SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 3: Vector Spaces I

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Outline

- Recap: Last Lectures
- Maps between groups
- Vector Space theory I
  - Field
  - Vector space
  - Linear map

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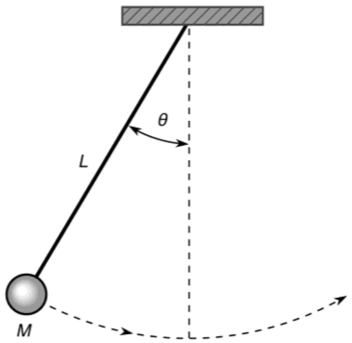
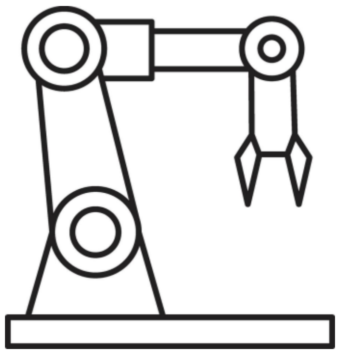
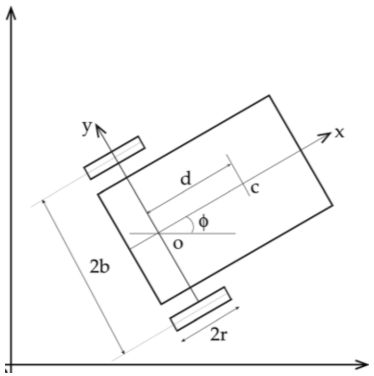
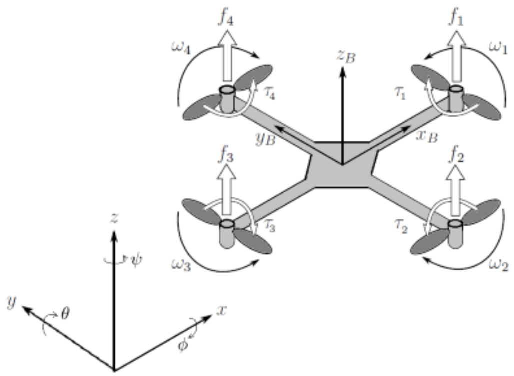
Recap: Why Geometric approach ?

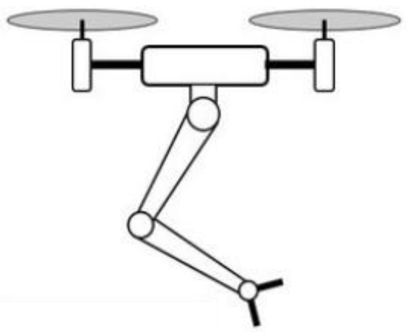
- Configuration space $\mathbb{Q}$ of (most) mechanical systems is not $\mathbb{R}^n$
  - Pendulum $\mathbb{Q} = S^1$
  - n-degree-of-freedom manipulator $\mathbb{Q} = T^n$
  - Planar mobile robot $\mathbb{Q} = SE(2)$
  - Multirotor aerial vehicle $\mathbb{Q} = SE(3)$
  - Aerial manipulator $\mathbb{Q} = SE(3) \times T^n$











Recap: Structure hierarchy

- A recurrent theme in mathematics is the classification of spaces by means of *structure-preserving maps* between them.
- Space = set + some structure

Set
$\mathbb{S}$

Group
$(\boldsymbol{G}, \odot)$

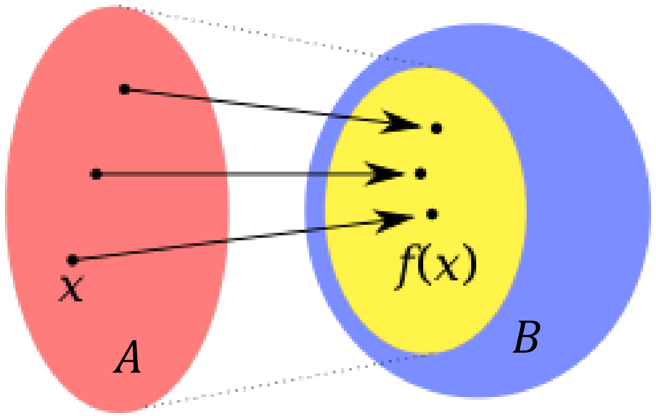
Vector space over a **Field**
$(\boldsymbol{V}, \oplus, \odot)$

