SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 9: Rigid body dynamics

جامعة الملك فهد للبترول والمعادن
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Outline

- Recap last lectures
- Point mass dynamics
- Rigid body rotation dynamics
- Rigid body motion dynamics

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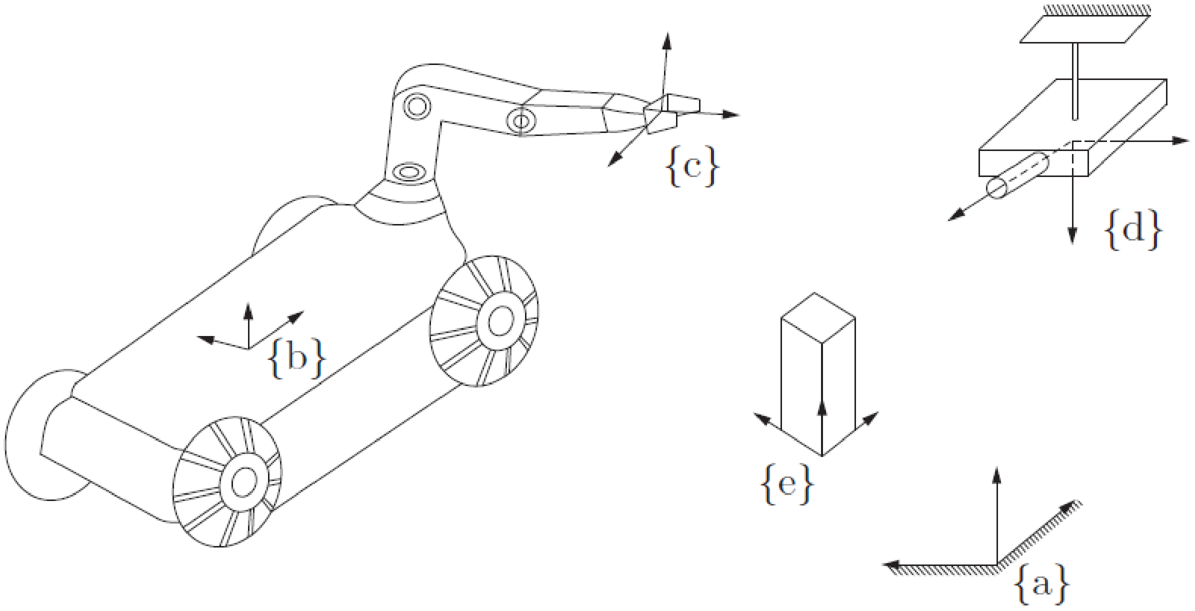
Recap: Kinematic Modeling Notation

- A frame will be denoted by Ψ_i or $\{i\}$.
- The relative pose of $\{i\}$ with respect to $\{k\}$ is described by

$$H_i^k = \begin{pmatrix} R_i^k & \xi_i^k \\ 0 & 1 \end{pmatrix} \in SE(3), \qquad R_i^k \in SO(3), \qquad \xi_i^k \in \mathbb{R}^3$$



Recap: Kinematic Relations

1. Point mass translation :

- Configuration:

$$\xi_b^s \in \mathbb{R}^3$$

- Rate-of-change of configuration:

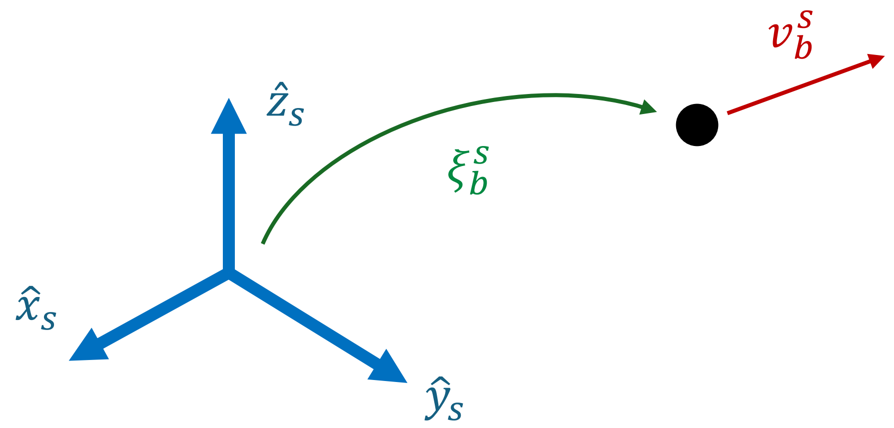
$$\dot{\xi}_b^s \in \mathbb{R}^3$$

- Velocity expressed in $\{s\}$:

