# SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 7: Rigid Body Kinematics

جامعة الملك فهد للبترول والمعادن
King Fahd University of Petroleum & Minerals

**Ramy Rashad**
**Assistant Professor**
**CIE Department**

# Outline

- Recap Last Lectures
- Kinematic Modeling
- Lie group structure of $SO(3)$
- Lie group structure of $SE(3)$

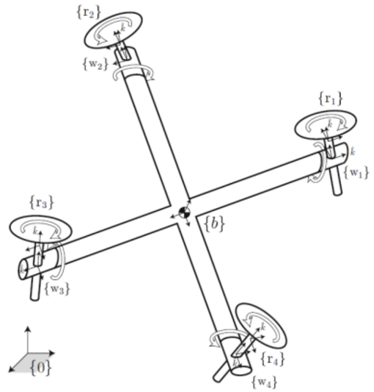
# Outline

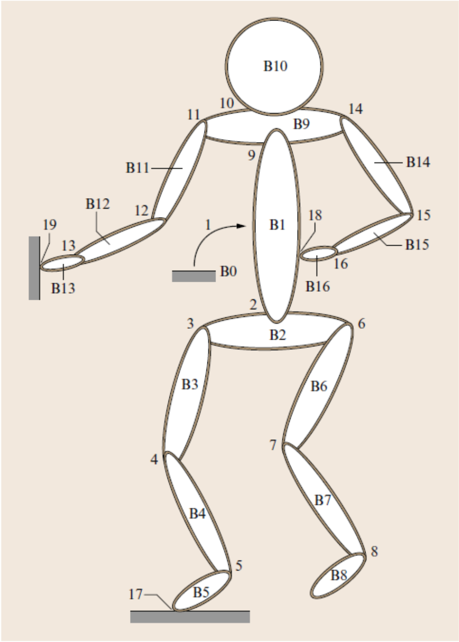
- Recap Last Lectures
- Kinematic Modeling
- Lie group structure of $SO(3)$
- Lie group structure of $SE(3)$

# Recap: Rigid Body Modeling

- Most robotic mechanisms are systems of rigid bodies connected by joints.

- Understanding how to model and interconnect rigid bodies is fundamental !



# Recap: Special Euclidean Group

- The special Euclidean group $(SE(n), \blacksquare)$ is defined as

$$\boxed{SE(n) := \{h = (R, \xi) | R \in SO(n), \xi \in \mathbb{R}^n\}}$$

  - Group operation $\blacksquare: SE(n) \times SE(n) \to SE(n)$ defined as

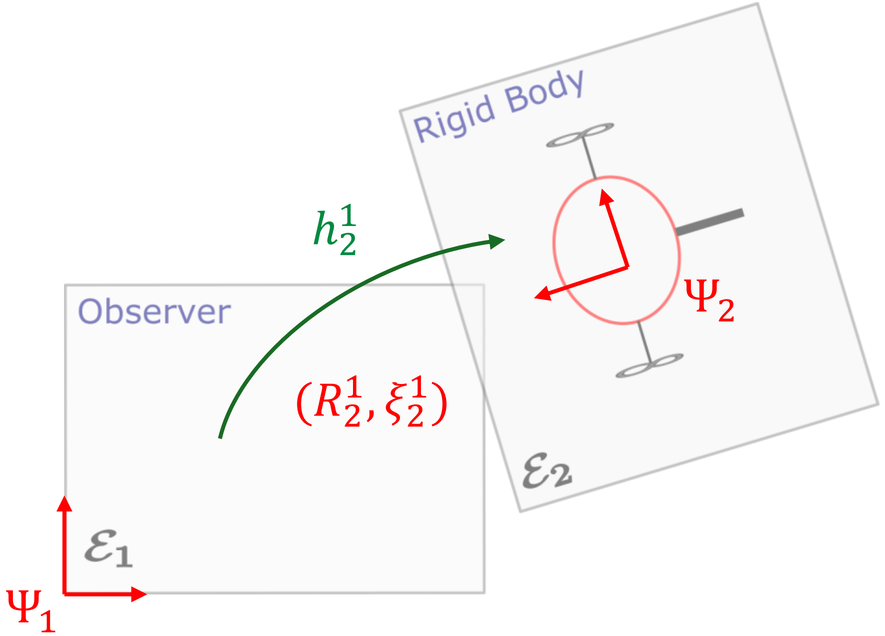
$$\boxed{h_2 \blacksquare h_1 \ := (R_2 \cdot R_1, R_2 \cdot \xi_1 + \xi_2)}$$

