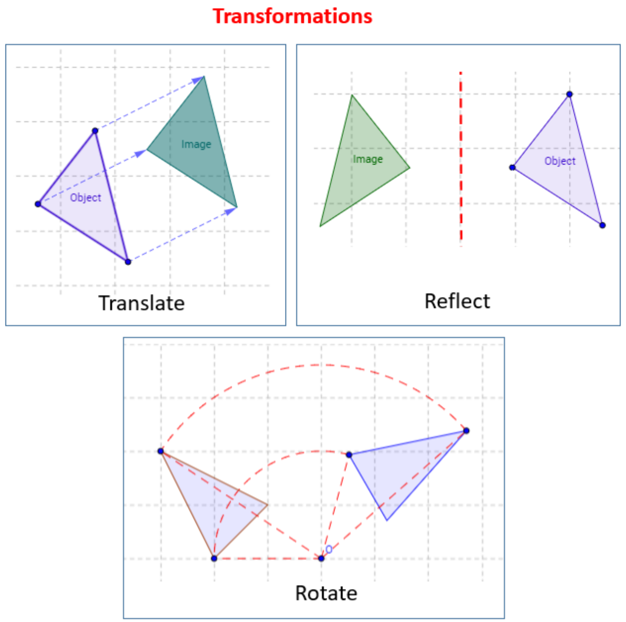


Isometries on $\mathbb{R}^n$

- For $\varphi_h: \mathbb{R}^n \to \mathbb{R}^n$ to represent a rigid body transformation, it needs to preserve distances between points.
- Such a map is called an isometry on $\mathbb{R}^n$:

$$\boxed{\|\varphi_h(v) - \varphi_h(w)\|_{\mathbb{R}^n} = \|v - w\|_{\mathbb{R}^n}, \qquad \forall v, w \in \mathbb{R}^n}$$

- Isometries on $\mathbb{R}^n$ include:
  - Translation
  - Rotation
  - Reflection



Isometries on $\mathbb{R}^n$

- In general, an isometry $\varphi_h : \mathbb{R}^n \to \mathbb{R}^n$ can be written as

$$\boxed{\varphi_h(v) = \boldsymbol{M}v + \boldsymbol{\xi}}$$

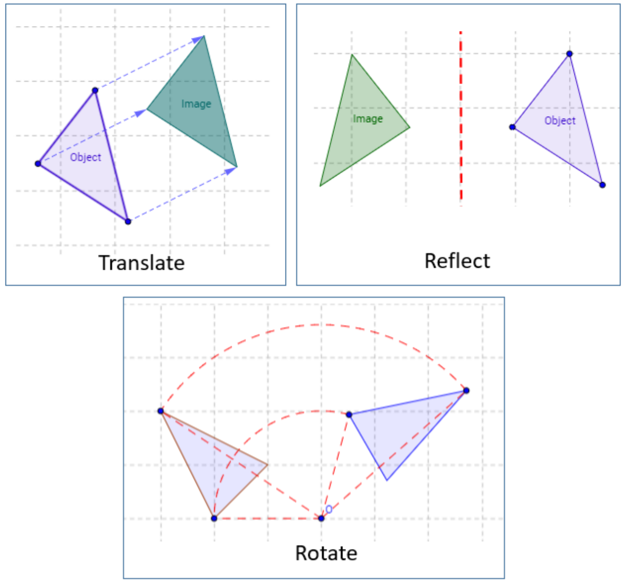
where $\boldsymbol{M} \in O(n)$ is an orthogonal matrix and $\boldsymbol{\xi} \in \mathbb{R}^3$ is a vector.

- The set $O(n)$ is defined by

$$\boxed{O(n) \coloneqq \{\boldsymbol{M} \in R^{n\times n} \,|\boldsymbol{M}\boldsymbol{M}^\top = \boldsymbol{I_n}\}}$$

$O(n)$ has a group structure equipped with matrix multiplication

Transformations



- M represents rotation or reflection
- ξ represents translation

Special Orthogonal Group

- An element $\boldsymbol{M} \in O(n)$ can have determinant*:
  - $\det \boldsymbol{M} = +1$ → Rotation
  - $\det \boldsymbol{M} = -1$ → Reflection

$$\boxed{O(n) := \{\boldsymbol{M} \in R^{n\times n} \,|\, \boldsymbol{M}\boldsymbol{M}^{\top} = \boldsymbol{I_n}\}}$$

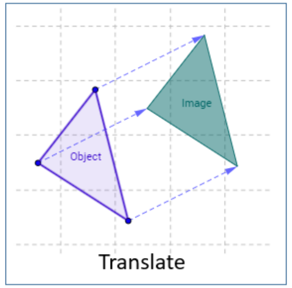
- To exclude reflections, we restrict $O(n)$ to the subgroup:

$$\boxed{SO(n) := \{\boldsymbol{R} \in O(n) \,|\, \det \boldsymbol{R} = 1\}}$$

which is called the special orthogonal group.

