SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 6: Manifolds and Lie groups I

جامعة الملك فهد للبترول والمعادن
King Fahd University of Petroleum & Minerals

Ramy Rashad
Assistant Professor
CIE Department

Outline

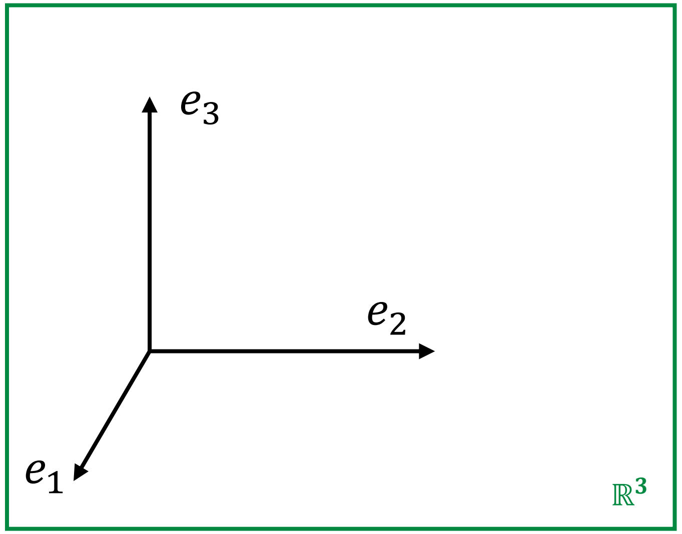
- Why differentiable structure ?
- Atlas of the world
- Manifold theory
- Maps on a manifold

Outline

- Why differentiable structure ?
- Atlas of the world
- Manifold theory
- Maps on a manifold

Different views of $\mathbb{R}^n$

- As a set $\mathbb{R}^n := \mathbb{R} \times \cdots \times \mathbb{R}$
- As a vector space $(\mathbb{R}^n, \oplus, \odot)$ over $(\mathbb{R}, +, \cdot)$



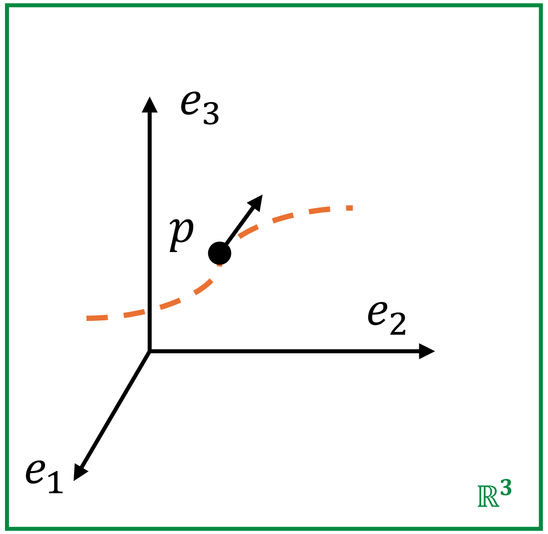
Different views of $\mathbb{R}^n$

- As a set $\mathbb{R}^n \coloneqq \mathbb{R} \times \cdots \times \mathbb{R}$
- As a vector space $(\mathbb{R}^n, \oplus, \odot)$ over $(\mathbb{R}, +, \cdot)$

- To do calculus, analysis, and describe dynamical systems we need more structure.
- This is called a differentiable structure.

- A set M along with this differentiable structure

is called a differentiable manifold.



Differentiability Class of functions

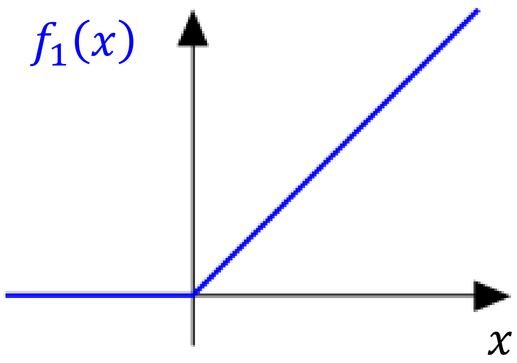
- In mathematical analysis, the smoothness of a function is a property measured by the number of continuous derivatives (differentiability class) it has over its domain.

- Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a map from an open interval of $\mathbb{R}$ to $\mathbb{R}$. Then the function f is said to be of:
  - Class C^0: if f is continuous on I
  - Class C^1: if its derivative f' exists and both f, f' are continuous on I
  - Class C^k: if its derivatives $f', f'', \cdots, f^{(k)}$ exist and are all continuous on I
  - Class C^∞: if it has derivatives of all orders on I

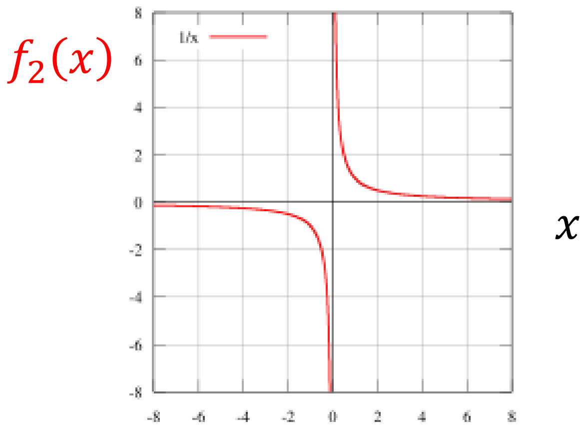
C^∞ functions are called smooth / infinitely-differentiable functions

