SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 5: Vector Spaces III

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Outline

- Recap: Last Lectures
- Bases and components
  - Components of vectors
  - Components of co-vectors
  - Components of endomorphisms and bilinear maps
  - Change of bases

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Recap: Structure hierarchy

- A recurrent theme in mathematics is the classification of spaces by means of *structure-preserving maps* between them.
- Space = set + some structure

	Set	**Group**	**Vector space over a Field**
Mathematical object	$\mathbb{S}$	$(G,\oplus)$	$(V,\oplus,\odot)$ over $(K,+,\cdot)$
Structure-preserving map	Map $f: \mathbb{S} \to \mathbb{T}$	Group homomorphism $\rho: G \to H$	Linear map $A: V \xrightarrow{\sim} W$
Isomorphic spaces	Bijection $\mathbb{S} \cong_{\text{set}} \mathbb{T}$	Group isomorphism $G \cong_{\text{grp}} H$	Linear isomorphism $V \cong_{\text{vec}} W$

Recap: Vector space

- A vector space $(V,\oplus,\odot)$ over a field $(K,+,\cdot)$ is the set V equipped with two operations:
  - $\oplus : V \times V \to V$ called vector addition
  - $\odot : K \times V \to V$ called scalar multiplication

that should satisfy the rules:

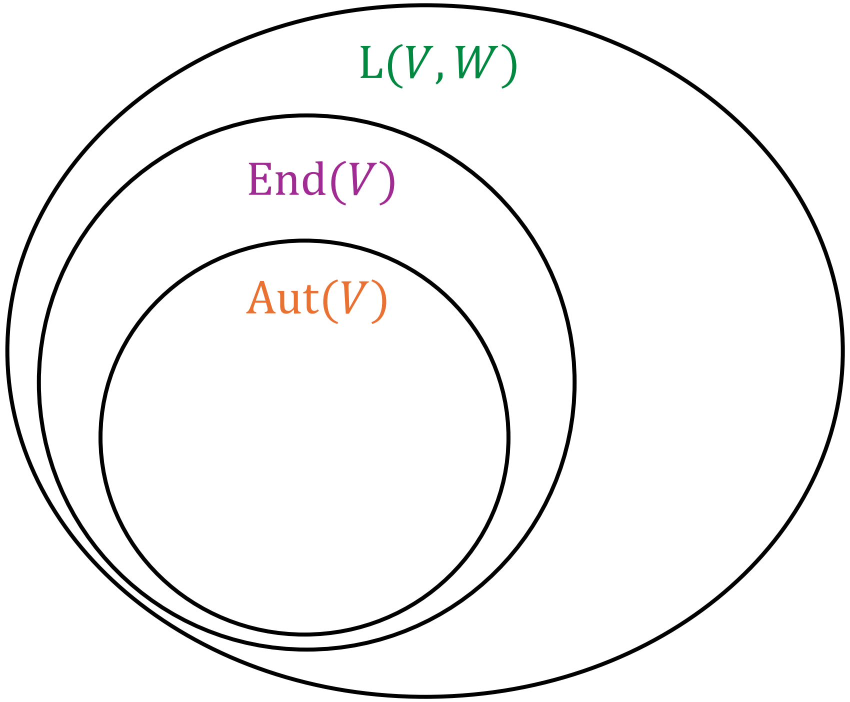
  - $(V,\oplus)$ is an Abelian group
  - The map $\odot$ is an action of K on $(V,\oplus)$

- An element of $\mathrm{v} \in V$ is called a **vector**.

Recap: Subsets of $\mathrm{L}(V, W)$

- $L(V; W) \coloneqq \{A: V \to W \mid A \text{ is a linear map}\}$
- $\mathrm{End}(V) \coloneqq L(V, V) \coloneqq \{A \in L(V, W) \mid W = V\}$
- $\mathrm{Aut}(V) \coloneqq \{A \in \mathrm{End}(V) \mid A \text{ is a bijective map}\}$

$$= \{A: V \to W \mid A \text{ is a linear isomorphism}\}$$



Even if a **set** S can be given a structure, a **subset** or **superset** of it does **not** automatically inherit that same structure with the same operations !