$$v_b^{s,s} \in \mathbb{R}^3$$

- Kinematic relation:

$$\boxed{\dot{\xi}_b^s = v_b^{s,s}}$$



Recap: Kinematic Relations

2. Rigid body rotation :

- Configuration:

$$R_b^s \in SO(3)$$

- Rate-of-change of configuration:

$$\dot{R}_b^s \in T_{R_b^s} SO(3)$$

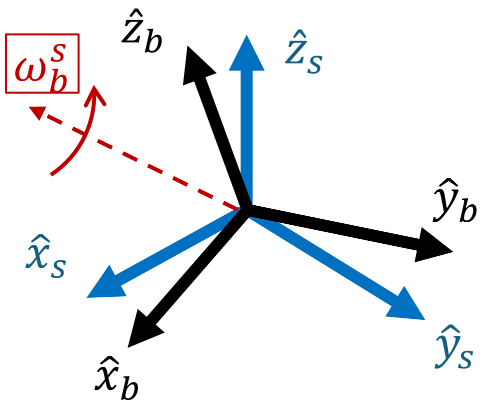
- Angular velocity expressed in $\{*\}$:

$$\widetilde{\omega}_b^{*,s} \in T_I SO(3) =: so(3)$$

- Kinematic relation:

$$\boxed{\dot{R}_b^s = R_b^s \widetilde{\omega}_b^{b,s}}$$

$$\boxed{\dot{R}_b^s = \widetilde{\omega}_b^{s,s} R_b^s}$$



Recap: Kinematic Relations

3. Rigid body motion :

- Configuration:

$$H_b^s \in SE(3)$$

- Rate-of-change of configuration:

$$\dot{H}_b^s \in T_{H_b^s} SE(3)$$

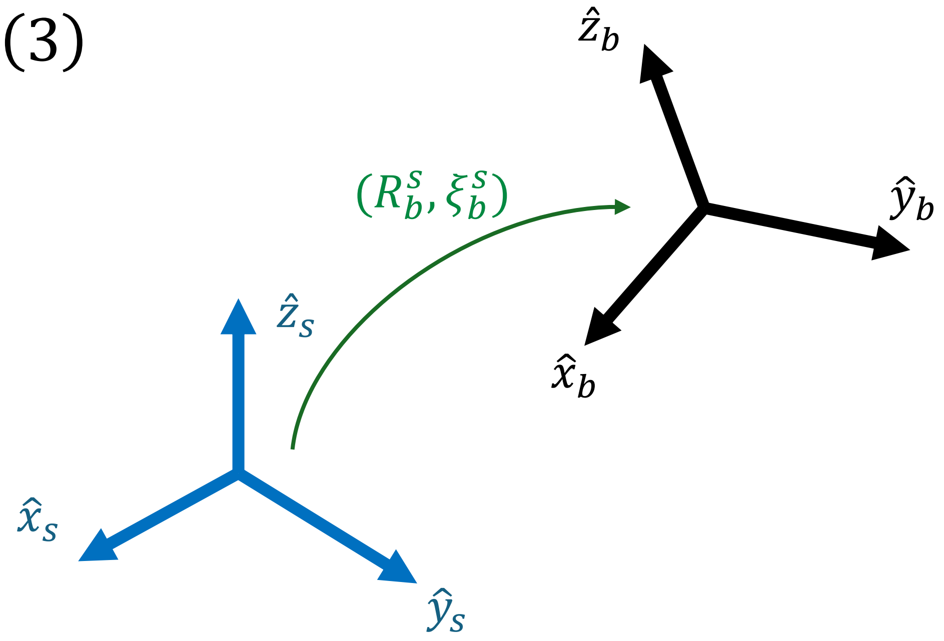
- Twist expressed in $\{*\}$:

$$\tilde{\mathcal{V}}_b^{*,s} \in T_I SE(3) =: se(3)$$

- Kinematic relation:

