# SCE594: Special Topics in Intelligent Automation & Robotics

## Homework 4

**Question 1:**

a) Let $\mathrm{Rot}(\hat{n}, \theta) := e^{\tilde{n}\theta} \in SO(3)$, where $\tilde{n} := S(\hat{n}) \in so(3)$ and $\hat{n} \in \mathbb{S}^2$ is a unit vector. Show using Rodrigues' formula that rotation operations about the coordinate-frame axes $\hat{x} := (1,0,0), \hat{y} := (1,0,0), \hat{z} := (1,0,0)$ are given by

$$\mathrm{Rot}(\hat{\mathrm{x}}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \mathrm{Rot}(\hat{\mathrm{y}}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\mathrm{z}}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

b) Show using Rodrigues' formula that the rotation operation for a general $\widehat{\omega} = (\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3) \in \mathbb{S}^2$ is given by

$$\mathrm{Rot}(\hat{\omega}, \theta) =$$
$$\begin{bmatrix} \mathrm{c}_\theta + \hat{\omega}_1^2(1-\mathrm{c}_\theta) & \hat{\omega}_1\hat{\omega}_2(1-\mathrm{c}_\theta) - \hat{\omega}_3\mathrm{s}_\theta & \hat{\omega}_1\hat{\omega}_3(1-\mathrm{c}_\theta) + \hat{\omega}_2\mathrm{s}_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-\mathrm{c}_\theta) + \hat{\omega}_3\mathrm{s}_\theta & \mathrm{c}_\theta + \hat{\omega}_2^2(1-\mathrm{c}_\theta) & \hat{\omega}_2\hat{\omega}_3(1-\mathrm{c}_\theta) - \hat{\omega}_1\mathrm{s}_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-\mathrm{c}_\theta) - \hat{\omega}_2\mathrm{s}_\theta & \hat{\omega}_2\hat{\omega}_3(1-\mathrm{c}_\theta) + \hat{\omega}_1\mathrm{s}_\theta & \mathrm{c}_\theta + \hat{\omega}_3^2(1-\mathrm{c}_\theta) \end{bmatrix},$$

**Tip: You can write a MATLAB script that computes for you symbolically.**

**Question 2:**

Consider the following properties for $\tilde{\mathcal{V}} \in se(3)$ and $H \in SE(3)$

- $e^{H_i^k\ \tilde{\mathcal{V}}_*^{i,\circ}\ H_k^i} = H_i^k\ e^{\tilde{\mathcal{V}}_*^{i,\circ}} H_k^i,$
- $H_i^k\ \tilde{\mathcal{V}}_*^{i,\circ}\ H_k^i = \tilde{\mathcal{V}}_*^{k,\circ}$
- $H_n^0 = H_1^0 H_2^1 \dots H_n^{n-1}$
- $H_i^{i-1}(\theta_i) = e^{\tilde{\mathcal{S}}_i^{i-1,i-1}\theta_i} H_i^{i-1}(0)$

Using these properties, show that

$$H_1^0(\theta_1)H_2^1(\theta_2)H_3^2(\theta_3) = e^{\tilde{\mathcal{S}}_1^{0,0}\theta_1} e^{\tilde{\mathcal{S}}_2^{0,1}\theta_2} e^{\tilde{\mathcal{S}}_3^{0,2}\theta_3} H_3^0(0)$$

**Question 3:**

Consider the following properties for $\tilde{\mathcal{V}} \in se(3)$ and $H \in SE(3)$

- $H_n^0 = H_1^0 H_2^1 \dots H_n^{n-1}$,
- $(AB)^{-1} = B^{-1}A^{-1}, \ \forall A, B \in \mathbb{R}^{n\times n}$
- $\left(H_i^k\right)^{-1} = H_k^i$
- $\tilde{\mathcal{V}}_i^{j,j} := \dot{H}_i^j H_j^i$

Show that

$$\tilde{\mathcal{V}}_n^{0,0} = \tilde{\mathcal{V}}_1^{0,0} + \tilde{\mathcal{V}}_2^{0,1} + \cdots + \tilde{\mathcal{V}}_n^{0,n-1}$$

and explain why it follows that

$$\mathcal{V}_n^{0,0} = \mathcal{V}_1^{0,0} + \mathcal{V}_2^{0,1} + \cdots + \mathcal{V}_n^{0,n-1}$$

**Question 4:**
Solve examples 4.3 and 4.6 (6R spatial open chain), 4,4 (RRPRRR spatial open chain), 4.5 (UR5 6R robot arm) in Lynch and Park Chapter 4.

**Question 5:**

For the 3R planar robot arm shown below:

a) expand the 3 equations corresponding to

$$\mathcal{V}_i^{i,s} = \mathrm{Ad}_{H_{i-1}^i} \mathcal{V}_{i-1}^{i-1,s} + S_i^{i,i-1} \dot{q}_i$$

b) Solve the 3 equations recursively such that you can write the right hand side as some function of the form

$$\mathcal{V}_1^{1,s} = f_1(q)\, \dot{q},$$
$$\mathcal{V}_2^{2,s} = f_2(q)\, \dot{q},$$
$$\mathcal{V}_3^{3,s} = f_3(q)\, \dot{q}$$

with $q := \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$.

c) Then show you get the same result by using the compact formula $\mathcal{V} = \mathcal{L}(q)\mathcal{S}\, \dot{q}$ defined in Lect. 15 slides 37 and 42.