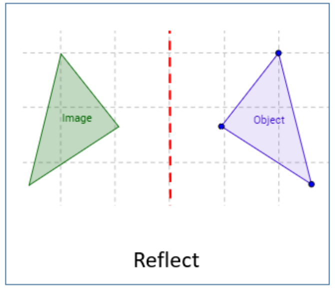
- An element $\boldsymbol{R}$ of $SO(n)$ is called a rotation matrix.

$\xi \in \mathbb{R}^3$

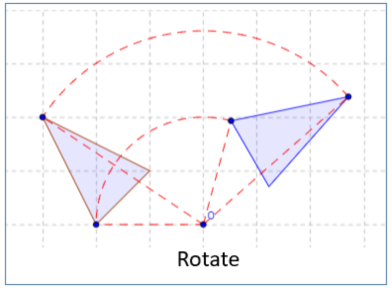


Translate

$\det \boldsymbol{M} = -1$



Reflect


Rotate

$\det \boldsymbol{M} = +1$

* Homework 2 problem

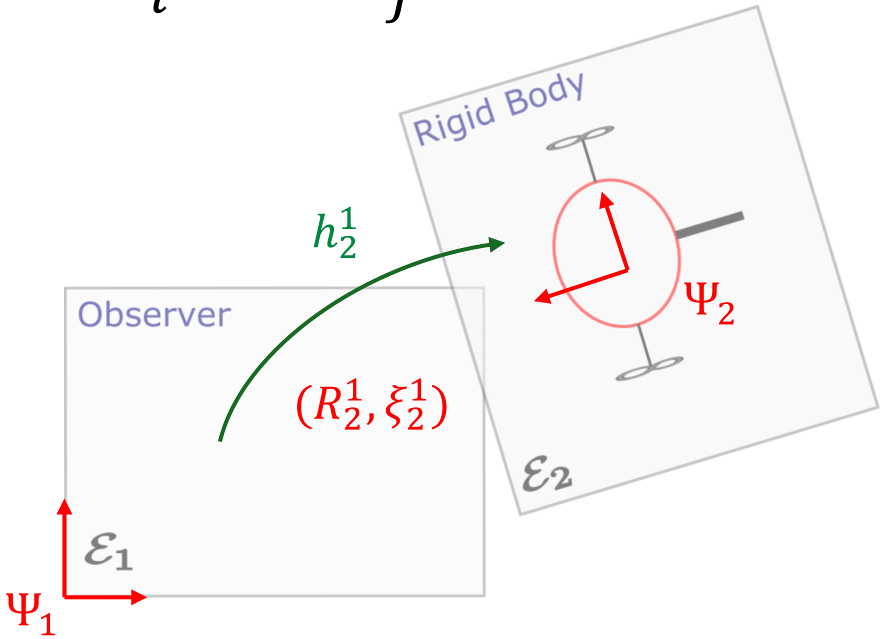
Special Euclidean Group

- Therefore, the isometries $\varphi_{h_i^j} : \mathbb{R}^n \to \mathbb{R}^n$ that represent rigid body rotations and translations are identified by the pair

$$\left(\mathbf{R}_i^j, \boldsymbol{\xi}_i^j\right) \in SO(n) \times \mathbb{R}^n$$

where $\mathbf{R}_i^j \in SO(n)$ represents the relative orientation of Ψ_i and Ψ_j and $\boldsymbol{\xi}_i^j \in \mathbb{R}^n$ represents the relative displacement of Ψ_i and Ψ_j.

- The product $SO(n) \times \mathbb{R}^n =: SE(n)$ is called the special Euclidean group.