Recap: Maps between sets

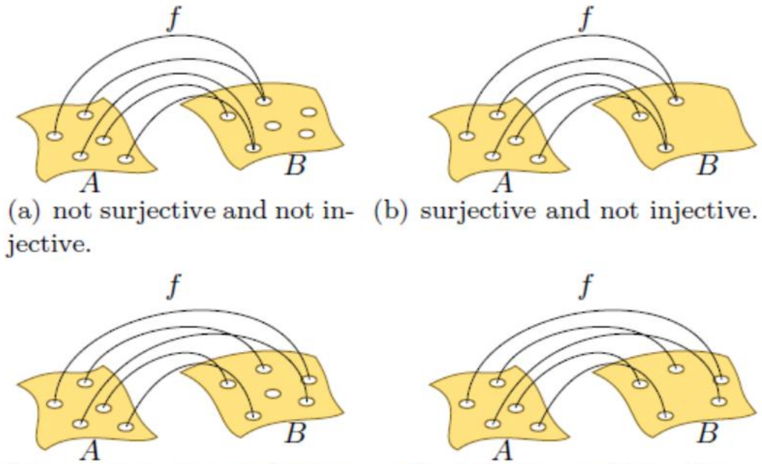
- The standard notation for a map is:

$$f : A \to B$$
$$x \mapsto f(x)$$

- We call:
  - A the **domain** of f.
  - B the **codomain/target** of f.
- A map can be either:
  - Surjective
  - Injective
  - Both
  - None



$f : A \to B$



(a) not surjective and not injective. (b) surjective and not injective.
(c) not surjective and injective. (d) surjective and injective.

Recap: Group

- A group is a pair $(G,\circ)$ where G is a set and $\circ: G \times G \to G$ is a map (called binary operation) that satisfies:
  i. $\forall a, b, c \in G$ we have that $(a \circ b) \circ c = a \circ (b \circ c)$ (Associativity)
  ii. $\exists\ e \in G$ such that $\forall g \in G$ we have that $e \circ g = g \circ e = g$ (Identity element)
  iii. $\forall g \in G, \exists\ g^{-1} \in G$ such that $\mathrm{g}^{-1} \circ g = g \circ \mathrm{g}^{-1} = e$ (Inverse element)

- A group $(G,\circ)$ is also called **abelian** if it satisfies:
  iv. $a \circ b = b \circ a, \quad \forall a, b \in G$

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Maps between groups

- Let $(G,\oplus)$ and $(H,\odot)$ be two groups.
- If there exists a map $\rho: G \to H$ that satisfies:
  - $\rho(a \oplus b) = \rho(\mathrm{a}) \odot \rho(b) \qquad \forall a, b \in G$

Then we call the map $\rho: G \to H$ a **group homomorphism**.

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- If there exists a map $\rho: G \to H$ that satisfies:
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Then we call the map $\rho: G \to H$ a **group homomorphism**.

- If ρ is also a bijective map, then we call ρ a (group) **isomorphism**
- If there exists an isomorphism between $(G,\oplus)$ & $(H,\odot)$, then we say that G and H are (group-theoretic) isomorphic to each other.

$$G \cong_{\text{grp}} H$$

Example 1

The map

$$\begin{aligned} \exp: \mathbb{R} &\to \mathbb{R}_+ \\ t &\mapsto e^t \end{aligned}$$

is a group *isomorphism* between $(\mathbb{R}, +)$ and $(\mathbb{R}_+, \cdot)$ because of the property $e^{t_1+t_2} = e^{t_1} \cdot e^{t_2}$.

Example 2

The map

$$\det : GL(n,\mathbb{R}) \to \mathbb{R}\backslash\{0\}$$
$$A \mapsto \det(A)$$

is a group *homomorphism* between $(GL(n,\mathbb{R}), \odot)$ and $(\mathbb{R}\backslash\{0\}, \cdot)$ because of the property $\det(A \odot B) = \det(A) \cdot \det(B)$

Recall: $GL(n,\mathbb{R}) := \{M \in \mathbb{R}^{n\times n} \,|\det(M) \neq 0\}$

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Vector Spaces

- Now we will turn attention to vector spaces (*aka* linear spaces)
- It is convenient to consider them in more abstract terms than to simply think of $\mathbb{R}^n$.
- A vector space $(V, \oplus, \odot)$ is a set that is equipped with two operations satisfying certain properties, not just a set of *n-tuples*.