Recap: Subsets of $\mathrm{L}(V, W)$

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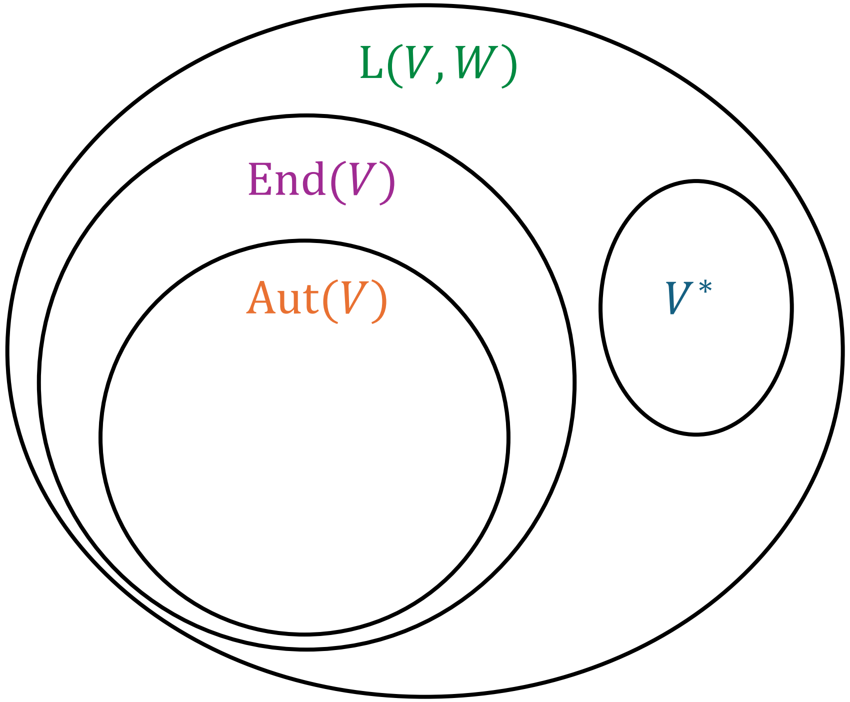
  $= \{A: V \to W \mid A \text{ is a linear isomorphism}\}$
- $V^* \coloneqq L(V; K) \coloneqq \{\alpha: V \to K, \alpha \text{ is a linear map}\}$



Elements of a **vector space** and its **dual space** are fundamentally different !!

Recap: Endomorphisms and Bilinear maps

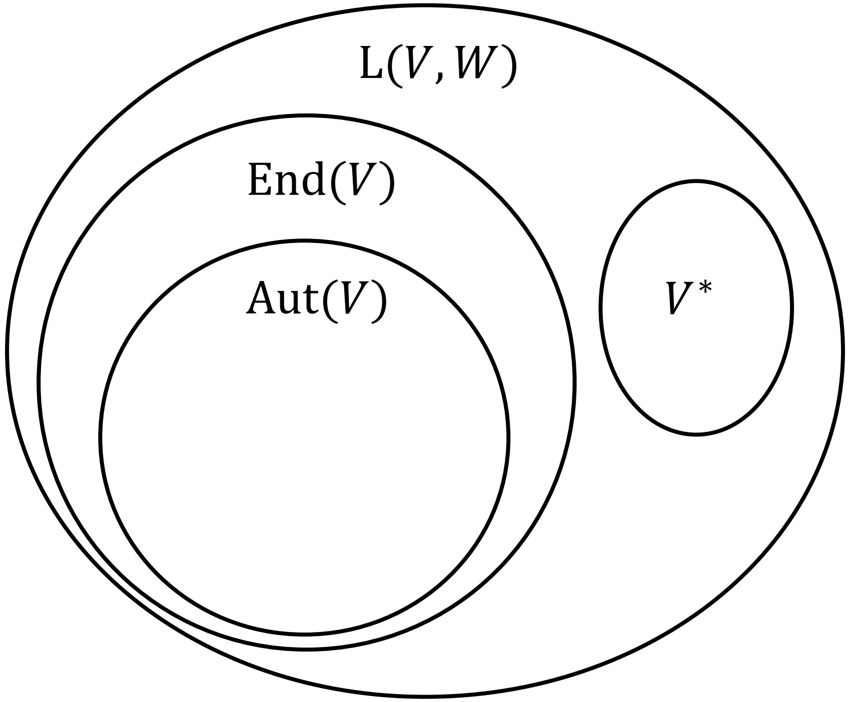
- One should be careful when defining maps between V and V^*.
- Endomorphisms
  - $\text{End}(V) \coloneqq L(V, V) = \{A: V \stackrel{\sim}{\rightarrow} V\}$
  - $\text{End}(V^*) \coloneqq L(V^*, V^*) = \{\Gamma: V^* \stackrel{\sim}{\rightarrow} V^*\}$
- Bilinear maps
  - $BL(V) \coloneqq L(V, V^*) = \{B: V \stackrel{\sim}{\rightarrow} V^*\}$
  - $BL(V^*) \coloneqq L(V^*, V) = \{\Pi: V^* \stackrel{\sim}{\rightarrow} V\}$