Special Euclidean Group

* Homework 2 problem

- The special Euclidean group $(SE(n), \blacksquare)$ is defined as

$$\boxed{SE(n) \coloneqq \{h = (R, \xi) | R \in SO(n), \xi \in \mathbb{R}^n\}}$$

  - Group operation $\blacksquare : SE(n) \times SE(n) \rightarrow SE(n)$ defined as

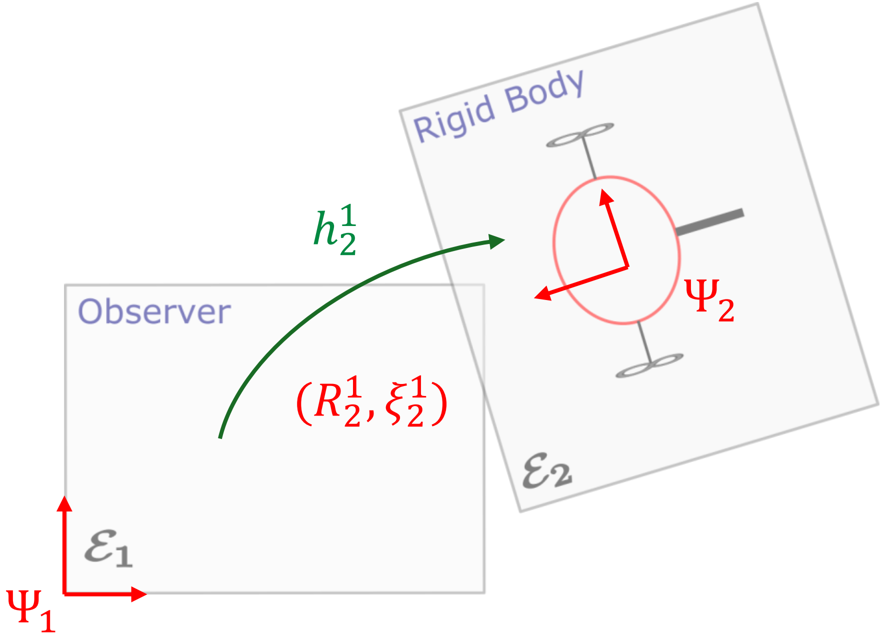
$$\boxed{h_2 \blacksquare h_1 \coloneqq (R_2 \cdot R_1, R_2 \cdot \xi_1 + \xi_2)}$$

  - Identity element $e \in SE(n)$:

$$\boxed{e = (I_n, 0)}$$

  - Inverse element of any $h \in SE(n)$:

$$\boxed{h^{-1} = (R^\top, -R^\top \cdot \xi)}$$



Special Euclidean Group

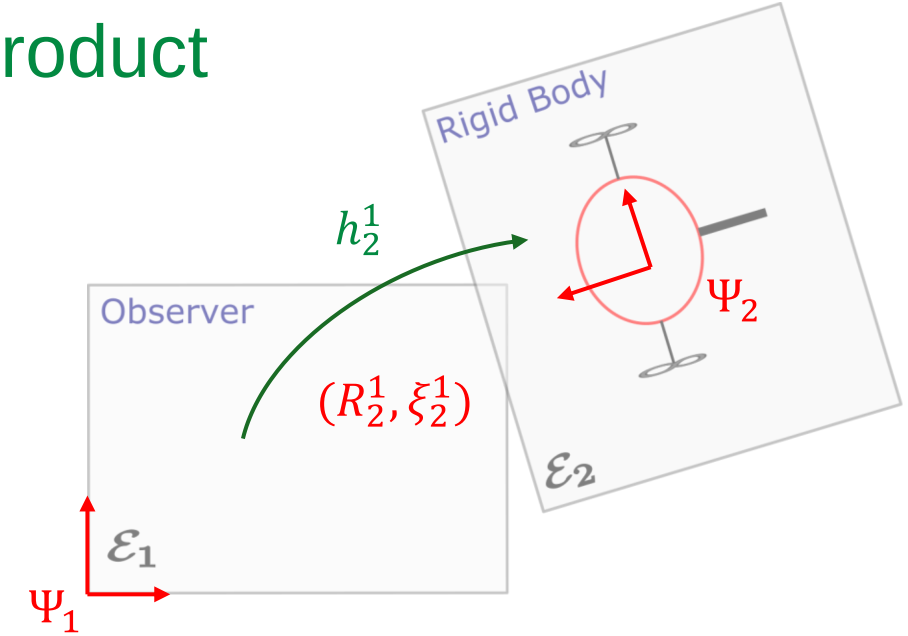
* Homework 2 problem

- The special Euclidean group $(SE(n), \blacksquare)$ is defined as

$$\boxed{SE(n) := \{h = (R, \xi) | R \in SO(n), \xi \in \mathbb{R}^n\}}$$

- The above construction implies that $SE(n)$ is isomorphic to $SO(n) \times \mathbb{R}^n$ as sets but not as groups.
- We say that $(SE(n), \blacksquare)$ is the semi-direct product of the groups $(SO(n), \cdot)$ and $(\mathbb{R}^n, +)$:

$$\boxed{SE(n) = SO(n) \ltimes \mathbb{R}^n}$$



$SE(n)$ is the **configuration space** of rigid body pose in n-dimensional space

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Homogeneous transformations

- Using concepts from projective geometry, we can represent $SE(n) = SO(n) \ltimes \mathbb{R}^n$ using matrices in dimension $n + 1$.