$$\boxed{\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}}$$

$$\boxed{\dot{H}_b^s = \tilde{\mathcal{V}}_b^{s,s} H_b^s}$$



Recap: Matrix form of velocities in 3D

- Angular velocity

$$\omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \in \mathbb{R}^3$$

$$S \downarrow \qquad \uparrow S^{-1}$$

$$\widetilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \in so(3)$$

- Twist

$$\mathcal{V} = \begin{pmatrix} \omega \\ v \end{pmatrix} \in \mathbb{R}^6$$

$$\tilde{S} \downarrow \qquad \uparrow \tilde{S}^{-1}$$

$$\tilde{\mathcal{V}} = \begin{pmatrix} \widetilde{\omega} & v \\ 0_{3\times1} & 0 \end{pmatrix} \in se(3)$$

Recap: Spatial and body velocities

- Spatial Twist

$$\mathcal{V}_b^{s,s} = \begin{pmatrix} \omega_b^{s,s} \\ v_b^{s,s} \end{pmatrix} \in \mathbb{R}^6$$

- Body Twist

$$\mathcal{V}_b^{b,s} = \begin{pmatrix} \omega_b^{b,s} \\ v_b^{b,s} \end{pmatrix} \in \mathbb{R}^6$$

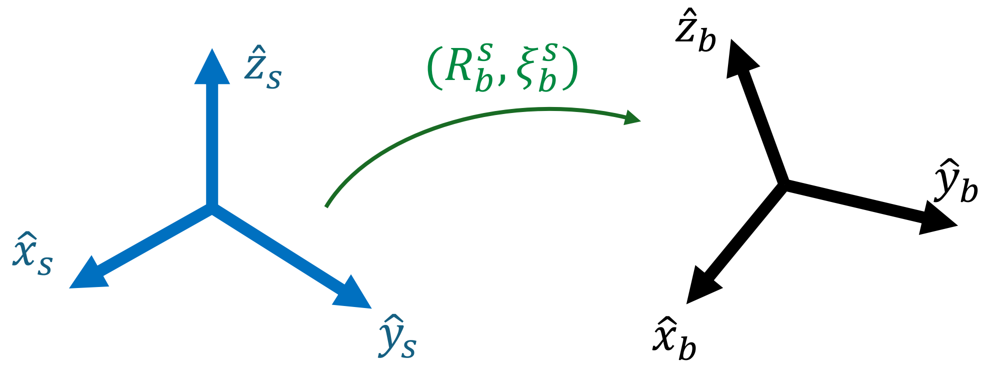
$$\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s} = \tilde{\mathcal{V}}_b^{s,s} H_b^s$$

$$\tilde{\mathcal{V}}_b^{s,s} = H_b^s \tilde{\mathcal{V}}_b^{b,s} H_s^b$$

$$\mathcal{V}_b^{s,s} = \mathrm{Ad}_{H_b^s}\, \mathcal{V}_b^{b,s}$$

$$\mathrm{Ad}_H = \begin{pmatrix} R & 0 \\ \tilde{\xi} R & R \end{pmatrix}$$

called the Adjoint matrix of $SE(3)$



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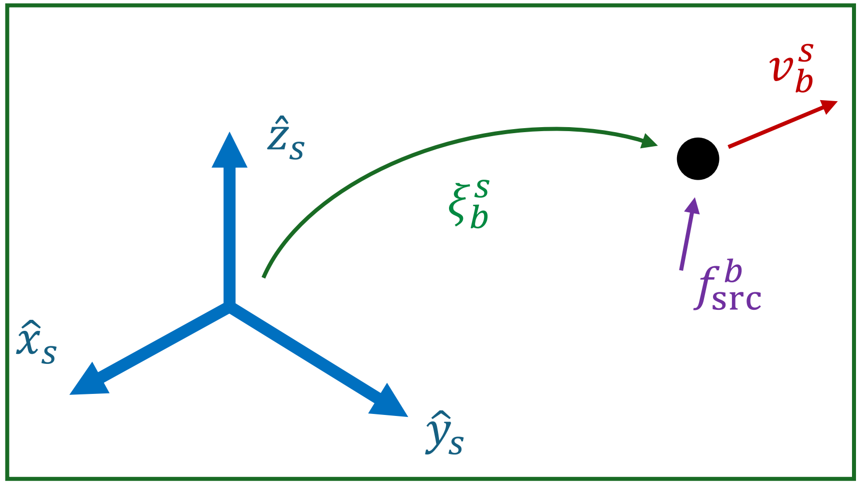
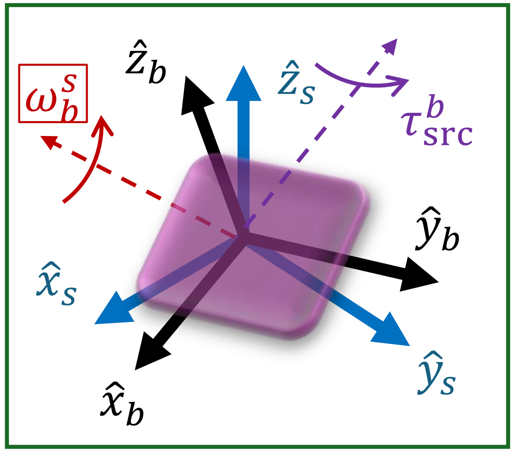
Dynamic modeling

