Homework Problem: Deriving Manipulator Dynamics in the Task Space

Consider an n-DOF manipulator with joint-space dynamics

$$M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta) = \tau + J^T(\theta)\mathcal{W}_{\text{int}}^{e,e}$$

where:

- $\theta \in \mathbb{R}^n$: joint coordinates
- $M(\theta)$: joint-space inertia matrix
- $c(\theta, \dot{\theta})$: Coriolis/centrifugal vector
- $g(\theta)$: gravity vector
- $J(\theta)$: manipulator Jacobian
- $\mathcal{W}_{\text{int}}^{e,e}$: wrench applied at the end-effector
- τ: actuator torque vector

Show that the dynamics can be written in the form

$$\Lambda(\theta)\dot{\mathcal{V}}_e^{e,0} + \eta(\theta, \mathcal{V}_e^{e,0})\mathcal{V}_e^{e,0} + \gamma(\theta) = \mathcal{W}_{\text{int}}^{e,e}$$

and derive explicit expressions for:

1. The task-space inertia matrix:

$$\Lambda(\theta)$$

2. The task-space gravity term:

$$\gamma(\theta)$$

3. The velocity-dependent term:

$$\eta(\theta, \mathcal{V}_e^{e,0})\mathcal{V}_e^{e,0}$$

Solution

Let

$$\mathcal{V}_e^{e,0} = J(\theta)\dot{\theta}$$

be the end-effector velocity/twist expressed in frame e. Start from the joint-space dynamics with no actuator torque:

$$M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + b(\dot{\theta}) + g(\theta) = J^T(\theta)\mathcal{W}_{\text{int}}^{e,e}$$

1. Relate task acceleration to joint acceleration

Differentiate

$$\mathcal{V}_e^{e,0} = J(\theta)\dot{\theta}$$

to get

$$\dot{\mathcal{V}}_e^{e,0} = J(\theta)\ddot{\theta} + \dot{J}(\theta, \dot{\theta})\dot{\theta}$$

so

$$\ddot{\theta} = J^{-1}(\theta)\left(\dot{\mathcal{V}}_e^{e,0} - \dot{J}(\theta, \dot{\theta})\dot{\theta}\right)$$

assuming J is square and nonsingular.

2. Substitute into joint-space dynamics

$$MJ^{-1}\left(\dot{\mathcal{V}}_e^{e,0} - \dot{J}\dot{\theta}\right) + c + b + g = J^T\mathcal{W}_{\text{int}}^{e,e}$$

Expand:

$$MJ^{-1}\dot{\mathcal{V}}_e^{e,0} - MJ^{-1}\dot{J}\dot{\theta} + c + b + g = J^T\mathcal{W}_{\text{int}}^{e,e}$$

3. Convert generalized forces to task-space wrench

Premultiply both sides by J^{-T}:

$$J^{-T}MJ^{-1}\dot{\mathcal{V}}_e^{e,0} + J^{-T}\left(c + b + g - MJ^{-1}\dot{J}\dot{\theta}\right) = \mathcal{W}_{\text{int}}^{e,e}$$

4. Define the task-space inertia matrix

$$\Lambda(\theta) = J^{-T}(\theta)M(\theta)J^{-1}(\theta)$$

This is the apparent inertia “felt” at the end-effector.

5. Group velocity-dependent terms

Since

$$\dot{\theta} = J^{-1}(\theta)\mathcal{V}_e^{e,0}$$

all terms depending on $\dot{\theta}$, $\dot{J}$, Coriolis/centrifugal effects, and damping can be written as task-space velocity terms:

$$\eta(\theta, \mathcal{V}_e^{e,0})\mathcal{V}_e^{e,0} = J^{-T}\left(c(\theta, \dot{\theta}) + b(\dot{\theta}) - MJ^{-1}\dot{J}\dot{\theta}\right)$$

6. Define the task-space gravity term

$$\gamma(\theta) = J^{-T}(\theta)g(\theta)$$

Conclusion

Therefore, the joint-space equation

$$\boxed{M\ddot{\theta} + c + b + g = J^T \mathcal{W}_{\text{int}}^{e,e}}$$

is converted into the task-space equation

$$\boxed{\Lambda(\theta)\dot{\mathcal{V}}_e^{e,0} + \eta(\theta, \mathcal{V}_e^{e,0})\mathcal{V}_e^{e,0} + \gamma(\theta) = \mathcal{W}_{\text{int}}^{e,e}}$$

by using the kinematic relation $\mathcal{V} = J\dot{\theta}$, differentiating it, substituting for $\ddot{\theta}$, and premultiplying by J^{-T}.