$$SE(n) \to HM(n+1) \subset GL(n+1)$$
$$\boldsymbol{h} = (\boldsymbol{R}, \boldsymbol{\xi}) \mapsto \begin{pmatrix} \boldsymbol{R} & \boldsymbol{\xi} \\ \boldsymbol{0} & 1 \end{pmatrix} =: \boldsymbol{H}.$$

- The matrix $\boldsymbol{H} \in HM(n+1)$ is called the homogeneous representation of $\boldsymbol{h} = (\boldsymbol{R}, \boldsymbol{\xi}) \in SE(n)$.

Homogeneous transformations

- The group and inverse operations of SE(n) given earlier can be represented now as:

$$H_2 H_1 = \begin{pmatrix} R_2 & \xi_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 & \xi_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_2 R_1 & R_2 \xi_1 + \xi_2 \\ 0 & 1 \end{pmatrix},$$

$$H^{-1} = \begin{pmatrix} R & \xi \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^\top & -R^\top \xi \\ 0 & 1 \end{pmatrix}.$$

Homogeneous transformations

- Similarly, the action of $SE(n)$ on $\mathbb{R}^n$ is now identified by the action of $HM(n+1)$ on $\mathbb{R}^{n+1}$

$$\boxed{\boldsymbol{HV} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{\xi} \\ \boldsymbol{0} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{Rv} + \boldsymbol{\xi} \\ 1 \end{pmatrix}}$$

where $V \in \mathbb{R}^{n+1}$ is called the homogeneous coordinates of $v \in \mathbb{R}^n$

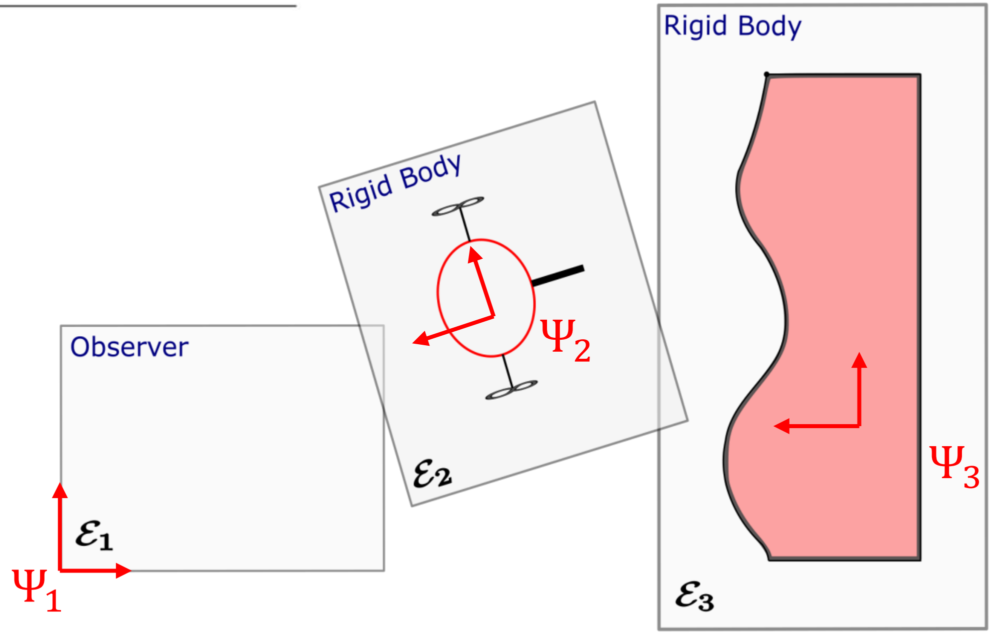
Summary

- We introduced different mathematical objects that describe the configuration of a rigid body.