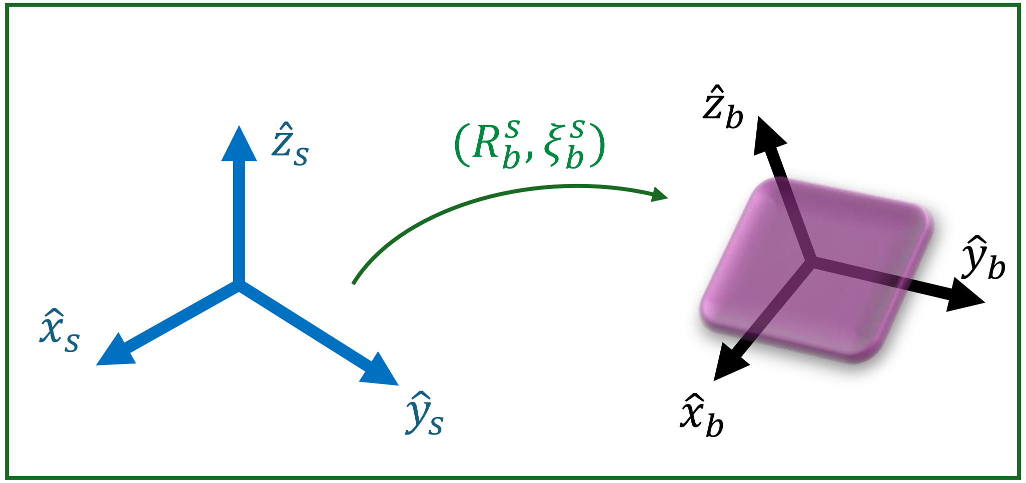
- A dynamic model describes the motion of a system while considering the forces and torques that cause the motion.
- It includes both kinematics and conservation laws



<u>Forces</u> on translating point mass

<u>Torques</u> on rotating rigid body

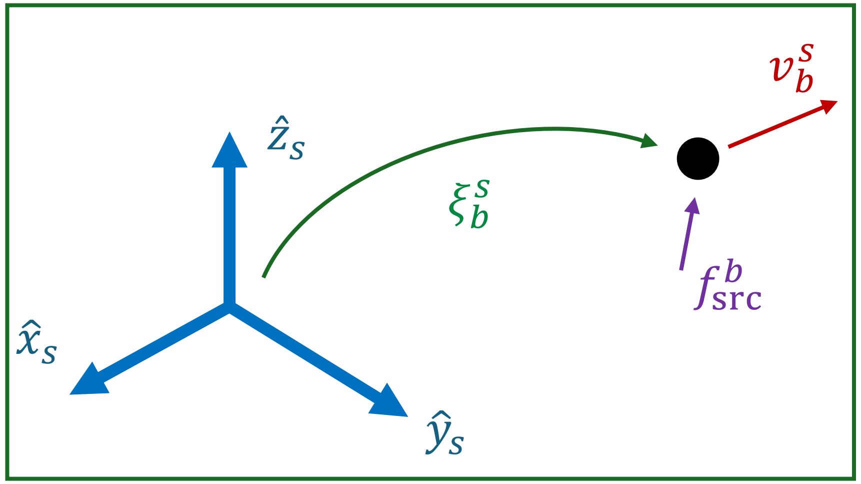




<u>Wrenches</u> on moving rigid body

Point mass dynamics

- We will denote by:
  - $f_{\text{src}}^{b} \in \mathbb{R}^3$: the abstract force from source src acting on point mass b
  - $f_{\text{src}}^{s,b} \in \mathbb{R}^3$: the force from source src acting on point mass b, expressed in $\{s\}$.



