SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 4: Vector Spaces II

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Outline

- Recap: Last Lectures
- Dual spaces
- Special case: $\mathbb{R}^n$
- Bases and components

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Recap: Structure hierarchy

- A recurrent theme in mathematics is the classification of spaces by means of *structure-preserving maps* between them.
- Space = set + some structure

	Set	**Group**	**Vector space over a Field**
Mathematical object	$\mathbb{S}$	$(G,\oplus)$	$(V,\oplus,\odot)$ over $(K,+,\cdot)$
Structure-preserving map	Map $f\colon \mathbb{S} \to \mathbb{T}$	Group homomorphism $\rho\colon G \to H$	Linear map $A\colon V \xrightarrow{\sim} W$
Isomorphic spaces	Bijection $\mathbb{S} \cong_{\text{set}} \mathbb{T}$	Group isomorphism $G \cong_{\text{grp}} H$	Linear isomorphism $V \cong_{\text{vec}} W$

Recap: Vector space

- A vector space $(V, \oplus, \odot)$ over a field $(K, +, \cdot)$ is the set V equipped with two operations:
  - $\oplus : V \times V \rightarrow V$ called vector addition
  - $\odot : K \times V \rightarrow V$ called scalar multiplication

that should satisfy the rules:

  - $(V, \oplus)$ is an Abelian group
  - The map $\odot$ is an action of K on $(V, \oplus)$

- An element of $\mathrm{v} \in V$ is called a **vector**.

Recap: Vector space

- **Example is** $(\mathbb{R}^{\boldsymbol{n}}, \oplus, \odot)$:
- $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$
- $\lambda \odot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \lambda \cdot x_1 \\ \vdots \\ \lambda \cdot x_n \end{pmatrix}$
- Identity element of $(\mathbb{R}^{\boldsymbol{n}}, \oplus)$ is the zero **vector** $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

- **Example is** $(\mathbb{R}^{\boldsymbol{m \times n}}, \oplus, \odot)$:
- $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \oplus \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} := \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$
- $\lambda \odot \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} := \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & \ddots & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}$
- Identity element of $(\mathbb{R}^{\boldsymbol{m \times n}}, \oplus)$ is the zero **matrix** $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

Recap: Linear Maps

- A linear map $A: V \rightarrow W$ between the vector spaces $(V, \oplus, \odot)$ and $(W, \boxplus, \boxdot)$ over the same field K is defined such that:
  - $A(v_1 \oplus v_2) = A(v_1) \boxplus A(v_2), \qquad \forall\, v_1, v_2 \in V$
  - $A(\lambda \odot v) = \lambda \boxdot A(v)\,, \qquad \forall\, v \in V, \lambda \in K$

- If we drop the special notation for operators,

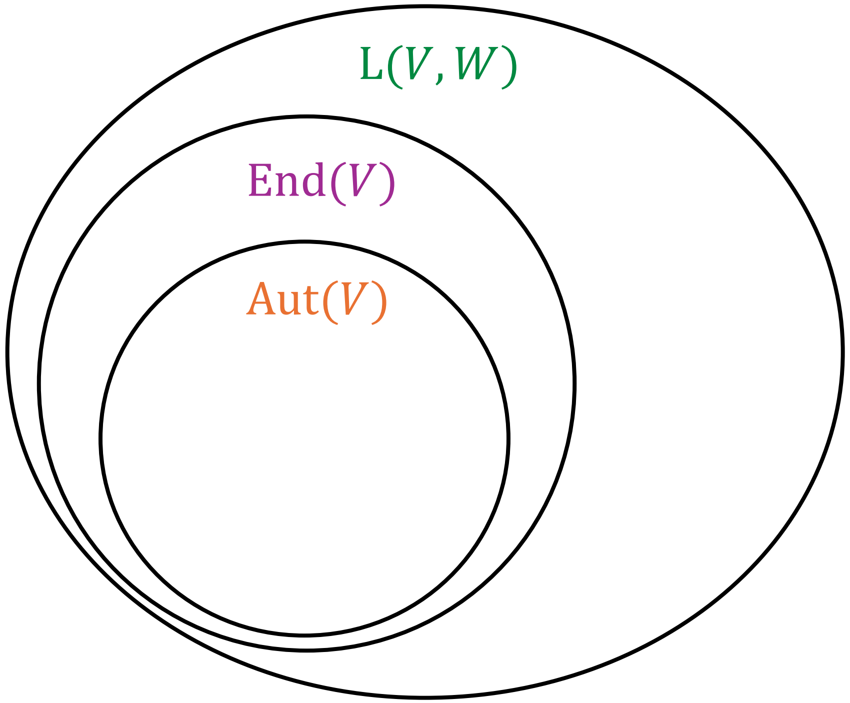
$$A(\lambda v_1 + v_2) = \lambda\, A(v_1) + A(v_2)$$