- If V is an n-dimensional vector space over $\mathbb{R}$.
- Then V^* is an n-dimensional vector space over $\mathbb{R}$.
- All these maps are represented by $n \times n$ matrices !!

Recap: Special Case $V = \mathbb{R}^n$

- $L(\mathbb{R}^n; \mathbb{R}^m) = \mathbb{R}^{m \times n}$
- $\mathrm{End}(\mathbb{R}^n) \coloneqq L(\mathbb{R}^n, \mathbb{R}^n) = \mathbb{R}^{n \times n}$
- $\mathrm{Aut}(\mathbb{R}^n) = \{A \in \mathbb{R}^{n \times n} | \det A \neq 0\}$
- $(\mathbb{R}^n)^* = \mathbb{R}^{1 \times n}$

$V = \mathbb{R}^n, W = \mathbb{R}^m$



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Components

- So far, the mathematical objects involved with vector spaces we introduced are abstractly defined.
  - Vectors, covectors, endomorphisms, and bilinear maps.
- All these objects can be written in components once a basis has been chosen.
- However, the geometric nature of these objects should be respected independent of the basis we choose.

Every finite-dimensional vector space V of dimension n, is isomorphic to $\mathbb{R}^n$ (as vector spaces).

Basis for a vector space

- Let V be a vector space over $\mathbb{R}$. A basis S for a vector space V is a collection of vectors in V that are:
  - Linearly independent from each other
  - Generate V

Example 1

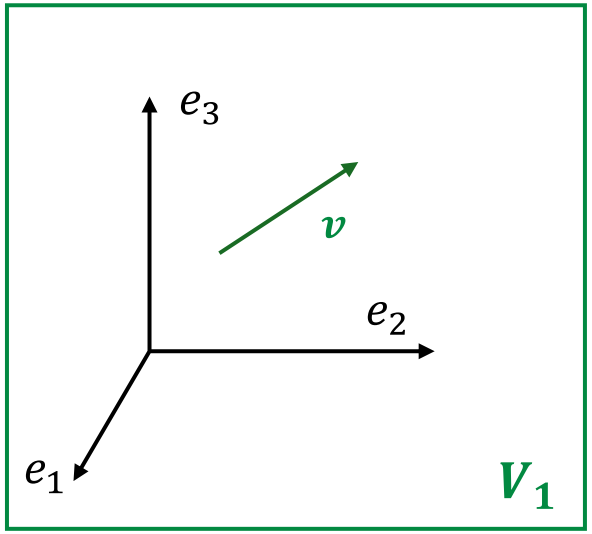
- Consider the vector space V_1
- Consider bases $e_i \in V_1$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- We have that any $v \in V_1$ can be expressed as

$$v = v^1 e_1 + v^2 e_2 + v^3 e_3$$

with $v^i \in \mathbb{R}$. Therefore, we have that $\dim V_1 = 3$.