Forces on translating point mass

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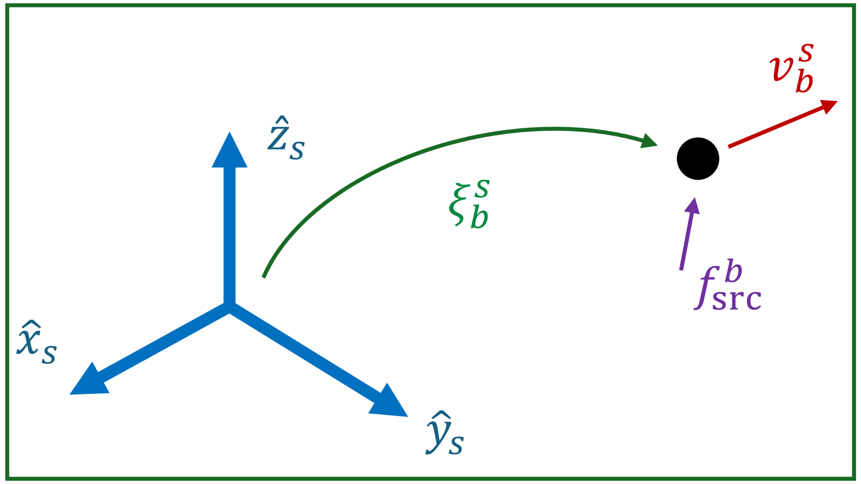
- The kinetic energy of point b, $E_k: \mathbb{R}^3 \to \mathbb{R}$ is given by:
  - $E_k\left(v_b^{s,s}\right) = \frac{1}{2}\,\mathrm{m}\left(v_b^{s,s}\right)^{\top} v_b^{s,s}$



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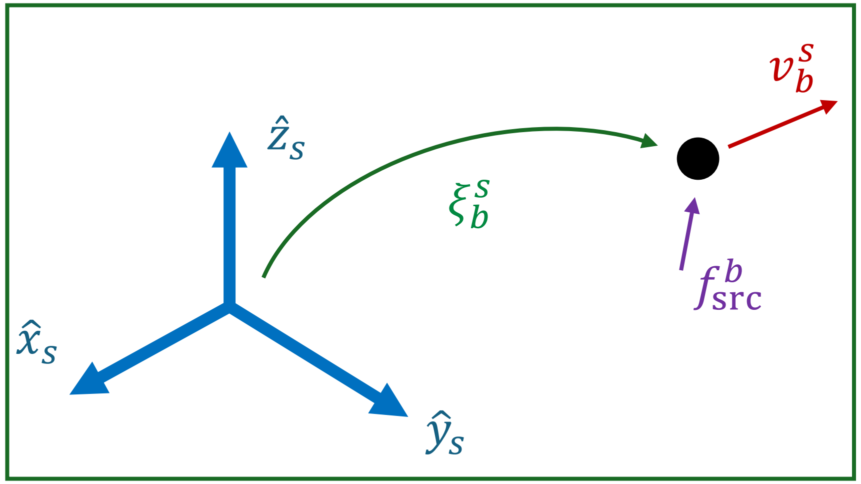
- The linear momentum of point b expressed in $\{s\}$, $P_v^{s,b} \in \mathbb{R}^3$ is given by:
  - $P_v^{s,b} \coloneqq \frac{\partial E_k}{\partial v_b^{s,s}}\left(v_b^{s,s}\right) = \mathrm{m} v_b^{s,s}$



