SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 2: Maps between Sets and Groups

جامعة الملك فهد للبترول والمعادن
King Fahd University of Petroleum & Minerals

Ramy Rashad
Assistant Professor
CIE Department

Outline

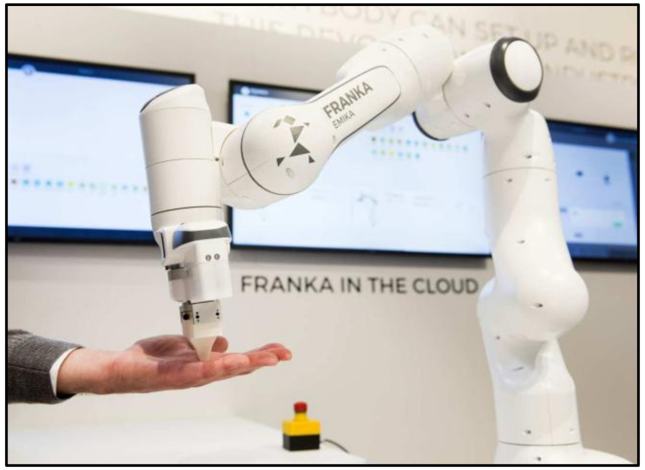
- Recap: Set theory basics
- Maps between sets
- Adding structure to sets
    - Groups

Outline

- Recap: Set theory basics
- Maps between sets
- Adding structure to sets
  - Groups

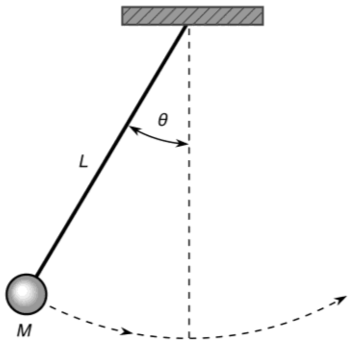
Recap: Robotic Systems

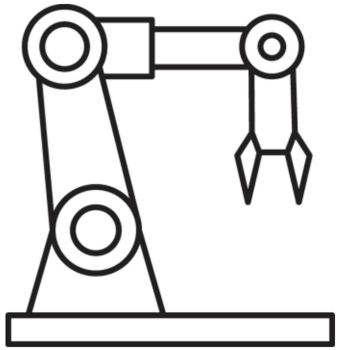
- Multi-rotor aerial vehicles
- Fixed-based manipulators

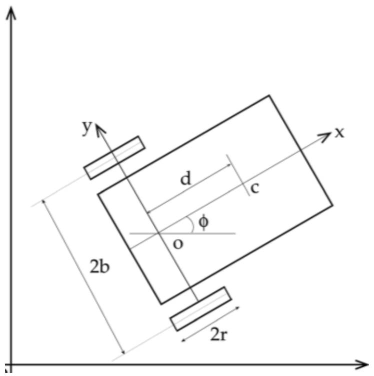


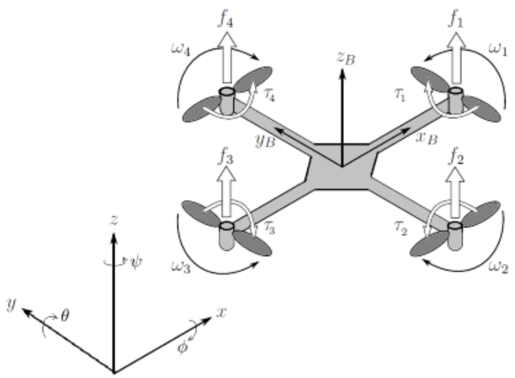
Recap: Why Geometric approach ?