- The set of all linear maps from a vector space V to W is itself a vector-space, denoted by $\mathrm{L}(V; W)$.

Recap: Subsets of $\mathrm{L}(V, W)$

- $L(V; W) \coloneqq \{A: V \to W \mid A \text{ is a linear map}\}$
- $\mathrm{End}(V) \coloneqq L(V, V) \coloneqq \{A \in L(V, W) \mid W = V\}$
- $\mathrm{Aut}(V) \coloneqq \{A \in \mathrm{End}(V) \mid A \text{ is a bijective map}\}$

$$= \{A: V \to W \mid A \text{ is a linear isomorphism}\}$$



Common Notation for linear maps

$$A: V \xrightarrow{\sim} W$$

Recap: Subsets of $\mathrm{L}(V, W)$

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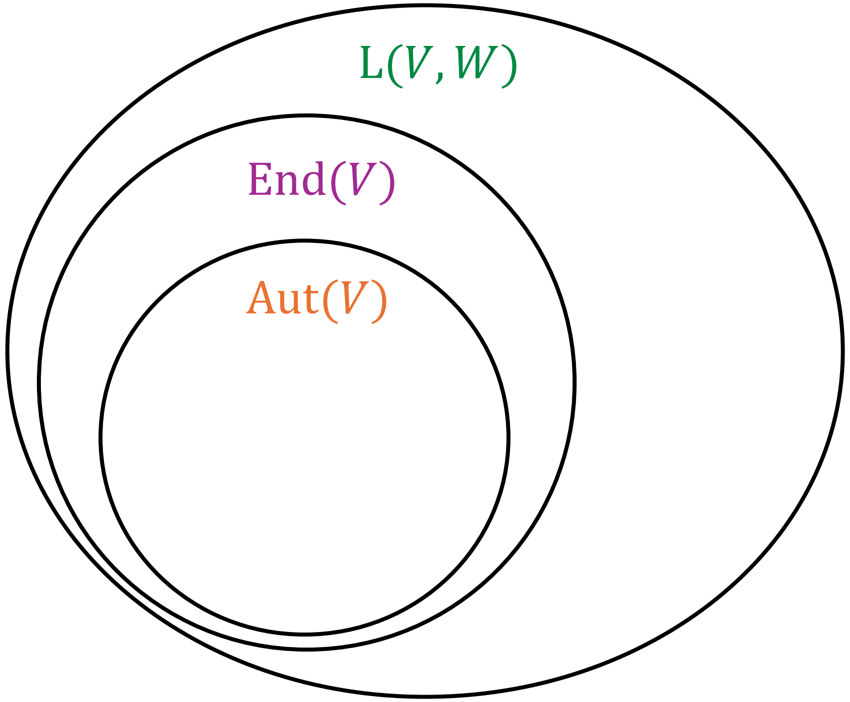
- $\mathrm{Aut}(V) \coloneqq \{A \in \mathrm{End}(V) \mid A \text{ is a bijective map}\}$

- $L(V; W)$ & $\mathrm{End}(V)$ can be made into a vector space equipped with pointwise addition and scalar multiplication.
- $\mathrm{Aut}(V)$ equipped with the same operations is not a vector space.