  - Identity element $e \in SE(n)$:

$$\boxed{e = (I_n, 0)}$$

  - Inverse element of any $h \in SE(n)$:

$$\boxed{h^{-1} = (R^\top, -R^\top \cdot \xi)}$$



# Recap: Homogeneous transformations

- Using concepts from projective geometry, we can represent $SE(n) = SO(n) \ltimes \mathbb{R}^n$ using matrices in dimension $n + 1$.

$$SE(n) \to HM(n+1) \subset GL(n+1)$$

$$\boldsymbol{h} = (\boldsymbol{R}, \boldsymbol{\xi}) \mapsto \begin{pmatrix} \boldsymbol{R} & \boldsymbol{\xi} \\ \boldsymbol{0} & 1 \end{pmatrix} =: \boldsymbol{H}.$$

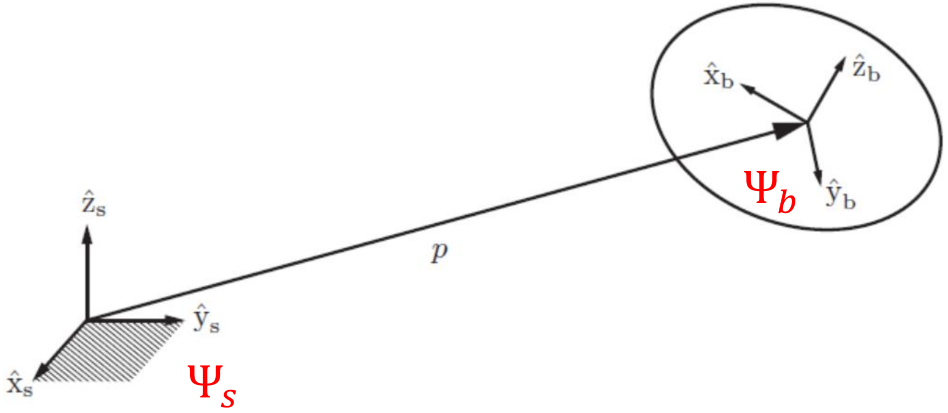
$$\boldsymbol{H}_2\boldsymbol{H}_1 = \begin{pmatrix} \boldsymbol{R}_2 & \boldsymbol{\xi}_2 \\ \boldsymbol{0} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{R}_1 & \boldsymbol{\xi}_1 \\ \boldsymbol{0} & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_2\boldsymbol{R}_1 & \boldsymbol{R}_2\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 \\ \boldsymbol{0} & 1 \end{pmatrix},$$

$$\boldsymbol{H}^{-1} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{\xi} \\ \boldsymbol{0} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{R}^\top & -\boldsymbol{R}^\top\boldsymbol{\xi} \\ \boldsymbol{0} & 1 \end{pmatrix}.$$

# Recap: Lie group $SE(n)$

- $SE(n)$ has the structure of a Lie group.
- $SO(n)$ itself is a Lie sub-group.