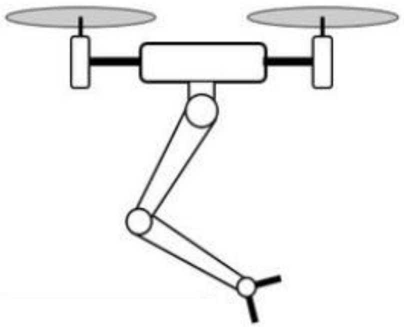
- Configuration space $\mathbb{Q}$ of (most) mechanical systems is not $\mathbb{R}^n$
  - Pendulum $\mathbb{Q} = S^1$
  - n-degree-of-freedom manipulator $\mathbb{Q} = T^n$
  - Planar mobile robot $\mathbb{Q} = SE(2)$
  - Multirotor aerial vehicle $\mathbb{Q} = SE(3)$
  - Aerial manipulator $\mathbb{Q} = SE(3) \times T^n$











Recap: Structure hierarchy

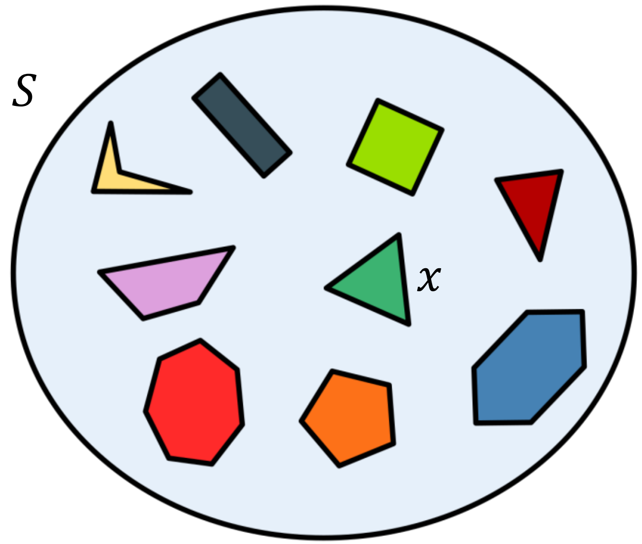
- A recurrent theme in mathematics is the classification of spaces by means of *structure-preserving maps* between them.
- Space = set + some structure

Lie Group	**Lie Algebra**
Symplectic manifold	**Algebra**
Riemannian manifold	**Vector space**
Smooth manifold	**Field**
Topological manifold	**Group**
Set	

Recap: What is a set ?

- A set is intuitively a collection of elements.
  - $x \in S$
  - $A := \{\text{square, rectangle, triangle, ...}\}$
  - $A := \{x \in S \mid \text{conditions on } x\} \subset S$

- We can define a set from two other ones by:
  - Cartesian product $A \times B$
  - Union $A \cup B$
  - Intersection $A \cap B$



Outline

- Recap: Set theory basics
- Maps between sets
- Adding structure to sets
  - Groups

Maps between sets

- Let A and B denote two sets, a **map** f from A to B assigns to each element $x \in A$ an element $f(x) \in B$.
- The standard notation for a map is:

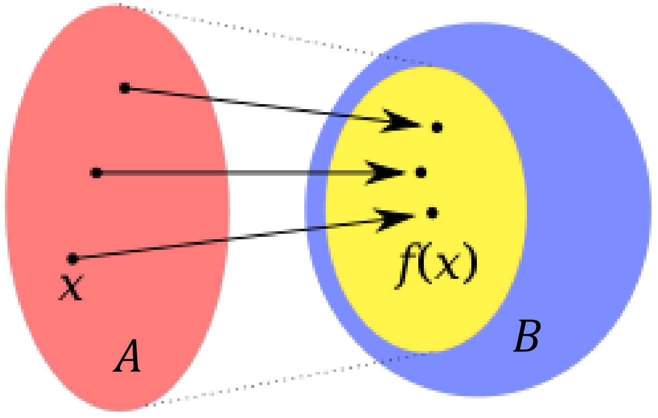
$$\begin{aligned} f: A &\to B \\ x &\mapsto f(x) \end{aligned}$$

Maps between sets

- Let A and B denote two sets, a **map** f from A to B assigns to each element $x \in A$ an element $f(x) \in B$.
- The standard notation for a map is:

$$f: A \to B$$
$$x \mapsto f(x)$$

- We call:
  - A the **domain** of f.
  - B the **codomain/target** of f.
  - $im_f(A) \coloneqq \{f(x) \in B \mid x \in A\}$ the **image** of A under f