Examples: $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$

- $f_1(x)$ is of Class C^0
- $f_2(x)$ is not of Class C^0



C^0 function on $\mathbb{R}$



Not a C^0 function on $\mathbb{R}$

Differentiability Class of functions

- The same concept can be extended to maps on $\mathbb{R}^n$

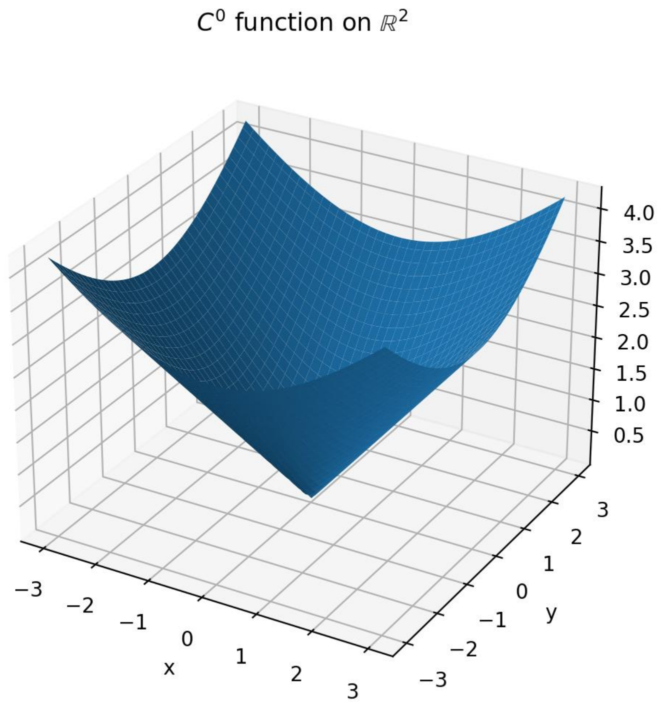
- Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ be a map from an open interval of $\mathbb{R}^n$ to $\mathbb{R}$. Then the function f is said to be of class C^k, for some positive integer k, if all partial derivatives

$$\frac{\partial^\alpha f}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}\, \cdots\, \partial x_n^{\alpha_n}}(y_1, y_2, \ldots, y_n)$$

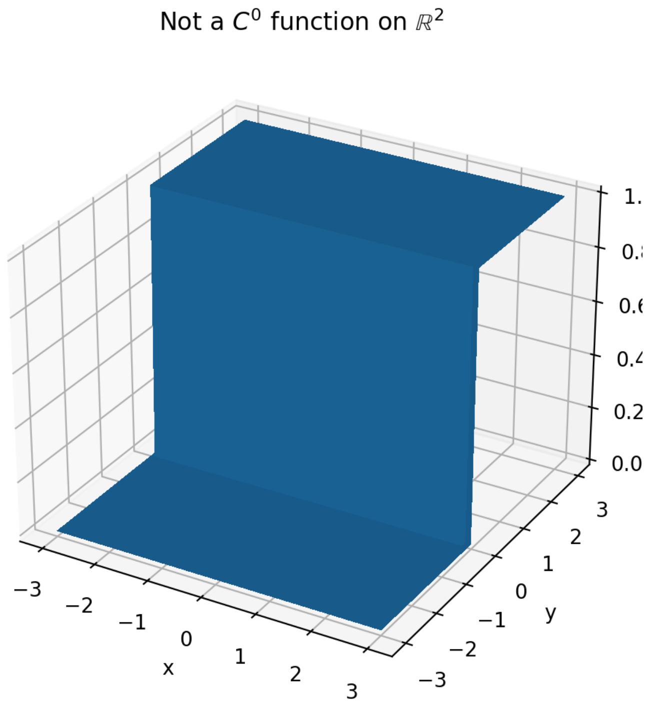
exist and are continuous on U.

Examples: $f: U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$

- $f_1(x_1, x_2)$ is of Class C^0
- $f_2(x_1, x_2)$ is not of Class C^0



$f_1(x_1, x_2)$



$f_2(x_1, x_2)$

What does a continuous function mean ?

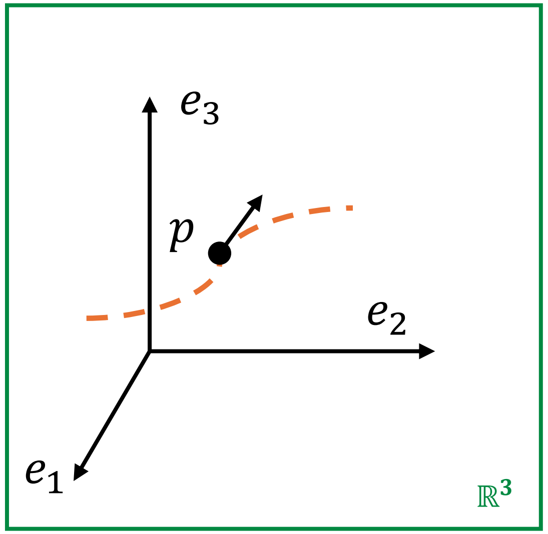
- The formal way of analyzing continuity of functions is by making a set M into a topological manifold by equipping it with a topology $\mathcal{O}$ satisfying certain conditions.