$$\boxed{SE(n) = SO(n) \ltimes \mathbb{R}^n}$$

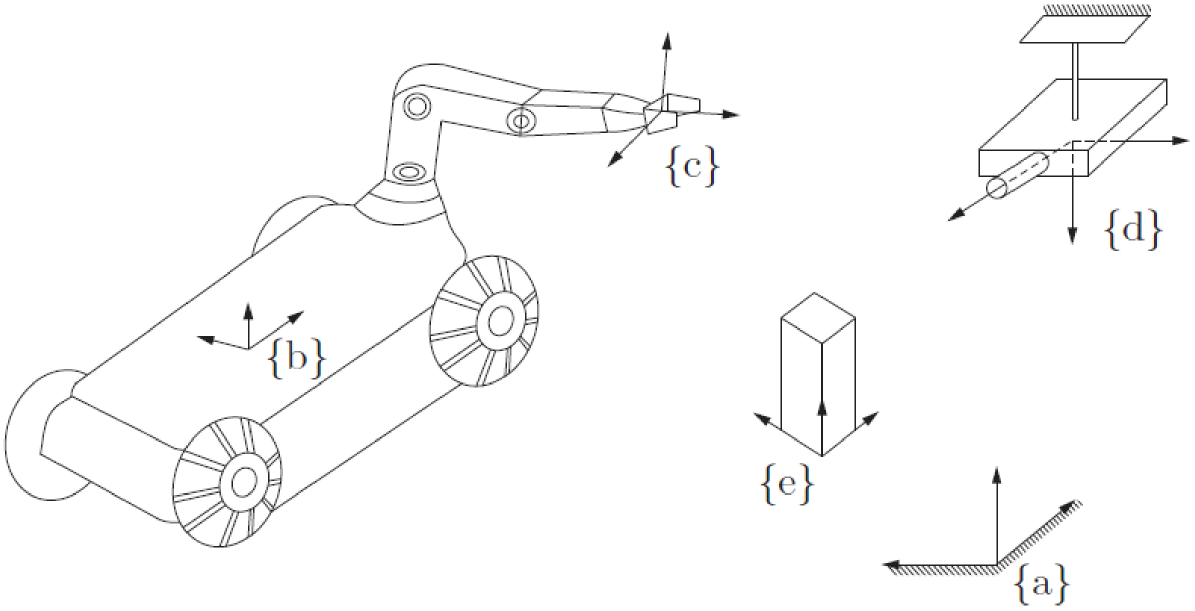


# Outline

- Recap Last Lectures
- Kinematic Modeling
- Lie group structure of $SO(3)$
- Lie group structure of $SE(3)$

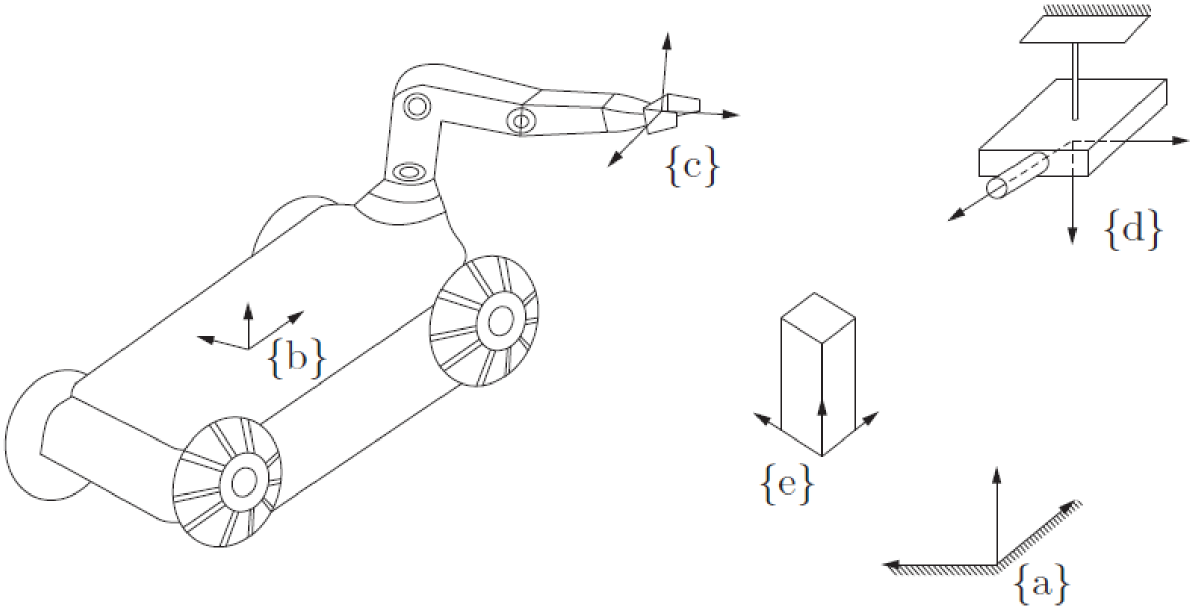
# Kinematic Modeling

- A kinematic model describes the motion of bodies in a robotic mechanism without regard to the forces/torques that cause the motion.
- It focuses purely on geometric and temporal relationships between position, velocity, and acceleration.