- To define a vector space, we need to define first what is a field

$$(K, +, \cdot)$$

Field

- An (algebraic) field is a triple $(K, +, \cdot)$ where K is a set equipped with the maps $+, \cdot : K \times K \to K$ satisfying:
  - $(K, +)$ is an abelian group
  - $(K \backslash \{0\}, \cdot)$ is an abelian group
  - The maps $+$ and $\cdot$ satisfy the distributive property i.e.

    $\forall a, b, c \in K$ we have that $(a + b) \cdot c = a \cdot \mathrm{c} + \mathrm{b} \cdot \mathrm{c}$

Recall $(K, +)$ is an abelian group

i.	$\forall a, b, c \in K$ we have that $(a + b) + c = a + (b + c)$
ii.	$\exists\, 0 \in K$ such that $\forall a \in K$ we have that $0 + a = a + 0 = a$
iii.	$\forall a \in K, \exists\, -a \in K$ such that $a + (-a) = (-a) + a = 0$
iv.	$a + b = b + a, \quad \forall a, b \in \mathrm{K}$

Field

- An (algebraic) field is a triple $(K,+,\cdot)$ where K is a set equipped with the maps $+,\cdot : K \times K \rightarrow K$ satisfying:
  - $(K,+)$ is an abelian group
  - $(K\backslash\{0\}, \cdot)$ is an abelian group
  - The maps $+$ and $\cdot$ satisfy the distributive property i.e.

    $\forall a,b,c \in K$ we have that $(a+b)\cdot c = a\cdot \mathrm{c} + \mathrm{b}\cdot \mathrm{c}$

Example:

- The sets $\mathbb{R},\mathbb{Q},\mathbb{C}$ are all fields under the usual addition and multiplication operations
- The triple $(\mathbb{Z},+,\cdot)$ is not a field

Recall $(K,+)$ is an abelian group

i.	$\forall a,b,c \in K$ we have that $(a+b)+c = a+(b+c)$
ii.	$\exists\ 0 \in K$ such that $\forall a \in K$ we have that $0+a = a+0 = a$
iii.	$\forall a \in K$, $\exists\ -a \in K$ such that $a+(-a) = (-a)+a = 0$
iv.	$a+b = b+a, \quad \forall a,b \in \mathrm{K}$

Vector space

- A vector space $(V, \oplus, \odot)$ over a field $(K, +, \cdot)$ is the set V equipped with two operations:
  - $\oplus : V \times V \to V$ called vector addition
  - $\odot : K \times V \to V$ called scalar multiplication

that should satisfy the rules:

- $(V, \oplus)$ is an Abelian group
- The map $\odot$ is an action of K on $(V, \oplus)$:

i) $\forall \lambda \in K : \forall v, w \in V : \lambda \odot (v \oplus w) = (\lambda \odot v) \oplus (\lambda \odot w)$;

ii) $\forall \lambda, \mu \in K : \forall v \in V : (\lambda + \mu) \odot v = (\lambda \odot v) \oplus (\mu \odot v)$;

iii) $\forall \lambda, \mu \in K : \forall v \in V : (\lambda \cdot \mu) \odot v = \lambda \odot (\mu \odot v)$;

iv) $\forall v \in V : 1 \odot v = v$.

- An element of $v \in V$ is called a **vector**.

Example 1

- $(\mathbb{R}^n, \oplus, \odot)$ is a vector space over the field $(\mathbb{R}, +, \cdot)$.

  - $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$

  - $\lambda \odot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \lambda \cdot x_1 \\ \vdots \\ \lambda \cdot x_n \end{pmatrix}$

  - Identity element of $(\mathbb{R}^{\boldsymbol{n}}, \oplus)$ is the zero **vector:** $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

The vector space $\mathbb{R}^n$(not the set !!) is frequently called the **n-dimensional Euclidean space**.

Example 2

- $(\mathbb{R}^{m\times n}, \oplus, \odot)$ is a vector space over the field $(\mathbb{R}, +, \cdot)$.

  - $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \oplus \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} := \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$

  - $\lambda \odot \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} := \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & \ddots & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}$

  - Identity element of $(\mathbb{R}^{\boldsymbol{m\times n}}, \oplus)$ is the zero **vector:** $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

Maps between vector spaces

Set $\mathbb{S}$	**Group** $(\boldsymbol{G}, \odot)$	**Vector space** over a **Field** $(\boldsymbol{V}, \oplus, \odot)$
Maps	**Group homomorphisms**	**Linear maps**
Bijections	**Group isomorphisms**	**Linear isomorphisms**