- When you do analysis on Euclidean space $\mathbb{R}^n$, you are using its standard topology $\mathcal{O}_{\text{std}}$.

- How do we do analysis on general topological spaces that are not Euclidean?

Equation of motion

$\dot{p} = f(p), \quad p \in \mathbb{R}^n$



Outline

- Why differentiable structure ?
- Atlas of the world
- Manifold theory
- Maps on a manifold

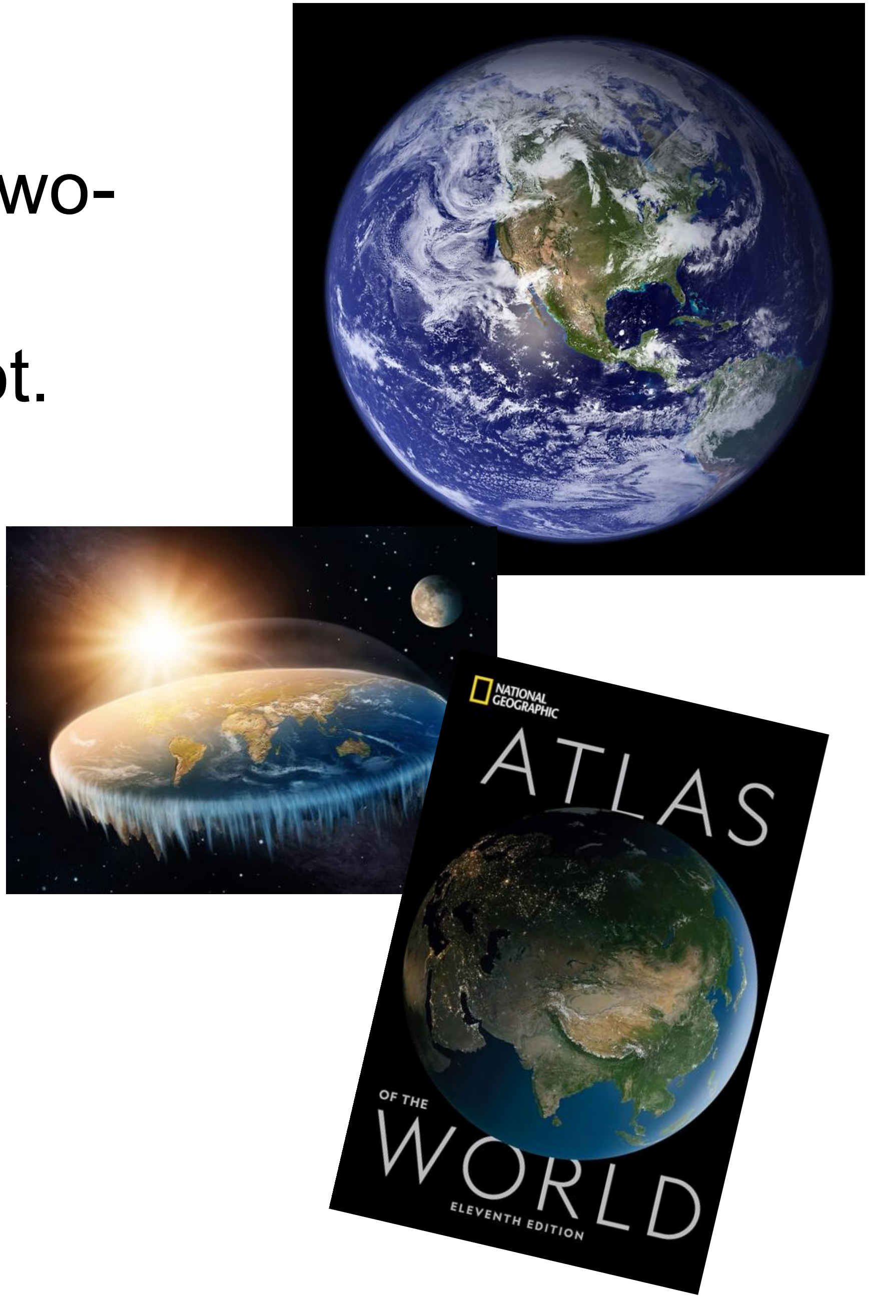
Atlas of the world

- The surface of the earth is an example of a two-dimensional non-Euclidean space.
- Locally, S^2 “looks-like” $\mathbb{R}^2$ but globally it is not.

$$S^2 \coloneqq \{(x, y, z) \in \mathbb{R}^3 \,|x^2 + y^2 + z^2 = c^2\}$$

- An atlas of Earth is a collection of charts.
- Each chart maps a “local” region of Earth

into a piece of paper $\mathbb{R}^2$.



Charts

- Consider two charts over Egypt and Saudi Arabia.