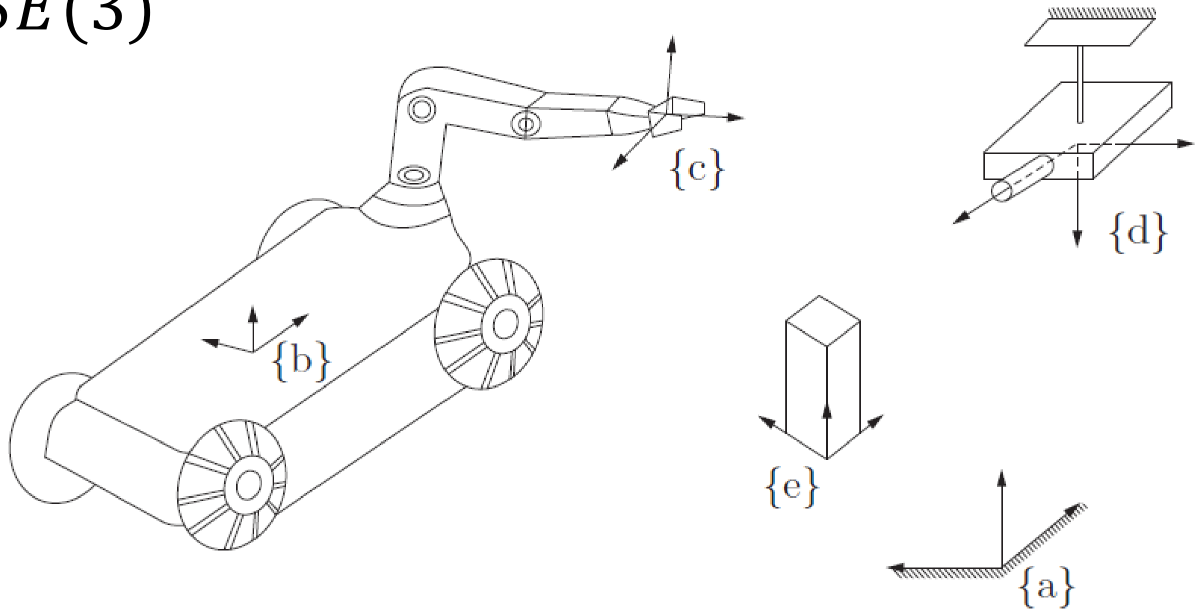
# Kinematic Modeling: Notation

- A frame will be denoted by $\Psi_i$ or $\{i\}$.
- A frame can be stationary or moving (body-fixed).
- All frames are right-handed, and its axes are orthonormal.
- A body-fixed frame can be arbitrarily specified.



# Kinematic Modeling: Notation

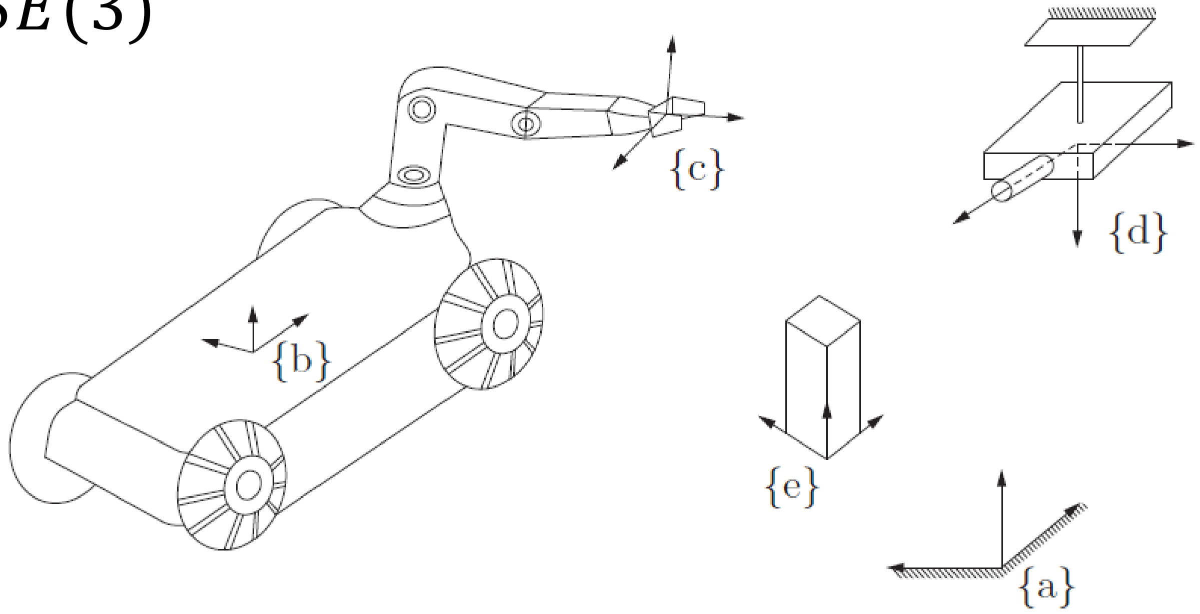
- The orientation of $\{i\}$ with respect to $\{k\}$ is described by
$$R_i^k \in SO(3)$$
- The displacement of the origin of $\{i\}$ expressed in $\{k\}$ is described by
$$\xi_i^k \in \mathbb{R}^3$$
- The relative pose of $\{i\}$ with respect to $\{k\}$ is described by
$$H_i^k \in SE(3)$$



