SCE594: Special Topics in Intelligent Automation & Robotics

Homework 2

Question 1:

Show that the vectorization map is defined in Lec. 2, Slide 19 is a linear isomorphism.

Question 2:

Search online or in the reading material for the Einstein summation convention and explain it in your own words.

Question 3:

For a given vector space V with dimension n, let $\{e_1, \cdots, e_n\}$ denote a basis B for V and $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ denote another basis $\tilde{B}$ for V such that any $v \in V$ can be expressed as $v = v^i e_i$ or $v = \tilde{v}^i \tilde{e}_i$.

Let $[v]_B = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \in \mathbb{R}^n$ and $[v]_{\tilde{B}} = \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \in \mathbb{R}^n$ denote the components of the vector $v \in V$ in the two bases.

a) For any endomorphism $A: V \to V$,show that the components of $A(v) \in V$ can be expressed as

$$[A(v)]_B = [A]_B [v]_B$$

where $[A]_B \in \mathbb{R}^{n \times n}$ is the matrix representation of A given by

$$[A]_B = \begin{pmatrix} A_1^1 & \cdots & A_n^1 \\ \vdots & \ddots & \vdots \\ A_1^n & \dots & A_n^n \end{pmatrix}$$

b) Let $\boldsymbol{P} \in GL(n, \mathbb{R}) \subset \mathbb{R}^{n \times n}$ be the change of basis matrix such that

$$[v]_B = \boldsymbol{P}\,[v]_{\tilde{B}}$$

Show that the coordinate change for endomorphisms is given by

$$[A]_{\tilde{B}} = \boldsymbol{P}^{-1}\,[A]_B\,\boldsymbol{P}$$

Concept: Dual basis
To answer the following questions (4 & 5), you will need the concept of a **dual basis**.

Let V^* denote the dual space to V which can be given the basis $\{\varphi^1, \cdots, \varphi^n\}$ such that any $\alpha \in V^*$ can be expressed as $\alpha = \alpha_k\, \varphi^k$, where $\varphi^k \in V^*$ is a linear map $\varphi^k : V \to \mathbb{R}$ defined by the property:

$$\varphi^k(e_i) = \delta_i^k = \begin{cases} 1 & , \quad i = k \\ 0 & , \quad i \neq k \end{cases}$$

The collection of co-vectors $\{\varphi^1, \cdots, \varphi^n\}$ is called the **dual basis**.

Question 4:
Using the concept of a dual basis, show that if $v = v^i e_i$ and $\alpha = \alpha_k\, \varphi^k$, we have that

$$\alpha(v) = \alpha_i v^i.$$

Question 5:

a) For any bilinear map $\beta: V \rightarrow V^*$,show that the components of $\beta(v) \in V^*$ can be expressed as

$$[\beta(v)]_B = [\beta]_B[v]_B$$

where $[\beta]_B \in \mathbb{R}^{n\times n}$ is the matrix representation of β given by

$$[\beta]_B = \begin{pmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \dots & \beta_{nn} \end{pmatrix}$$

b) Let $\boldsymbol{P} \in GL(n, \mathbb{R}) \subset \mathbb{R}^{n \times n}$ be the change of basis matrix such that

$$[v]_B = \boldsymbol{P}\,[v]_{\tilde{B}}$$

Show that the coordinate change for bilinear maps is given by

$$[\beta]_{\tilde{B}} = \boldsymbol{P}^\top\,[\beta]_B\,\boldsymbol{P}$$