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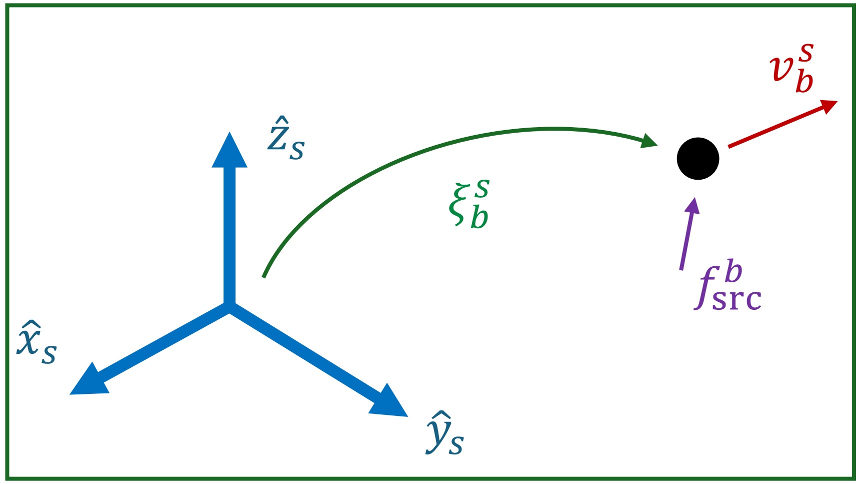
- Newton's law:
  - $\dot{P}_v^{s,b} = f_{\mathrm{tot}}^{s,b}$

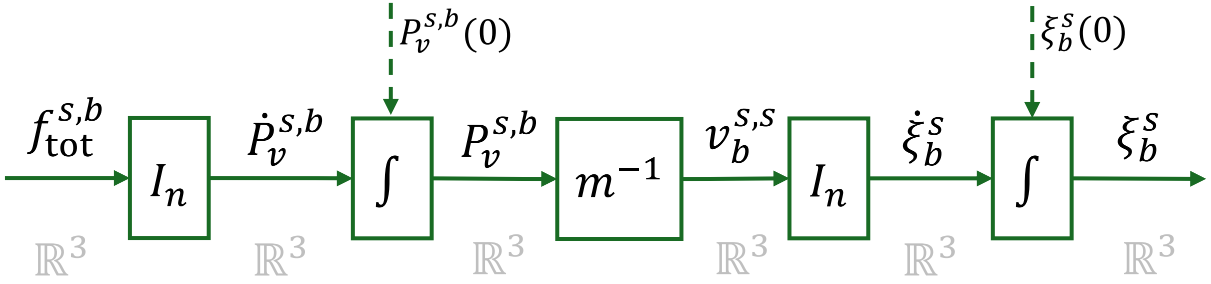
  or
  - $\dot{v}_b^{s,s} = \frac{1}{m} f_{\mathrm{tot}}^{s,b}$



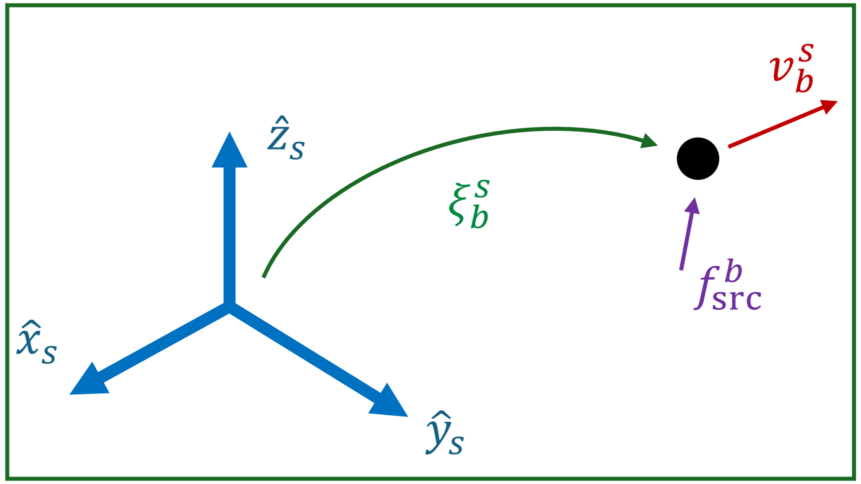
Forces on translating point mass

Point mass dynamics

- In summary,



- $\dot{\xi}_b^s = v_b^{s,s}$ Kinematic relation
- $\dot{P}_v^{s,b} = f_{\text{tot}}^{s,b}$ Momentum balance
- $v_b^{s,s} = m^{-1} P_v^{s,b}$ Constitutive relation