# Kinematic Modeling: Notation

- The orientation of $\{i\}$ with respect to $\{k\}$ is described by
$$R_i^k \in SO(3)$$
- The displacement of the origin of $\{i\}$ expressed in $\{k\}$ is described by
$$\xi_i^k \in \mathbb{R}^3$$
- The relative pose of $\{i\}$ with respect to $\{k\}$ is described by
$$H_i^k \in SE(3)$$

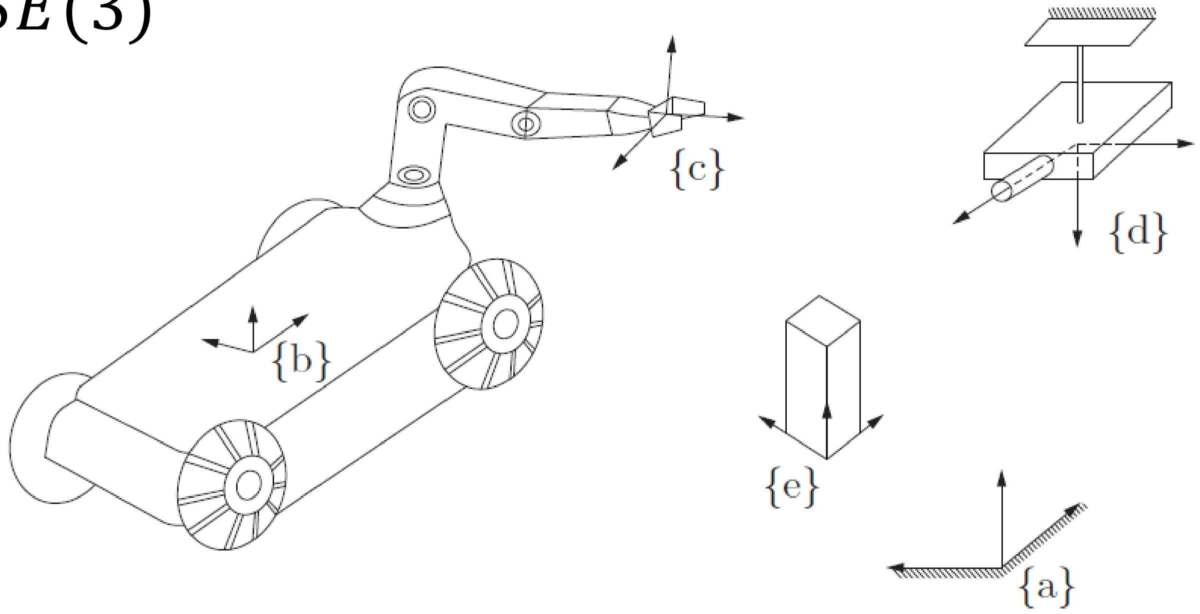
Question:
What does $H_i^j = I_4$ mean?



# Kinematic Modeling: Notation

- The orientation of $\{i\}$ with respect to $\{k\}$ is described by
$$R_i^k \in SO(3)$$
- The displacement of the origin of $\{i\}$ expressed in $\{k\}$ is described by
$$\xi_i^k \in \mathbb{R}^3$$
- The relative pose of $\{i\}$ with respect to $\{k\}$ is described by
$$H_i^k \in SE(3)$$