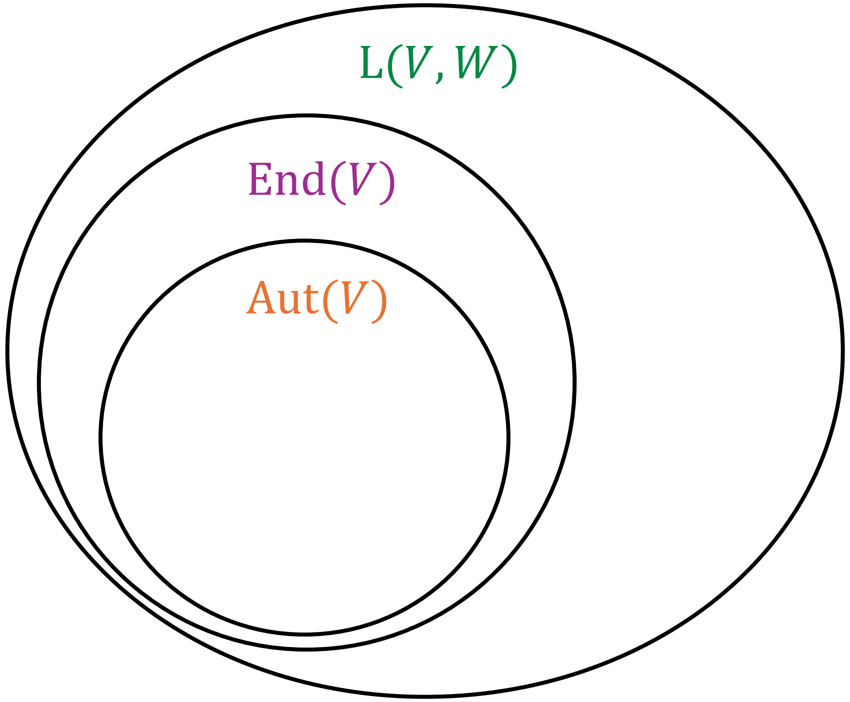


Recap: Subsets of $\mathrm{L}(V, W)$

- $L(V; W) \coloneqq \{A: V \to W \mid A \text{ is a linear map}\}$
- $\mathrm{End}(V) \coloneqq L(V, V) \coloneqq \{A \in L(V, W) \mid W = V\}$
- $\mathrm{Aut}(V) \coloneqq \{A \in \mathrm{End}(V) \mid A \text{ is a bijective map}\}$

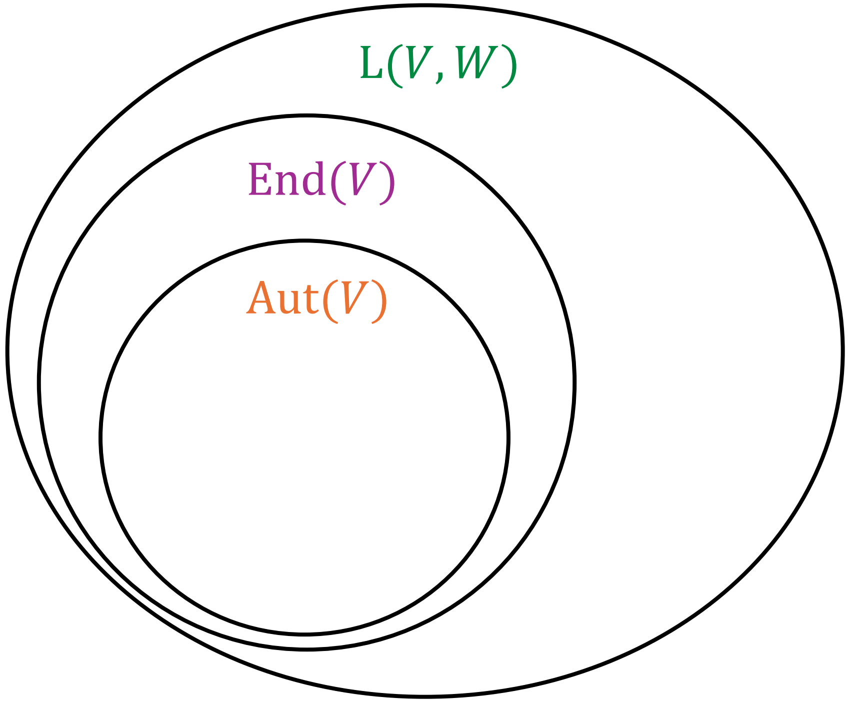
- $L(V; W)$ & $\mathrm{End}(V)$ can be made into a vector space equipped with pointwise addition and scalar multiplication.
- $\mathrm{Aut}(V)$ equipped with the same operations is not a vector space.