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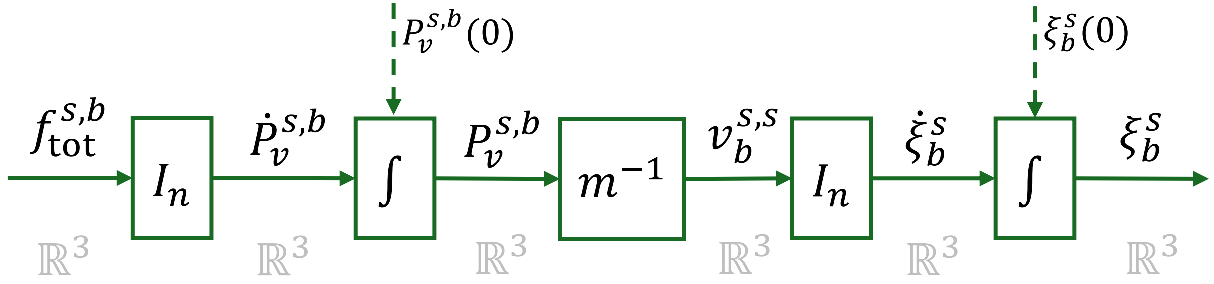

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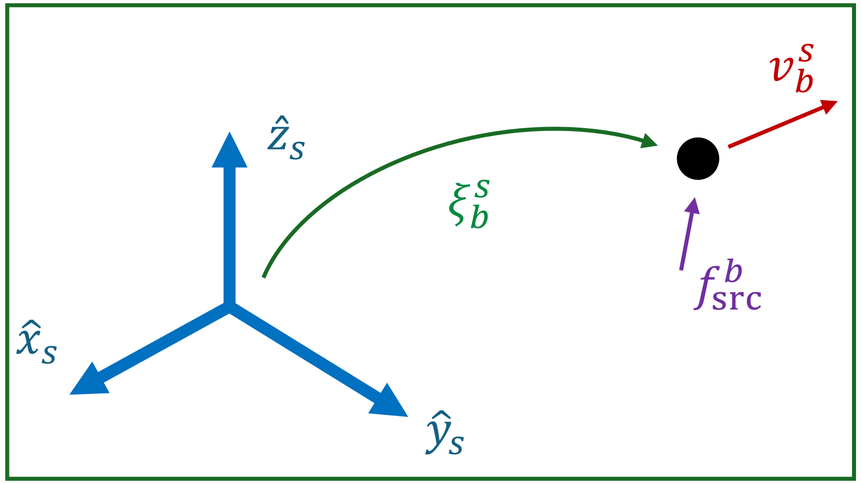
$\downarrow$

- $\dot{\xi}_b^s = v_b^{s,s}$
- $\dot{v}_b^{s,s} = m^{-1} f_{\text{tot}}^{s,b}$

$\rightarrow$

- $\ddot{\xi}_b^s = m^{-1} f_{\text{tot}}^{s,b}$



[Forces](#) on translating point mass

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Rigid body rotation dynamics

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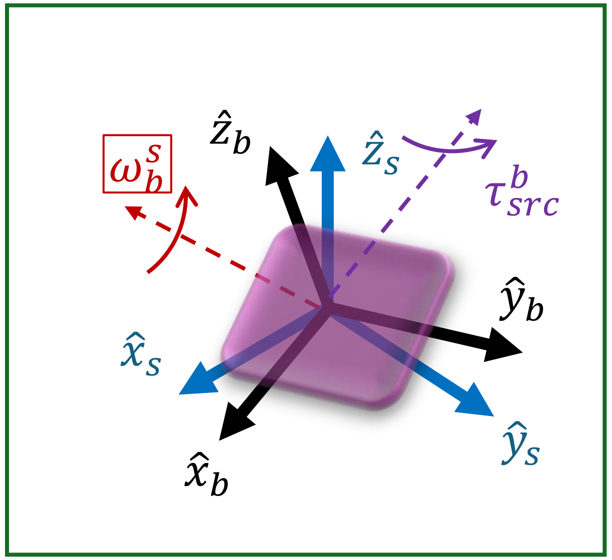