Abstract Configuration	$h_i \in SE_i(n)$
Abstract Relative Pose	$h_i^j \in SE_i^j(n)$
Rotation Matrix & Translation Vector	$(\boldsymbol{R}_i^j, \boldsymbol{\xi}_i^j) \in SE(n)$
Homogeneous Matrix	$\boldsymbol{H}_i^j \in HM(n+1)$

From now on:

- We will refer to the configuration space of a rigid body as $SE(n)$.
- We will work with homogeneous matrices for calculations.
- We will abusively refer to the space of homogeneous matrices as $SE(n)$, and thus write $\boldsymbol{H} \in SE(n)$.



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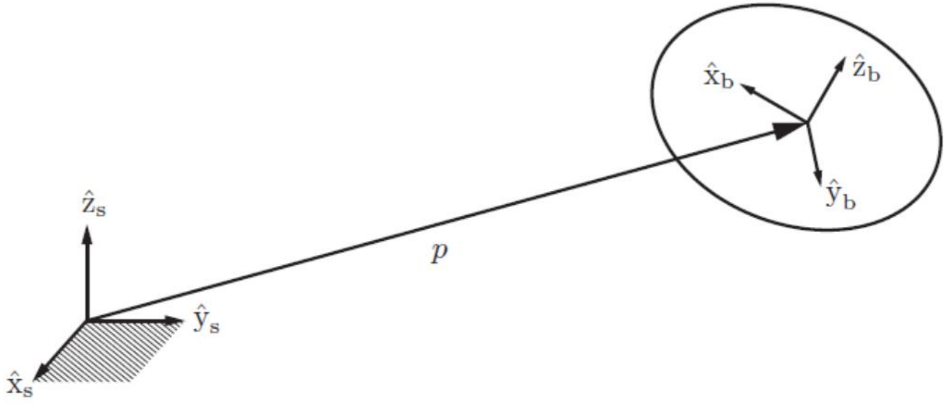
Lie group $SE(n)$

- So far we've seen that $SE(n)$ is a group made up of the semi-direct product of two groups $SO(n)$ and $\mathbb{R}^n$.
- Now to talk about the kinematics and dynamics of rigid body motion, we need to start considering $SE(n)$ as a manifold.
- Therefore, $SE(n)$ has the structure of a Lie group.
- Moreover, $SO(n)$ itself is a Lie sub-group.

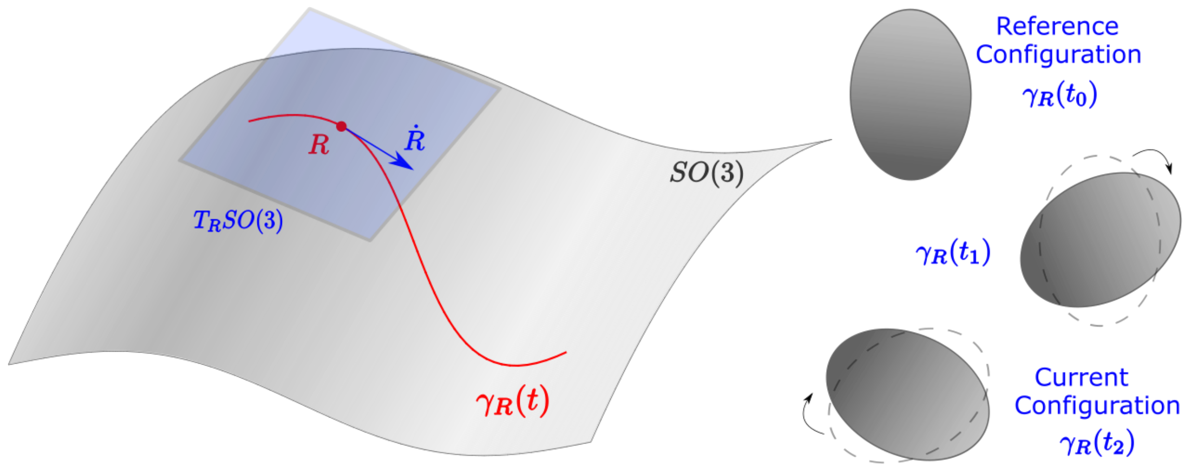
$$\boxed{SE(n) = SO(n) \ltimes \mathbb{R}^n}$$



Rigid body rotations

- In what follows, we focus on the Lie group structure of $SO(3)$.
- A rigid body rotational motion is represented mathematically by a curve

$$\gamma_R : I \subset \mathbb{R} \rightarrow SO(3)$$
$$t \mapsto \gamma_R(t) =: \boldsymbol{R}_t$$



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Next Up!

How do we represent the relation between rate-of-change of orientation and angular velocity ?