- $\mathrm{Aut}(V)$ equipped with the composition operation $\circ$ is a group $GL(V)$.
- $L(V; W)$ & $\mathrm{End}(V)$ equipped with $\circ$ is not a group.



Recap: Subsets of $\mathrm{L}(V, W)$

- **General Conclusion:**
  - Even if a **set** S can be given a structure (e.g., "vector space" or "group"), a **subset** $A \subset S$ or a **superset** $B \supset S$ does **not** automatically inherit that same structure with the same operations.



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- Bases and components

Dual space

- The notion of a **dual (vector) space** $\boldsymbol{V}^*$ to a vector space $\boldsymbol{V}$ is extremely important in mechanics.
- However, it is usually overlooked !!
- While an element of $\boldsymbol{V}$ is called a vector, an element of $\boldsymbol{V}^*$ is called a covector*.
  - Velocity-like variables are vectors.
  - Force-like variables are covectors.

* Also referred to as one-form.

Dual space

- Let V be a vector space over the field K
- The **dual space** to V is:

$$V^* := L(V; K),$$

where $(K, +, \cdot)$ is considered a vector space over itself.

- An element of $\alpha \in V^*$ is a linear map from V to K called a **covector** or **one-form**.

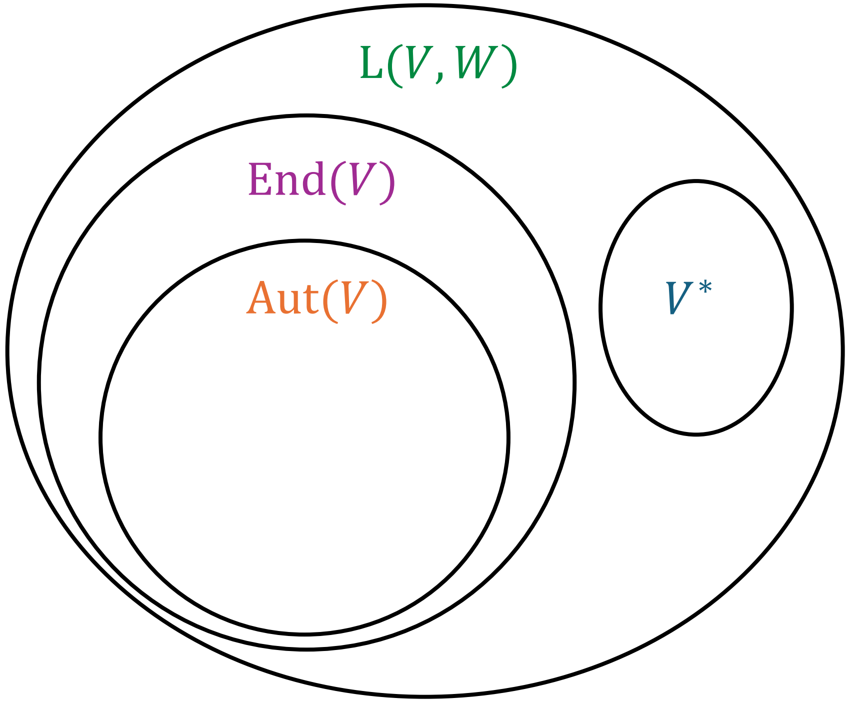
$$\begin{aligned} \alpha: V &\overset{\sim}{\rightarrow} K \\ v &\mapsto \alpha(v) \end{aligned}$$

Dual space

- The **dual space** to a vector space V is:

$$V^* := L(V; K)$$

- $\mathrm{L}(V, W)$ is a vector space in general, so V^* is also a vector space.
- In fact, both V and V^* have the same *dimension**.



** We did not formally introduce what is the dimension of a vector space.*

Endomorphisms and Bilinear maps

- Since fundamentally a vector space V and its dual V^* are different spaces, one should be careful when defining maps between them.
- Endomorphisms
  - $\text{End}(V) \coloneqq L(V, V) = \{A: V \xrightarrow{\sim} V\}$
  - $\text{End}(V^*) \coloneqq L(V^*, V^*) = \{\Gamma: V^* \xrightarrow{\sim} V^*\}$
- Bilinear maps
  - $BL(V) \coloneqq L(V, V^*) = \{B: V \xrightarrow{\sim} V^*\}$
  - $BL(V^*) \coloneqq L(V^*, V) = \{\Pi: V^* \xrightarrow{\sim} V\}$

- If V is an n-dimensional vector space over $\mathbb{R}$.
- Then V^* is an n-dimensional vector space over $\mathbb{R}$.
- All these maps are represented by $n \times n$ matrices !!

