1) Linear map between vector spaces:

Let $(V, \oplus, \odot)$ & $(W, \boxplus, \boxdot)$ be vector spaces over the same field K.

We say the map $f: V \to W$ is a linear map if:

$$\boxed{f((\lambda \odot v_1) \oplus v_2) = (\lambda \boxdot f(v_1)) \boxplus f(v_2)}, \forall \lambda \in K, \forall v_1, v_2 \in V$$

Usually,

$$\boxed{f(\lambda v_1 + v_2) = \lambda \cdot f(v_1) + f(v_2)}$$

2) linear isomorphism:

$\Longrightarrow$ if $f: V \to W$ is a linear map & bijective,

then we call it a linear isomorphism

3) Set of linear maps between vector spaces:

$$L(V, W) := \{ A: V \to W \mid A \text{ is a linear map} \}$$

$$\underbrace{\left(\mathcal{L}(V;W),\ \diamond\!\!\!+\ ,\ \diamond\!\!\!\cdot\ \right)}_{\text{vector space over } K}$$

$\boxed{\mathrm{Hom}(V;W)}$

Another notation

$A, B \in \mathcal{L}(V;W)$

- $\left(A \mathbin{\diamond\!\!\!+} B\right)(u) = A(u) \boxplus B(u),\ u \in V$
- $\left(a \mathbin{\diamond\!\!\!\cdot} A\right)(u) = a \boxdot \left(A(u)\right)$, $a \in K$

4) Endomorphism

Let V be a vector space. An endomorphism is a linear map $V \longrightarrow V$.

$$\boxed{\operatorname{End}(V) := \mathcal{L}(V; V)}$$

5) Automorphism:

$$\operatorname{Aut}(V) := \{ A \in \operatorname{End}(V) \mid A \text{ is isomorphic} \}$$

* $L(V;W) \Rightarrow$ Vector space
  $\not\Rightarrow$ not a group.
* $\mathrm{End}(V) \Rightarrow$ vector space
  $\not\Rightarrow$ not a group
* $\mathrm{Aut}(V) \not\Rightarrow$ vector space
  $\Rightarrow (\mathrm{Aut}(V), \circ) =: GL(V)$

Composition of maps.