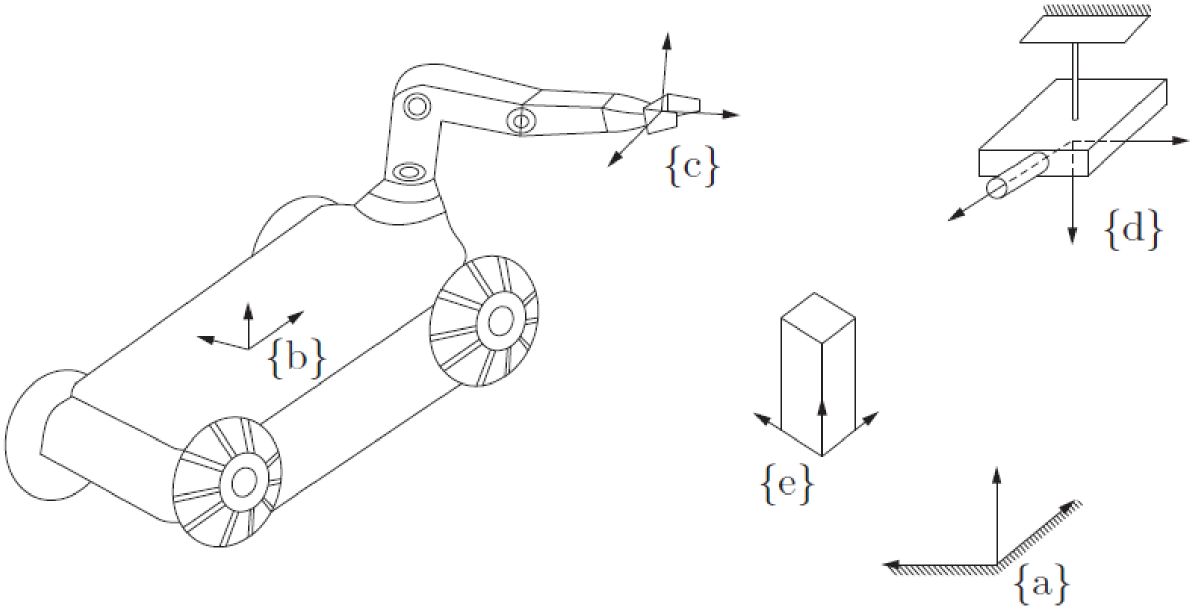
The relative pose of $\{k\}$ with respect to $\{i\}$ is described by
$$H_k^i = \left(H_i^k\right)^{-1} \in SE(3)$$
with
$$R_k^i = \left(R_i^k\right)^\top \in SO(3), \qquad \xi_k^i = -R_k^i \xi_i^k \in \mathbb{R}^3$$



# Kinematic Modeling: Notation

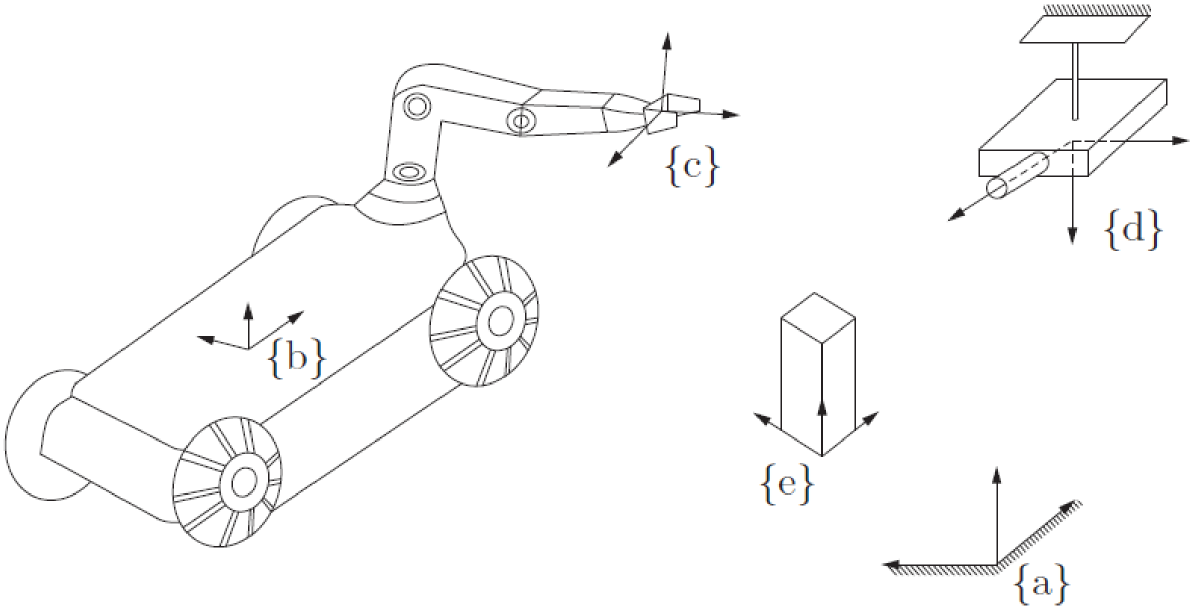
- Given the relative pose of $\{i\}$ with respect to $\{j\}$ and the relative pose of $\{j\}$ with respect to $\{k\}$, we have that

$$H_i^k = H_j^k H_i^j \in SE(3)$$



# Kinematic Modeling: Notation

- We will introduce later several velocity variables denoted by:
  - $\omega_i^{k,j}$ : angular velocity of body attached to $\{i\}$ w.r.t. $\{j\}$ expressed in $\{k\}$
  - $v_i^{k,j}$ : linear velocity of body attached to $\{i\}$ w.r.t. $\{j\}$ expressed in $\{k\}$
  - $\mathcal{V}_i^{k,j}$ : Twist (Combined velocity) of body attached to $\{i\}$ w.r.t. $\{j\}$ expressed in $\{k\}$




Note:
Velocity always has three indices !!