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Dual space $(\mathbb{R}^n)^*$

- Consider the vector spaces over the field $\mathbb{R}$:

$$V = \mathbb{R}^n,\ K = \mathbb{R}$$

equipped with vector addition and scalar multiplication.

- The dual space $(\mathbb{R}^n)^*$ consists of all linear maps $\alpha\colon \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}$.

- You can prove that any linear map α **must** have the form:

$$\alpha(v) = \sum_{i=1}^{n} \alpha_i v^i$$

with $\alpha_i \in \mathbb{R}$ for $i \in \{1, \cdots, n\}$,

In components

$$\alpha(v) = (\alpha_1 \quad \cdots \quad \alpha_n) \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

$(\mathbb{R}^n)^* = \mathbb{R}^{1\times n}$ is the set of row vectors

Linear Maps $\mathrm{L}(\mathbb{R}^n;\mathbb{R}^m)$

- Consider the vector spaces over the field $\mathbb{R}$:

$$V = \mathbb{R}^n \; , \; W = \mathbb{R}^m$$

equipped with vector addition and scalar multiplication.

- The set $\mathrm{L}(\mathbb{R}^n;\mathbb{R}^m)$ consists of all linear maps $A: \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^m$.

- You can prove that any linear map A **must** have the form:

$$A(v) = \left(\sum a_i^1 v^i, \cdots, \sum a_i^m v^i\right) \; ,$$

with $a_i^j \in \mathbb{R}$ for $i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}$.

In components

$$A(v) = \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \cdots & a_n^m \end{pmatrix} \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

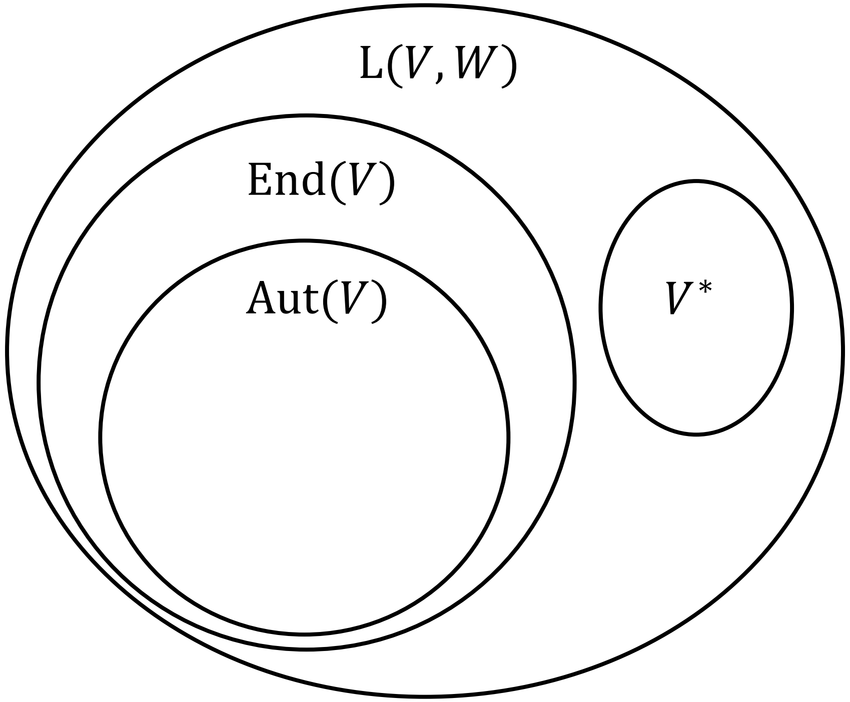
m – rows, n - columns

$\mathrm{L}(\mathbb{R}^n;\mathbb{R}^m) = \mathbb{R}^{m\times n}$, the set of $m \times n$ matrices.