S^2 Earth

$\mathbb{R}^2$ charts

Charts

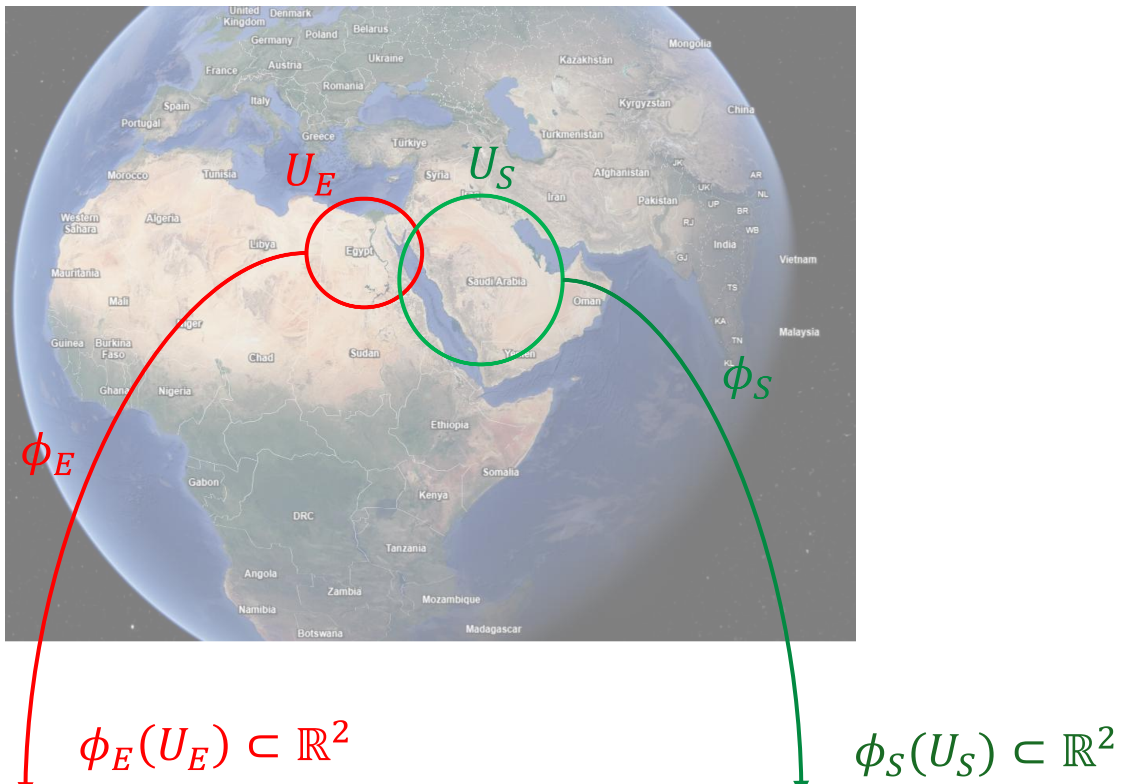
- A chart of Earth consists of the pair (U, ϕ) where $U \subset S^2$ and $\phi: U \rightarrow \mathbb{R}^2$ is a continuous map.

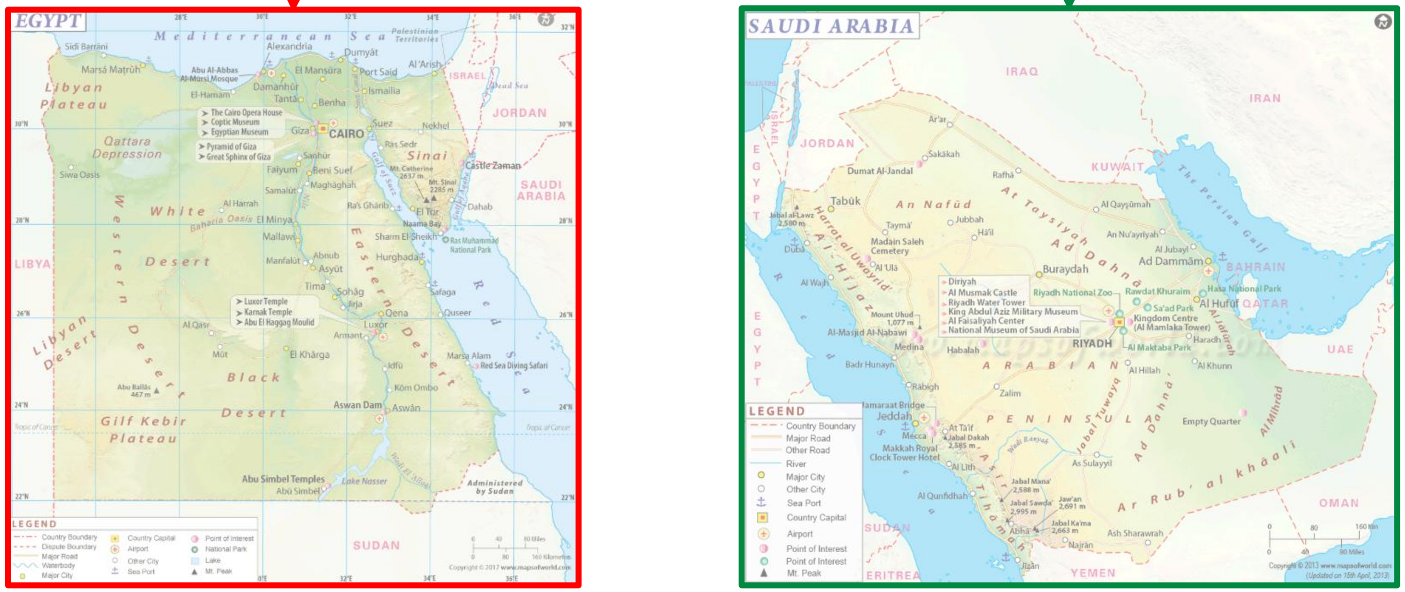
- A collection of charts that cover all of Earth S^2 is called an Atlas $\mathcal{A}$:

$$\mathcal{A} := \{(U_i, \phi_i)\}_{i \in A}$$

with the property that

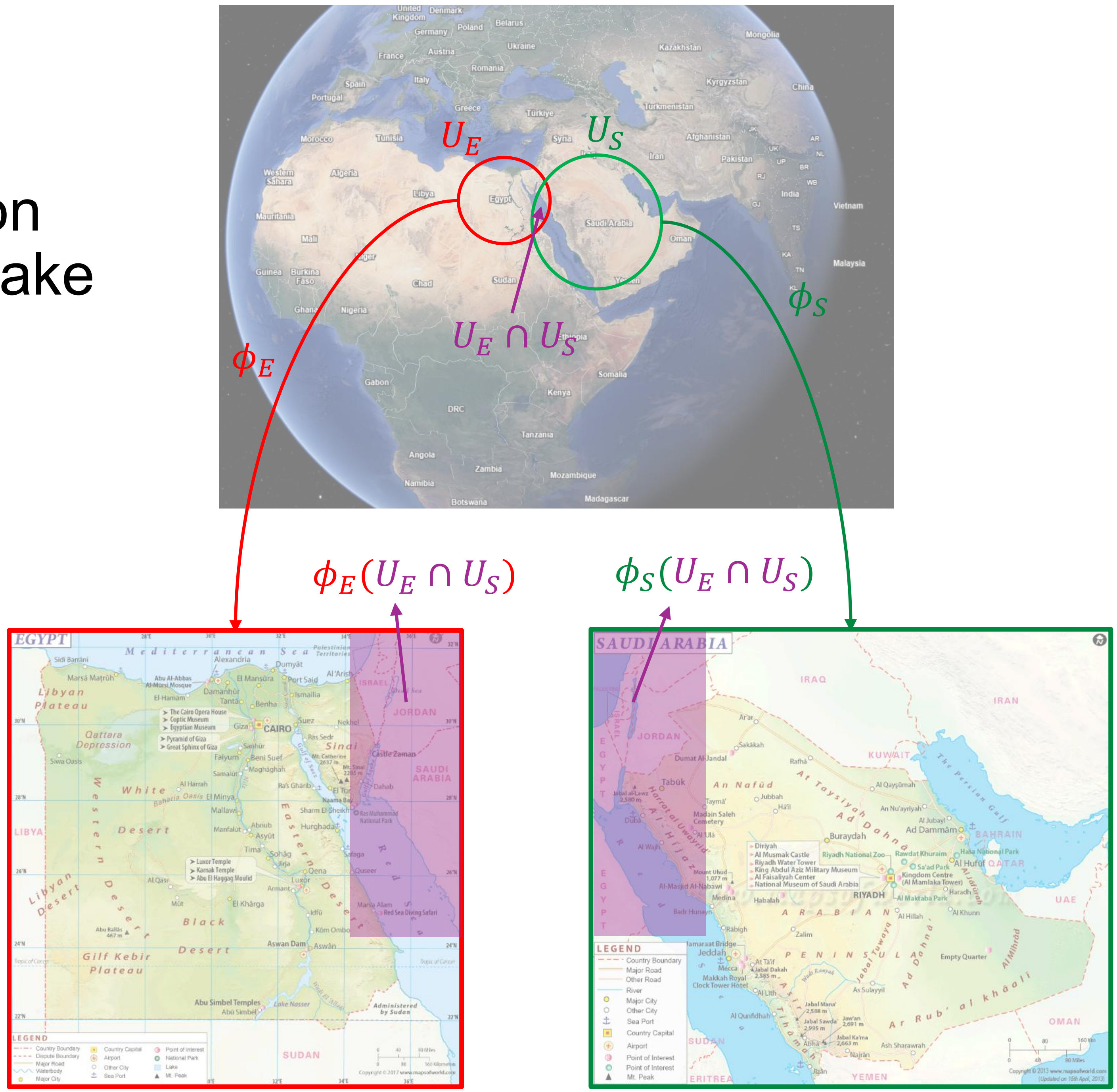
$$S^2 = \bigcup_{i \in A} U_i$$





Charts

- However, in an overlap region (e.g. $U_E \cap U_S$) we need to make sure that the charts are compatible.



Outline

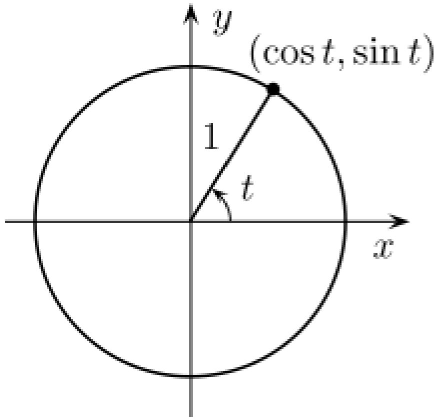
- Why differentiable structure ?
- Atlas of the world
- Manifold theory
- Maps on a manifold

Differentiable Manifold

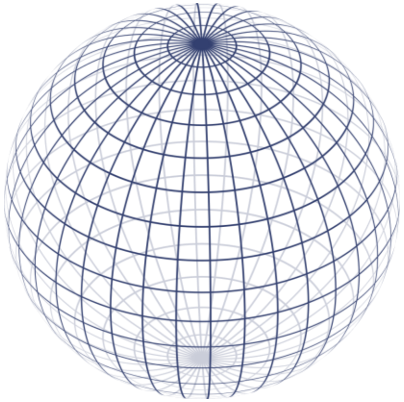
- The fundamental object in differential geometry is a differentiable manifold.
- Intuitively, an n-dimensional manifold is a set that locally “looks like” an open subset of Euclidean space $\mathbb{R}^n$.