Torques on rotating rigid body

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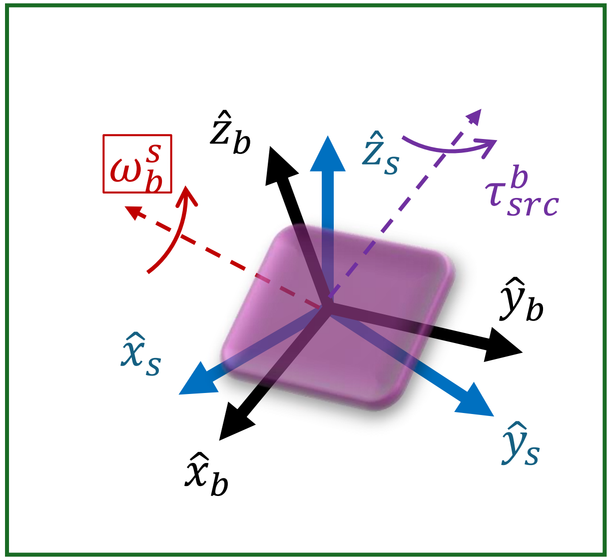
- The kinetic energy of the rigid body, $E_k: SO(3) \times \mathbb{R}^3 \to \mathbb{R}$ is given** by:
  - $E_k\left(R_b^s, \omega_b^{*,s}\right) = \frac{1}{2}\left(\omega_b^{*,s}\right)^\top J^{*,b}(R_b^s)\, \omega_b^{*,s}$



Torques on rotating rigid body

**The kinetic energy is formally a function on the tangent bundle $E_k: TSO(3) \to \mathbb{R}$

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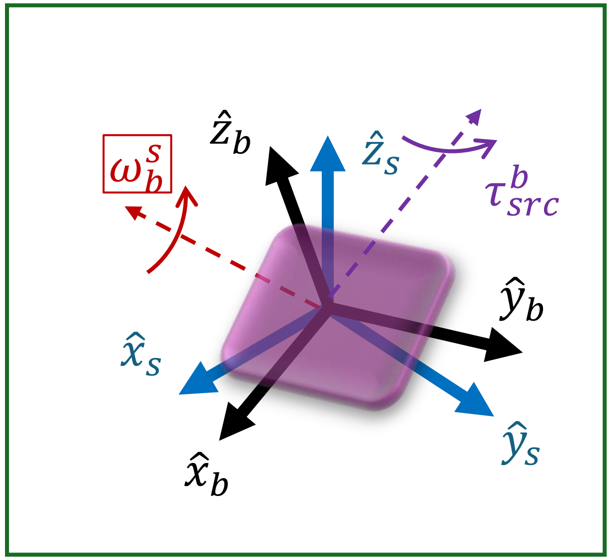
- The angular momentum of the rigid body expressed in $\{*\}$, $P_\omega^{*,b} \in \mathbb{R}^3$ is given by:
  - $P_\omega^{*,b} := \frac{\partial E_k}{\partial \omega_b^{*,s}}\left(R_b^s, \omega_b^{*,s}\right) = J^{*,b}(R_b^s)\, \omega_b^{*,s}$



Torques on rotating rigid body

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- Euler's law in $\{s\}$:
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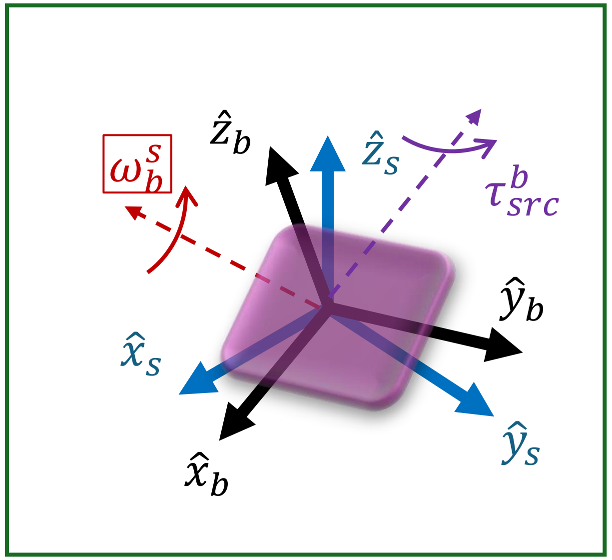

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