Subsets of $\mathrm{L}(\mathbb{R}^n; \mathbb{R}^m)$

- $(\mathbb{R}^n)^* = \mathbb{R}^{1\times n}$
- $L(\mathbb{R}^n; \mathbb{R}^m) = \mathbb{R}^{m\times n}$
- $\mathrm{End}(\mathbb{R}^n) \coloneqq L(\mathbb{R}^n, \mathbb{R}^n) = \mathbb{R}^{n\times n}$
- $\mathrm{Aut}(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times n} | \det A \neq 0\}$

$V = \mathbb{R}^n, W = \mathbb{R}^m$



Subsets of $\mathrm{L}(\mathbb{R}^n; \mathbb{R}^m)$

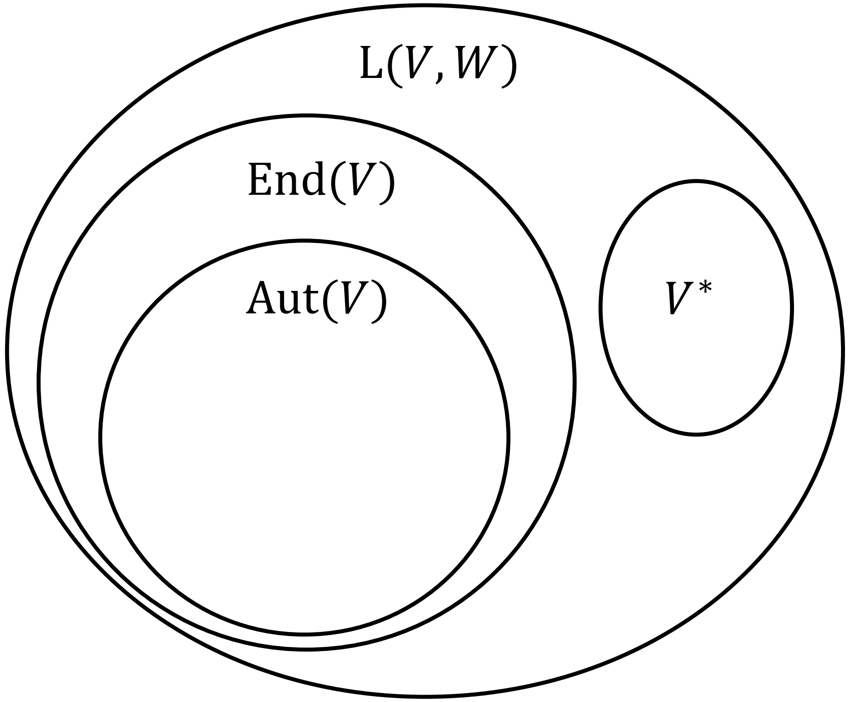
$\bigoplus$:= matrix addition
$\bigodot$:= scalar-matrix multiplication

- $(\mathbb{R}^n)^* = \mathbb{R}^{1\times n}$
- $L(\mathbb{R}^n; \mathbb{R}^m) = \mathbb{R}^{m\times n}$
- $\mathrm{End}(\mathbb{R}^n) \coloneqq L(\mathbb{R}^n, \mathbb{R}^n) = \mathbb{R}^{n\times n}$
- $\mathrm{Aut}(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times n} | \det A \neq 0\}$

- $(\mathbb{R}^{m\times n}, \bigoplus, \bigodot)$ has vector space structure
- $(\mathbb{R}^{n\times n}, \bigoplus, \bigodot)$ has vector space structure

- A vector space must have the zero linear map
- But the zero map $0(v) = \mathbf{0}$ is **not invertible**.
- So $0 \notin \mathrm{Aut}(\mathbb{R}^n)$
- Therefore, $(\mathrm{Aut}(\mathbb{R}^n), \bigoplus, \bigodot)$ is not a vector space.