$f: A \to B$

*A **map** is also known as a **function** or **mapping.**

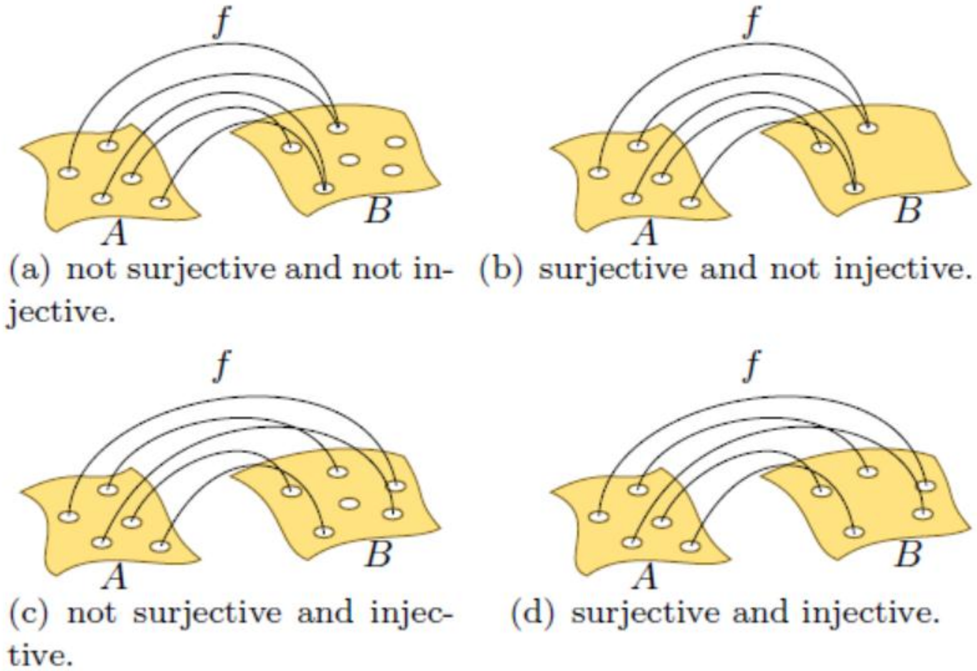
Classification of Maps

- A map $f: A \to B$ is said to be:
  - **Surjective**: if for each $y \in B$, there exists at least one $x \in A$ such that $f(x) = y$
  - **Injective**: if the equality $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$ implies that $x_1 = x_2$
  - **Bijective:** if it is both surjective and injective

*A **surjective** map is also called **onto,** while an **injective** map is also called **one-to-one**

Classification of Maps

- A map $f: A \to B$ is said to be:
  - **Surjective**: if for each $y \in B$, there exists at least one $x \in A$ such that $f(x) = y$
  - **Injective**: if the equality $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$ implies that $x_1 = x_2$
  - **Bijective:** if it is both surjective and injective



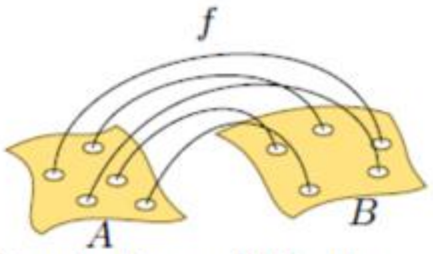
(a) not surjective and not injective.
(b) surjective and not injective.
(c) not surjective and injective.
(d) surjective and injective.

*A **surjective** map is also called **onto,** while an **injective** map is also called **one-to-one**

Classification of Maps

- A map $f: A \to B$ is said to be:
  - **Surjective**: if for each $y \in B$, there exists at least one $x \in A$ such that $f(x) = y$
  - **Injective**: if the equality $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$ implies that $x_1 = x_2$
  - **Bijective:** if it is both surjective and injective

- Examples:
  - $\mathrm{id}_A: A \to A, \qquad x \mapsto x$
  - $f: \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^3$

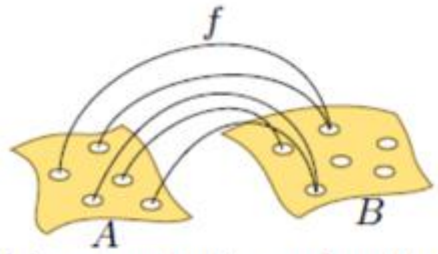


(d) surjective and injective.