Differentiable Manifold

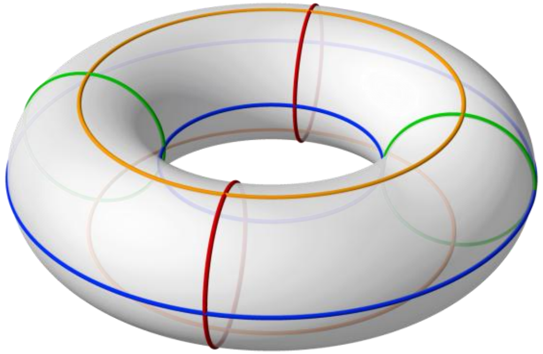
- The fundamental object in differential geometry is a differentiable manifold.
- Intuitively, an n-dimensional manifold is a set that locally "looks like" an open subset of Euclidean space $\mathbb{R}^n$.
- Examples:



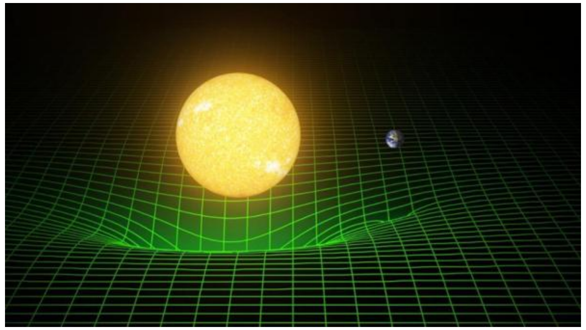
Circle S^1



Sphere S^2



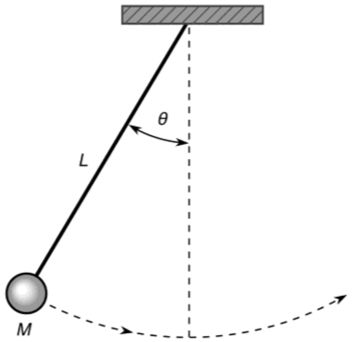
Torus T^2

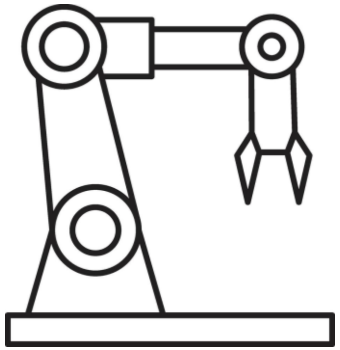


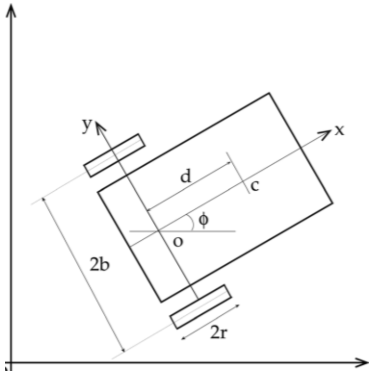
Spacetime

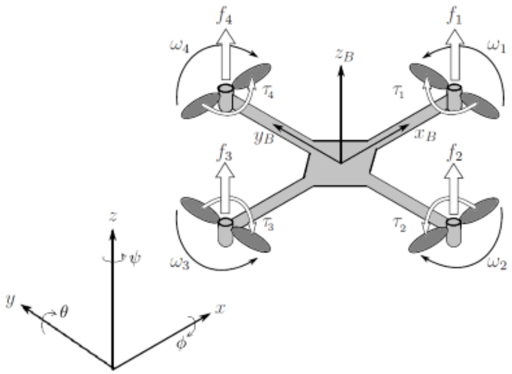
Manifolds in Robotics

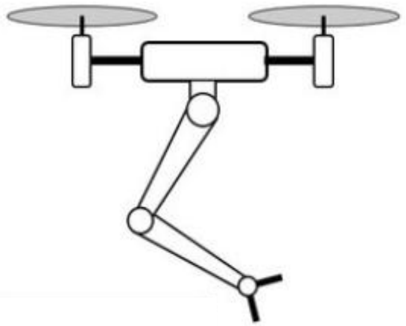
- Configuration space $\mathbb{Q}$ of (most) mechanical systems is a manifold
  - Pendulum $\mathbb{Q} = \mathbb{S}^1$ 1-dim. manifold
  - n-degree-of-freedom manipulator $\mathbb{Q} = \mathbb{T}^n$ n-dim. manifold
  - Planar mobile robot $\mathbb{Q} = SE(2)$ 3-dim. manifold
  - Multirotor aerial vehicle $\mathbb{Q} = SE(3)$ 6-dim. manifold
  - Aerial manipulator $\mathbb{Q} = SE(3) \times T^n$ $6 + n$-dim. manifold