$V = \mathbb{R}^n, W = \mathbb{R}^m$



Subsets of $\mathrm{L}(\mathbb{R}^n;\mathbb{R}^m)$

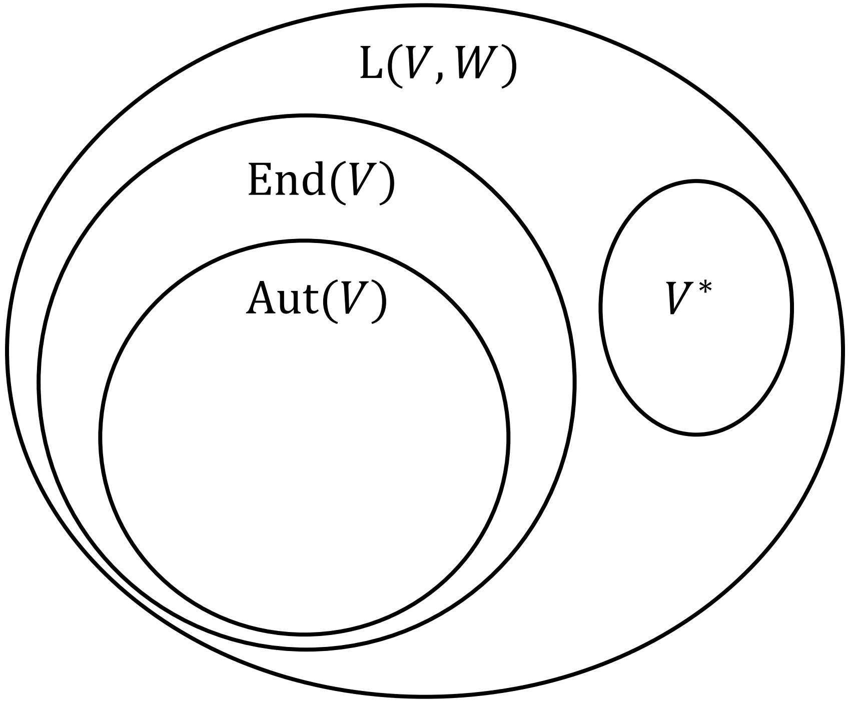
$\otimes :=$ matrix-matrix multiplication

- $(\mathbb{R}^n)^* = \mathbb{R}^{1\times n}$
- $L(\mathbb{R}^n;\mathbb{R}^m) = \mathbb{R}^{m\times n}$
- $\mathrm{End}(\mathbb{R}^n) \coloneqq L(\mathbb{R}^n,\mathbb{R}^n) = \mathbb{R}^{n\times n}$
- $\mathrm{Aut}(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\}$

- $(\mathrm{Aut}(\mathbb{R}^n),\otimes)$ has a group structure
- $(\mathrm{Aut}(\mathbb{R}^n),\otimes) =: GL(\mathbb{R}^n) \equiv GL(n,\mathbb{R})$ General Linear group

- A group requires every element to have an inverse.
- $\mathrm{End}(\mathbb{R}^n)$ includes **both** invertible and non-invertible matrices. Thus, $(\mathrm{End}(\mathbb{R}^n),\otimes)$ is not a group.

$V = \mathbb{R}^n, W = \mathbb{R}^m$



Subsets of $\mathrm{L}(\mathbb{R}^n; \mathbb{R}^m)$

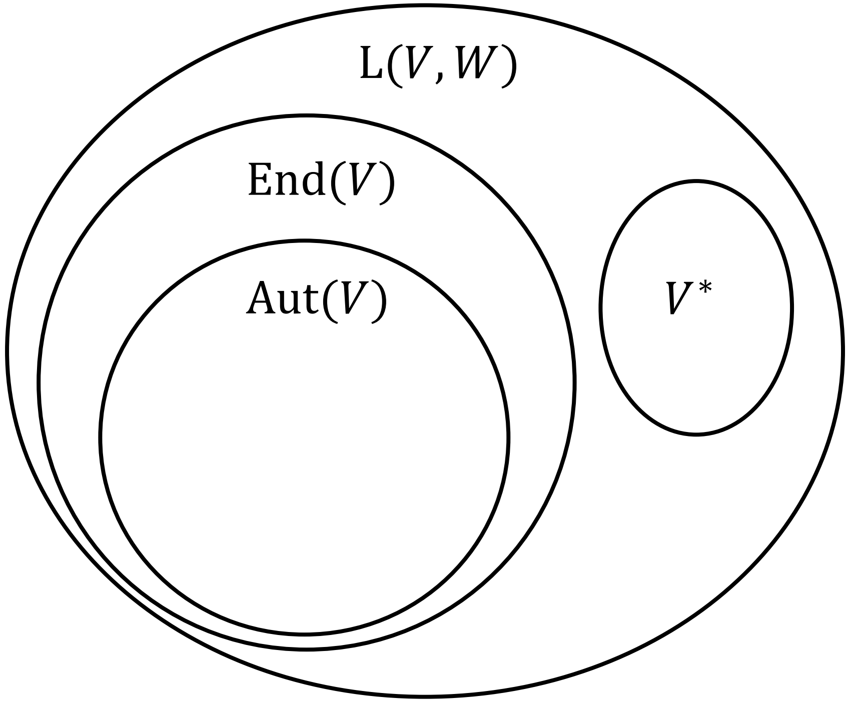
$\otimes :=$ matrix-matrix multiplication