# Outline

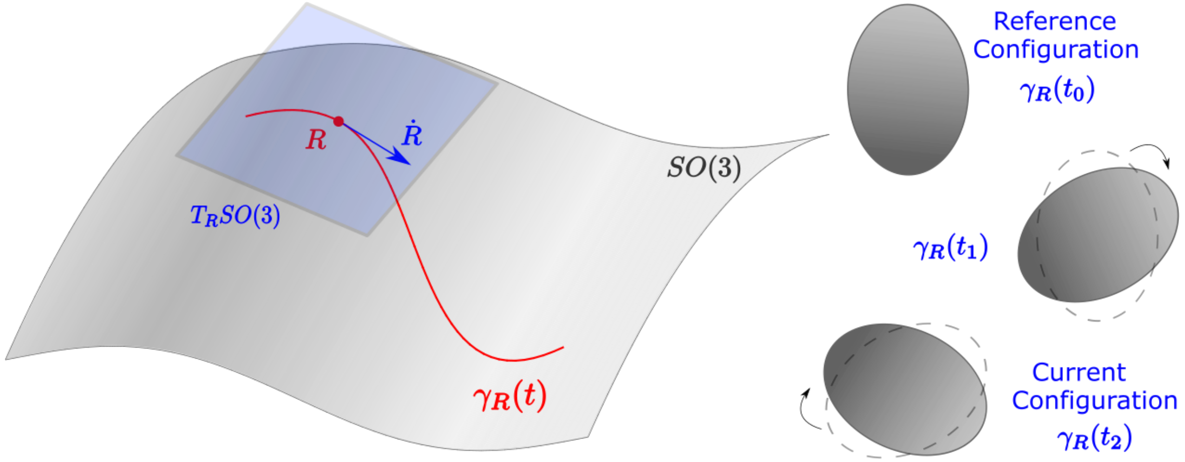
- Recap Last Lectures
- Kinematic Modeling
- Lie group structure of $SO(3)$
- Lie group structure of $SE(3)$

# Rigid body rotations

- In what follows, we focus on the Lie group structure of $SO(3)$.
- A rigid body rotational motion is represented mathematically by a curve

$$\begin{gathered} \gamma_R : I \subset \mathbb{R} \to SO(3) \\ t \mapsto \gamma_R(t) =: \boldsymbol{R}_t \end{gathered}$$

$$\boxed{\boldsymbol{R}_t \equiv \boldsymbol{R}(\boldsymbol{t})}$$



# Rigid body rotations

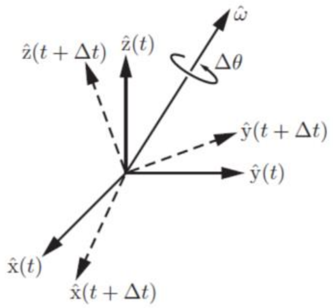
- In what follows, we focus on the Lie group structure of $SO(3)$.
- A rigid body rotational motion is represented mathematically by a curve

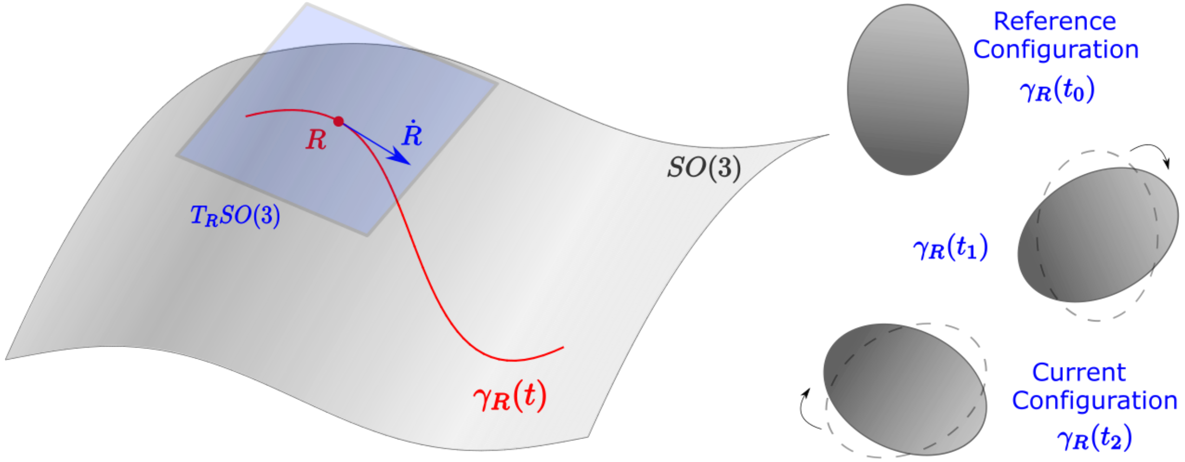
$$\gamma_R : I \subset \mathbb{R} \to SO(3)$$
$$t \mapsto \gamma_R(t) =: \boldsymbol{R}_t$$