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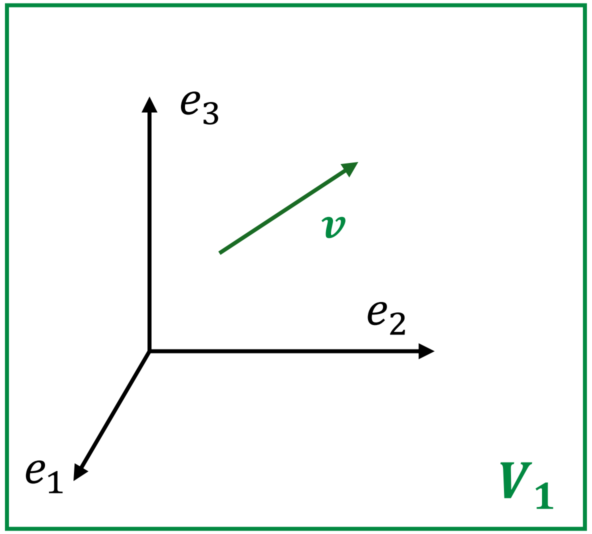
- The components of v are *usually* expressed as

$$[v] = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \in \mathbb{R}^3$$



Example 2

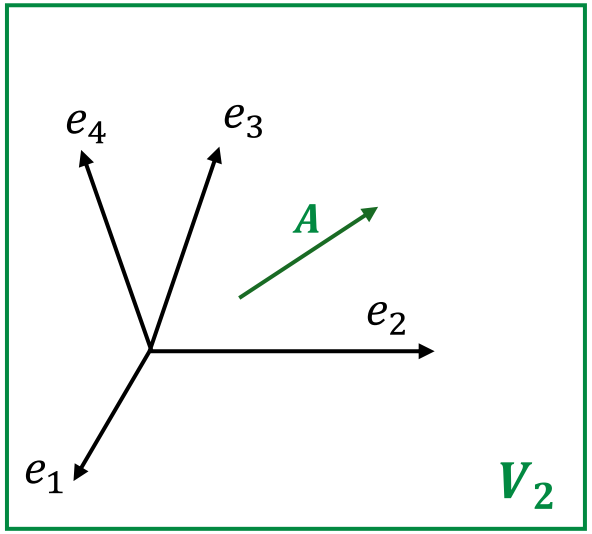
- Consider the vector space V_2
- Consider bases $e_i \in V_2$

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- We have that any $\mathrm{A} \in V_2$ can be expressed as

$$A = a^{11}e_1 + a^{12}e_2 + a^{21}e_3 + a^{22}e_4$$

with $a^{ij} \in \mathbb{R}$. Therefore, we have that $\dim V_2 = 4$.



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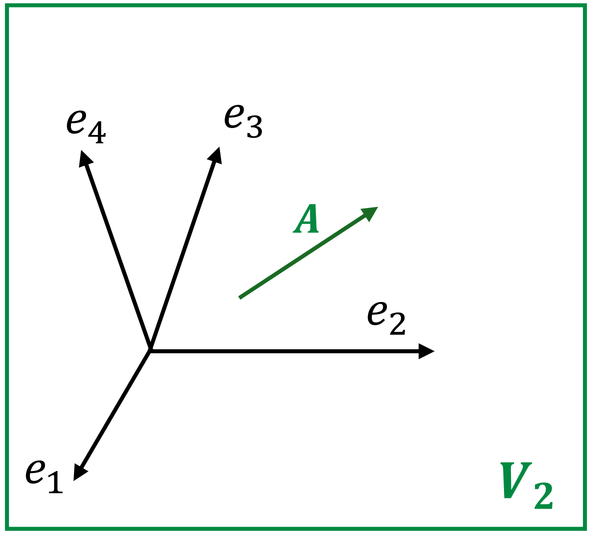
- The components of A are *usually* expressed as

$$[A] = \begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix} \in \mathbb{R}^{2\times 2} \cong \mathbb{R}^4$$