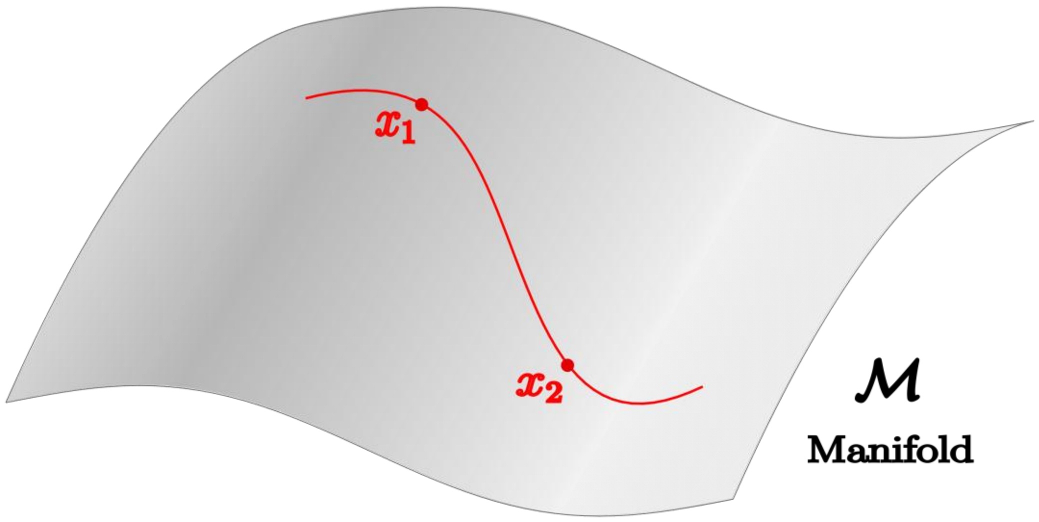






Differentiable Manifold

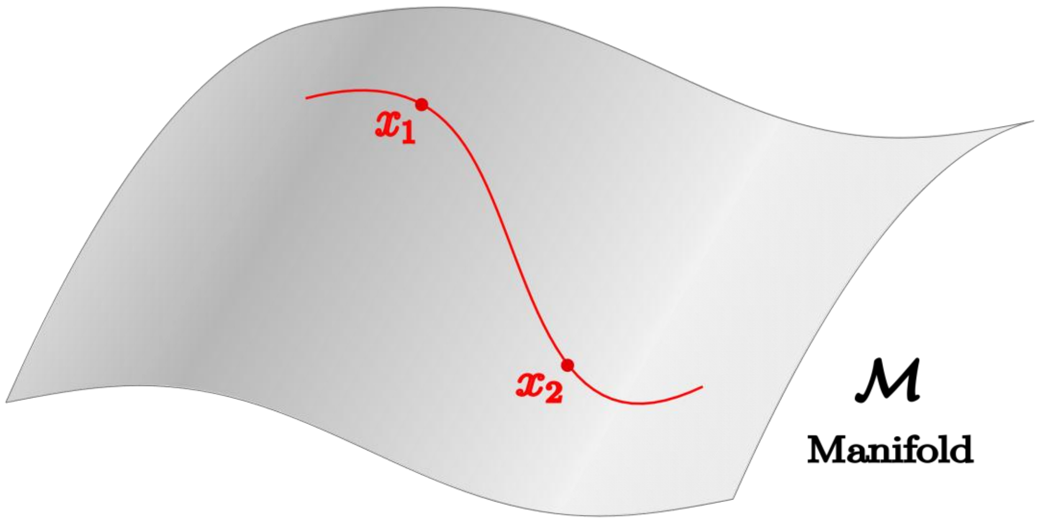
- In general, a manifold could not have a vector space structure
  - For $x_1, x_2 \in \mathcal{M}$, $x_1 + x_2$ or $x_1 - x_2$ is not defined !
  - For $x \in \mathcal{M}$, λx is not defined !
- In general, a manifold could not have a group structure
  - No binary operation $x_1 \odot x_2 \in \mathcal{M}$
  - No unique identity element $e \in \mathcal{M}$
  - No inverse element $x^{-1} \in \mathcal{M}$



Differentiable Manifold

- In general, a manifold could not have a vector space structure
  - For $x_1, x_2 \in \mathcal{M}$, $x_1 + x_2$ or $x_1 - x_2$ is not defined !
  - For $x \in \mathcal{M}$, λx is not defined !
- In general, a manifold could not have a group structure
  - No binary operation $x_1 \odot x_2 \in \mathcal{M}$
  - No unique identity element $e \in \mathcal{M}$
  - No inverse element $x^{-1} \in \mathcal{M}$

- A manifold that is also a group is called a Lie group.



Differentiable Manifold

- Intuitively, an n-dimensional manifold is a set that locally "looks like" an open subset of Euclidean space $\mathbb{R}^n$.
- Formally, a C^k differentiable manifold is the triple $(M, \mathcal{O}, \mathcal{A})$ where $(M, \mathcal{O})$ is a topological manifold* and $\mathcal{A}$ is a C^k-atlas for M.

*$(M, \mathcal{O})$ is a topological manifold is the formal way of saying it locally looks like $\mathbb{R}^n$.