$$\boxed{\boldsymbol{R}_t \equiv \boldsymbol{R}(\boldsymbol{t})}$$

Next Up!

How do we represent the relation between rate-of-change of orientation $\dot{\boldsymbol{R}}_t$ and angular velocity $\boldsymbol{\omega}$ ?



# Outline

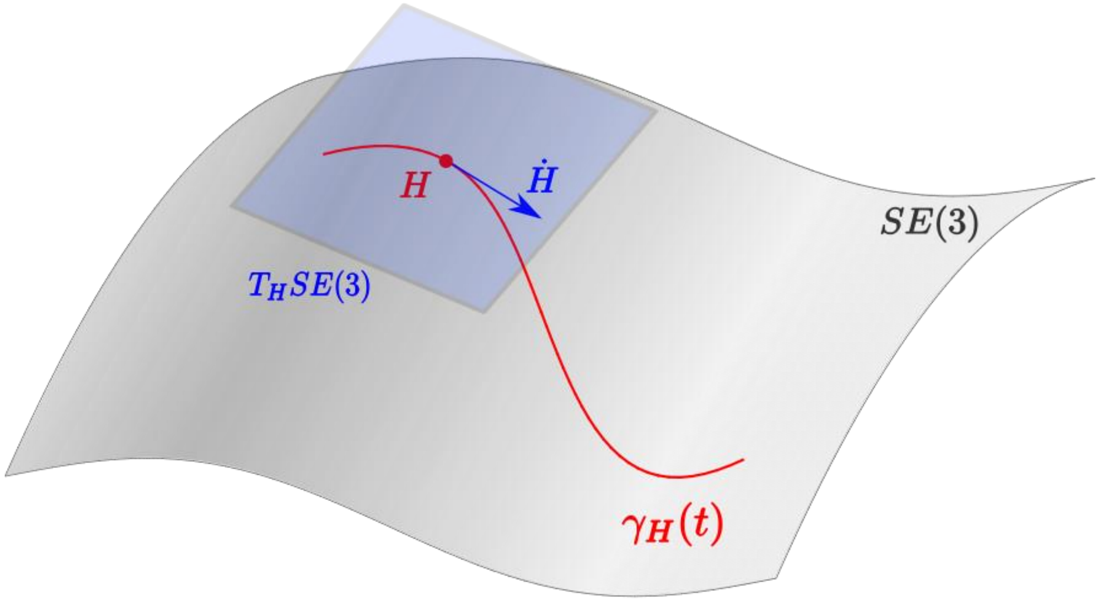
- Recap Last Lectures
- Kinematic Modeling
- Lie group structure of $SO(3)$
- Lie group structure of $SE(3)$

# Rigid body motion

- A rigid body general motion is represented mathematically by a curve

$$\gamma_H : I \subset \mathbb{R} \to SE(3)$$
$$t \mapsto \gamma_H(t) =: \boldsymbol{H}_t$$

$$\boxed{\boldsymbol{H}_t \equiv \boldsymbol{H}(\boldsymbol{t})}$$



# Rigid body motion

- A rigid body general motion is represented mathematically by a curve

$$\gamma_H : I \subset \mathbb{R} \to SE(3)$$
$$t \mapsto \gamma_H(t) =: \boldsymbol{H}_t$$

$$\boxed{\boldsymbol{H_t} \equiv \boldsymbol{H}(\boldsymbol{t})}$$

Next Up!

- If $\boldsymbol{H} \in SE(3)$ represents the combined orientation and displacement of some rigid body with respect to some frame, can we represent angular and linear velocity with a single variable ?
- How do we represent the relation between rate-of-change of pose $\dot{\boldsymbol{H}}_t$ and that generalized velocity variable ?