Classification of Maps

- A map $f: A \to B$ is said to be:
  - **Surjective**: if for each $y \in B$, there exists at least one $x \in A$ such that $f(x) = y$
  - **Injective**: if the equality $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$ implies that $x_1 = x_2$
  - **Bijective:** if it is both surjective and injective

- Examples:
  - $f: \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^2$
  - $f: \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \sin(x)$

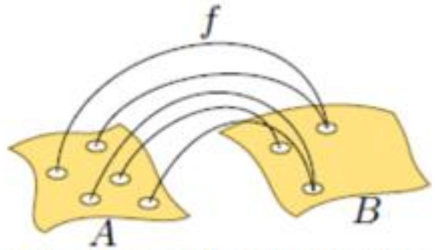


(a) not surjective and not injective.

Classification of Maps

- A map $f: A \to B$ is said to be:
  - **Surjective**: if for each $y \in B$, there exists at least one $x \in A$ such that $f(x) = y$
  - **Injective**: if the equality $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$ implies that $x_1 = x_2$
  - **Bijective:** if it is both surjective and injective

- Examples:
  - $f: \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \tan(x)$
  - $f: \mathbb{R} \to [-1,1], \qquad x \mapsto \sin(x)$

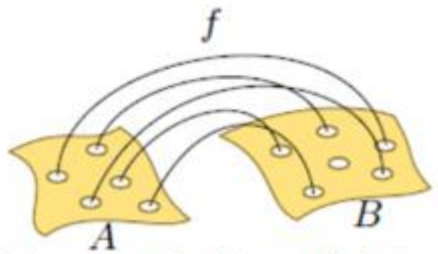


(b) surjective and not injective.

Classification of Maps

- A map $f: A \to B$ is said to be:
  - **Surjective**: if for each $y \in B$, there exists at least one $x \in A$ such that $f(x) = y$
  - **Injective**: if the equality $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$ implies that $x_1 = x_2$
  - **Bijective:** if it is both surjective and injective

- Examples:
  - $\exp: \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$



(c) not surjective and injective.

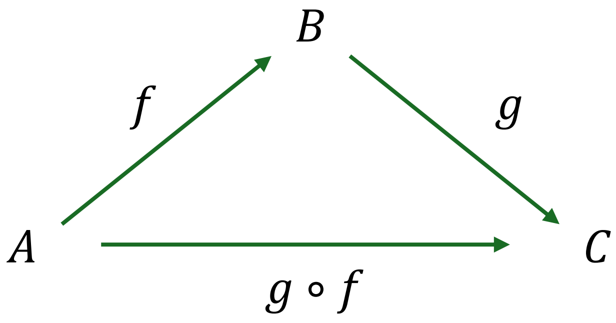
* But $\exp: \mathbb{R} \to \mathbb{R}_+$ is bijective
In general, if you restrict the codomain of an injective map to its image, the resulting function becomes bijective.

Composition of Maps

- Given two maps $f: A \to B$ and $g: B \to C$, we can construct a third map, called the **composition** of f and g, denoted by $g \circ f$ and defined as:

$$g \circ f: A \to C$$
$$x \mapsto g\big(f(x)\big).$$

- This is often represented by drawing the following **commutative diagram:**



Inverse of a map

- Let $f: A \to B$ be a bijective map. Then the **inverse** of f, denoted by $f^{-1}: B \to A$ is defined by:
  - $f^{-1} \circ f = \mathrm{id}_A$
  - $f \circ f^{-1} = \mathrm{id}_B$

- Two sets A and B are called (set-theoretic) **isomorphic** if there exists a bijective map $f: A \to B$. In this case, we write that

$$A \cong_{\mathrm{set}} B$$

Example: Bijection between $\mathbb{R}^{n^2}$ and $\mathbb{R}^{n\times n}$

- The vectorization map is defined as

$$\text{vec}: \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$$

such that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \longmapsto \text{vec}(A) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \\ a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \\ \vdots \\ a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} \in \mathbb{R}^{n^2}$$

```matlab
v = A(:);                % vec(A)
A_back = reshape(v,n,n); % inverse
```

Outline

- Recap: Set theory basics
- Maps between sets
- Adding structure to sets
    - Groups

Adding structure to sets

- So far, we've considered sets with no additional structure
- Now we'll add more structure to define:
  - Groups
  - Fields (Next Lecture)
  - Vector Space (Next Lecture)