Differentiable Manifold

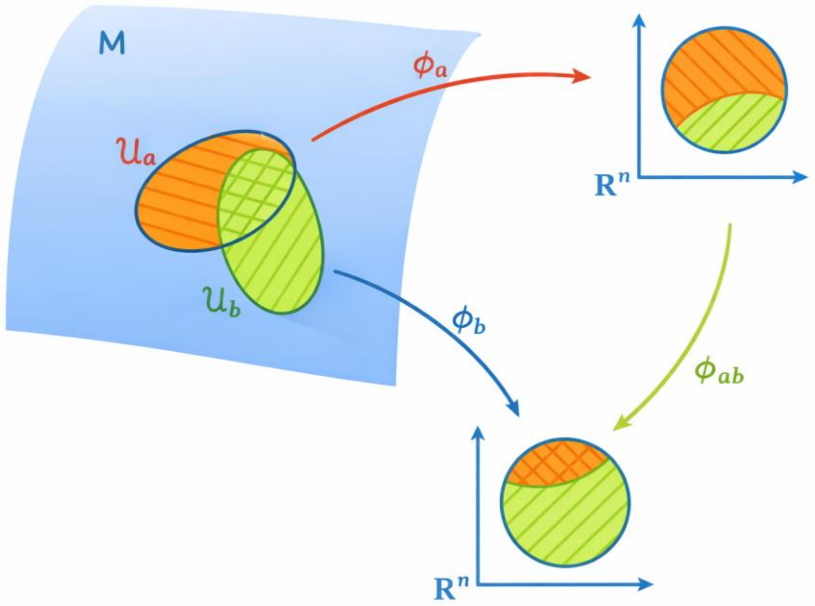
- Intuitively, an n-dimensional manifold is a set that locally "looks like" an open subset of Euclidean space $\mathbb{R}^n$.
- Formally, a C^k differentiable manifold is the triple $(M, \mathscr{O}, \mathcal{A})$ where $(M, \mathscr{O})$ is a topological manifold* and $\mathcal{A}$ is a C^k-atlas for M.

- An C^k-atlas for M is a collection of charts that cover the entire manifold while satisfying certain overlap conditions.
- Given this C^k-differentiable structure on M, we can then talk about curves on manifolds, maps between manifolds, differentiability of maps,… etc.

*$(M, \mathscr{O})$ is a topological manifold is the formal way of saying it locally looks like $\mathbb{R}^n$.

Charts and Atlas of a manifold

- Let $(M, \mathcal{O})$ be a set equipped with a topology.
- A chart for M is the pair (U, ϕ) with $U \subset M$ an open subset* of M and $\phi: U \to \mathbb{R}^n$ with $\phi(U) \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$.



*open subset $U \subset M$ is equivalent to $U \in \mathcal{O}$

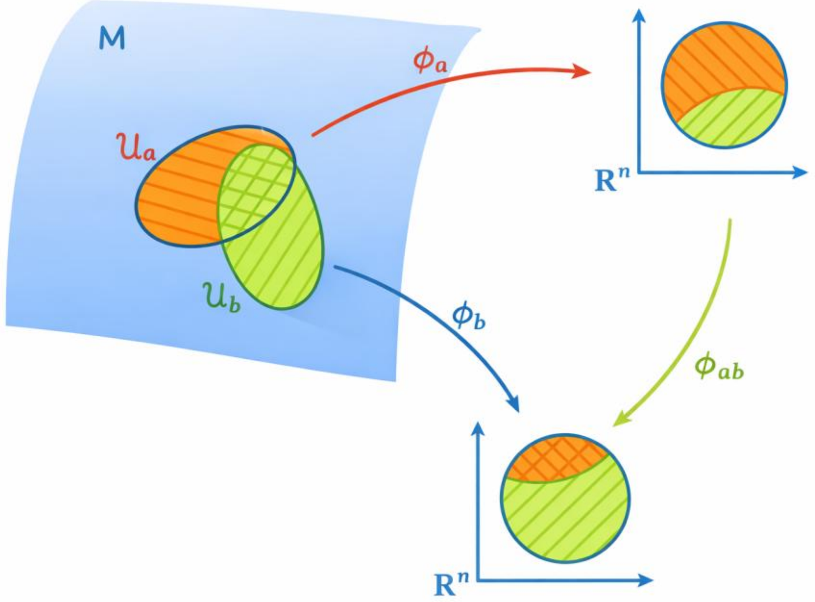
Charts and Atlas of a manifold

- Let $(M, \mathcal{O})$ be a set equipped with a topology.
- A chart for M is the pair (U, ϕ) with $U \subset M$ an open subset* of M and $\phi: U \to \mathbb{R}^n$ with $\phi(U) \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$.
- A C^k-atlas for M is the collection $\mathcal{A} \coloneqq \{(U_i, \phi_i)\}_{i \in A}$

with the properties that $M = \bigcup_{i \in A} U_i$ and whenever $U_a \cap U_b \neq \emptyset$ we have that the overlap/transition map

$$\phi_{ab} \coloneqq \phi_b \circ \phi_a^{-1}: \mathbb{R}^n \to \mathbb{R}^n$$

is of class C^k.



*open subset $U \subset M$ is equivalent to $U \in \mathcal{O}$

Outline

- Why differentiable structure ?
- Atlas of the world
- Manifold theory
- Maps on a manifold