- $(\mathbb{R}^n)^* = \mathbb{R}^{1\times n}$
- $L(\mathbb{R}^n; \mathbb{R}^m) = \mathbb{R}^{m\times n}$
- $\mathrm{End}(\mathbb{R}^n) := L(\mathbb{R}^n, \mathbb{R}^n) = \mathbb{R}^{n\times n}$
- $\mathrm{Aut}(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times n} | \det A \neq 0\}$

- $(\mathrm{Aut}(\mathbb{R}^n), \otimes)$ has a group structure
- $(\mathrm{Aut}(\mathbb{R}^n), \otimes) =: GL(\mathbb{R}^n) \equiv GL(n, \mathbb{R})$ General Linear group

- $(L(\mathbb{R}^n; \mathbb{R}^m), \otimes)$ is not even well-defined !

$V = \mathbb{R}^n, W = \mathbb{R}^m$



Example: Map that is not linear

- Example: $V = \mathbb{R}^n$, $W = \mathbb{R}^m$
- Consider the map $\Psi: \mathbb{R}^n \to \mathbb{R}^m$ between vector spaces defined by:

$$\Psi(v) = \left(\sum a_i^1 v^i + b^1, \cdots, \sum a_i^m v^i + b^m\right) ,$$

with $a_i^j, b^j \in \mathbb{R}$ for $i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}$.

In components

$$\Psi(v) = \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \cdots & a_n^m \end{pmatrix} \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} + \begin{pmatrix} b^1 \\ \vdots \\ b^m \end{pmatrix} = A(v) + b$$

- A map Ψ of the form above is called an **affine map** and is not a linear map.

$$\Psi(\lambda v) = A(\lambda v) + b = \lambda\, A(v) + b \neq \lambda\, \Psi(v)$$

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Components of tensors

- So far, the mathematical objects involved with vector spaces we introduced are abstractly defined.
  - Vectors, covectors, endomorphisms, and bilinear maps.
- All these objects can be written in components once a basis has been chosen.
- However, the geometric nature of these objects should be respected independent of the basis we choose.

Every finite-dimensional vector space V of dimension n, is isomorphic to $\mathbb{R}^n$ (as vector spaces).

Basis for a vector space

- Let V be a vector space over $\mathbb{R}$. A basis S for a vector space V is a collection of vectors in V that are:
  - Linearly independent from each other
  - Generate V

Basis for a vector space

- Let V be a vector space over $\mathbb{R}$. A basis S for a vector space V is a collection of vectors in V that are:
  - **Linearly independent from each other**
    - A set $S \subset V$ of vectors is linearly independent if, for every finite subset $\{e_1, \cdots, e_k\} \subset S$, the equality

$$\sum_{i=i}^{k} c^i e_k = c^1 e_1 + \cdots + c^k e_k = 0, \text{ for some constants } c^i \in \mathbb{R},$$

implies that these constants should be zero, i.e., $c^i = 0 \; \forall i \in \{1, \cdots, k\}$.

  - Generate V

Basis for a vector space

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      implies that these constants should be zero, i.e., $c^i = 0 \; \forall i \in \{1, \cdots, k\}$.

  - **Generate V**
    - A set $S \subset V$ of vectors generates a vector space V, if every vector $\mathrm{v} \in V$ can be written as the linear combination

      $v = c^1 e_1 + \cdots + c^k e_k$, for some constants $c^i \in \mathbb{R}$
    - We usually write that $V = \mathrm{span}_{\mathbb{R}}(S)$