Example 3

- Consider the vector space V_3 of skew-symmetric matrices

$$V_3 := \{\Omega \in \mathbb{R}^{3\times 3} \mid \Omega = -\Omega^\top\}$$

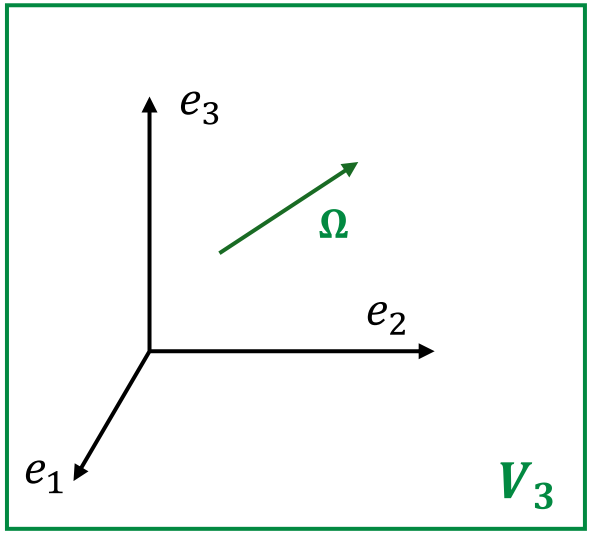
- Consider bases $e_i \in V_3$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- We have that any $\Omega \in V_3$ can be expressed as

$$\Omega = \Omega^1 e_1 + \Omega^2 e_2 + \Omega^3 e_3$$

with $\Omega^i \in \mathbb{R}$. Therefore, we have that $\dim V_3 = 3$.



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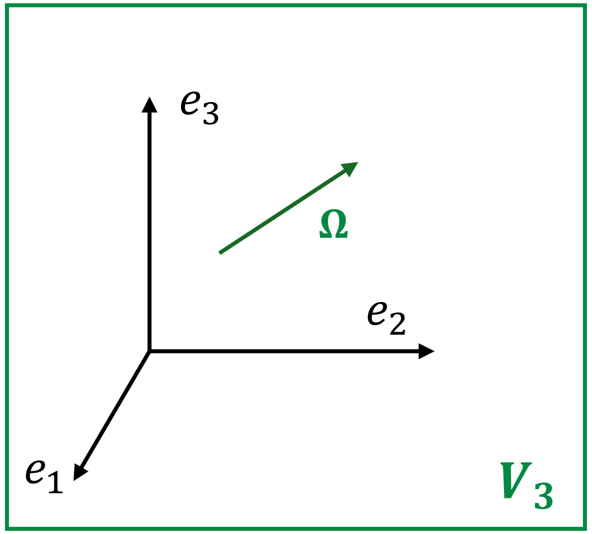
- We have that any $\Omega \in V_1$ can be expressed as

$$\Omega = \Omega^1 e_1 + \Omega^2 e_2 + \Omega^3 e_3$$

with $\Omega^i \in \mathbb{R}$. Therefore, we have that $\dim V_3 = 3$.

- The components of Ω are *usually* expressed as

$$[\Omega] = \begin{pmatrix} 0 & -\Omega^3 & \Omega^2 \\ \Omega^3 & 0 & -\Omega^1 \\ -\Omega^2 & \Omega^1 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3} \cong \mathbb{R}^3$$



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Components of a co-vector

- Consider a vector space V with bases $\{\mathrm{e}_1, \cdots, \mathrm{e}_n\}$.
- An element α of its dual V^* is a linear map $\alpha: V \to \mathbb{R}$.
- For any $v \in V$, we have that

$$\alpha(v) = \alpha(v^1 e_1 + \cdots + v^n e_n) = v^1 \alpha(e_1) + \cdots + v^n \alpha(e_n)$$

Recall Linear map:

$$\alpha(\lambda\, v_1 + v_2) = \lambda\, \alpha(v_1) + \alpha(v_2)$$

Components of a co-vector

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- Let $\alpha_i \coloneqq \alpha(e_i) \in \mathbb{R}$ denote the components of α.
- Therefore, we have that

$$\alpha(v) = v^1 \alpha_1 + \cdots + v^n \alpha_n = \sum_{i=1}^{n} v^i \alpha_i$$

Recall Lecture 4:

$$\alpha(v) = \sum_{i=1}^{n} v^i \alpha_i$$

Components of a co-vector

- Consider a vector space V with bases $\{\mathrm{e}_1, \cdots, \mathrm{e}_n\}$.
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- If we express $[v] \in \mathbb{R}^n$ and $[\alpha] \in \mathbb{R}^n$ then

$$\boxed{\alpha(v) = [\alpha]^{\mathsf{T}} [v]}$$

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What if $[v]$ are components of a matrix ?

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To be continued on Blackboard !