SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 7: Manifolds and Lie groups II

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Outline

- Recap last lecture
- Tangent space of a manifold
- Cotangent space of a manifold
- Matrix Lie groups

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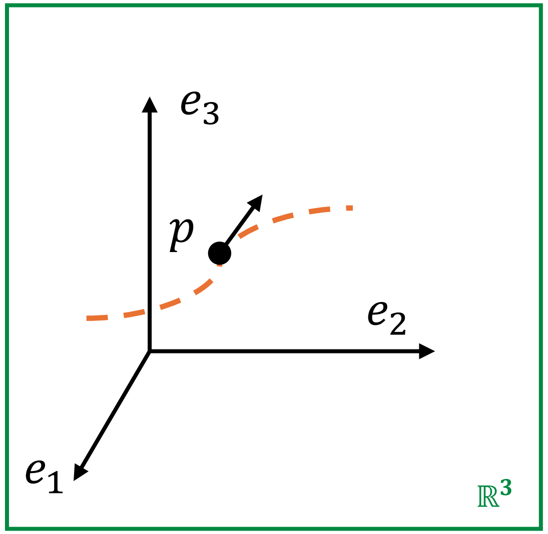
Recap: Different views of $\mathbb{R}^n$

- As a set $\quad \mathbb{R}^n \coloneqq \mathbb{R} \times \cdots \times \mathbb{R}$
- As a group $\quad (\mathbb{R}^n, \oplus)$
- As a vector space $\quad (\mathbb{R}^n, \oplus, \odot)$ over $(\mathbb{R}, +, \cdot)$
- As a manifold $\quad (\mathbb{R}^n, \mathcal{O}_{std}, \mathcal{A})$

- A set M along with this differentiable structure is called a differentiable manifold.

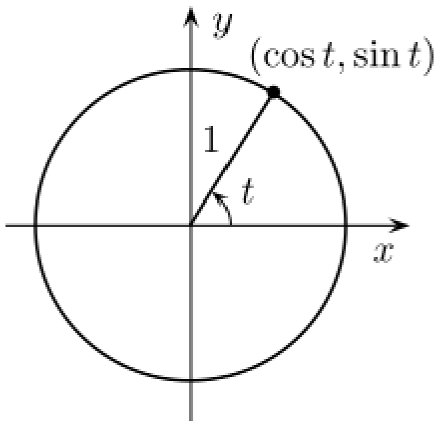
Equation of motion

$$\dot{p} = f(p), \quad p \in \mathbb{R}^n$$

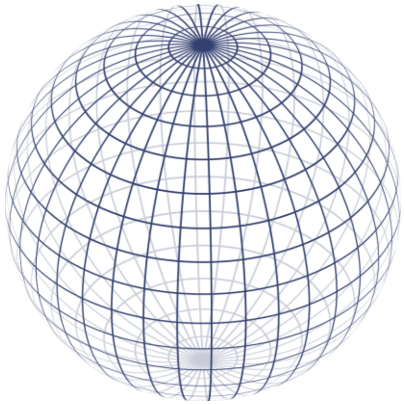


Recap: Differentiable Manifold

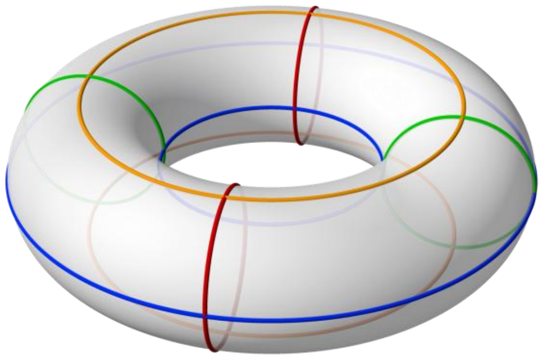
- The fundamental object in differential geometry is a differentiable manifold.
- Intuitively, an n-dimensional manifold is a set that locally "looks like" an open subset of Euclidean space $\mathbb{R}^n$.
- Examples:



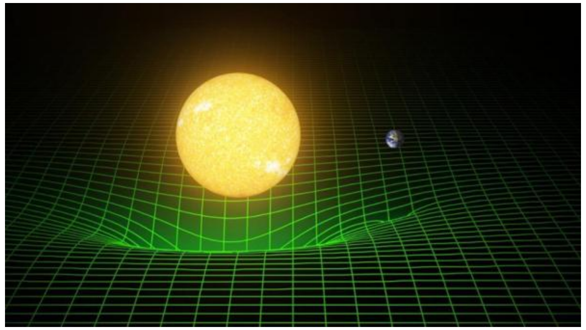
Circle S^1



Sphere S^2



Torus T^2

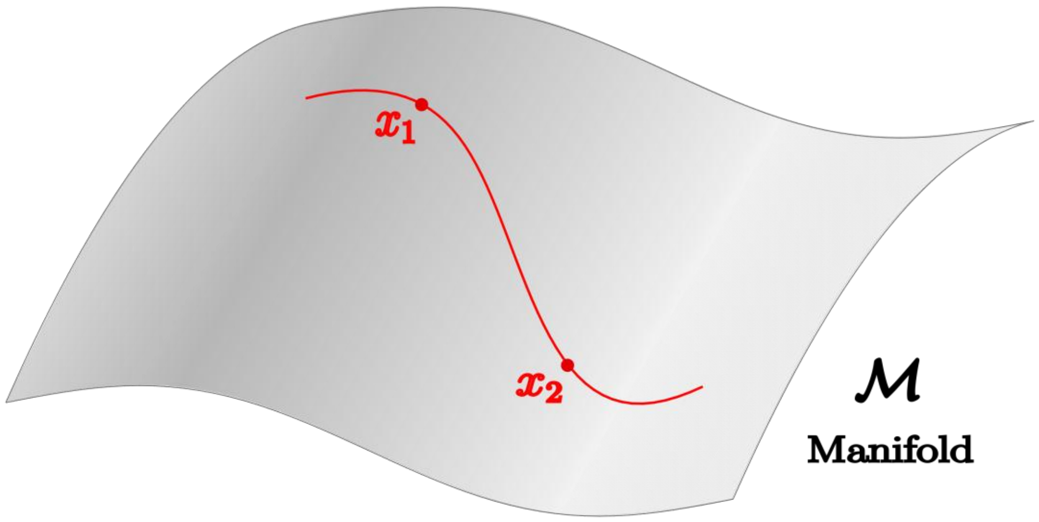


Spacetime

Recap: Differentiable Manifold

- In general, a manifold could not have a vector space structure
  - For $x_1, x_2 \in \mathcal{M}$, $x_1 + x_2$ or $x_1 - x_2$ is not defined !
  - For $x \in \mathcal{M}$, λx is not defined !
- In general, a manifold could not have a group structure
  - No binary operation $x_1 \odot x_2 \in \mathcal{M}$
  - No unique identity element $e \in \mathcal{M}$
  - No inverse element $x^{-1} \in \mathcal{M}$

- A manifold that is also a group is called a Lie group.



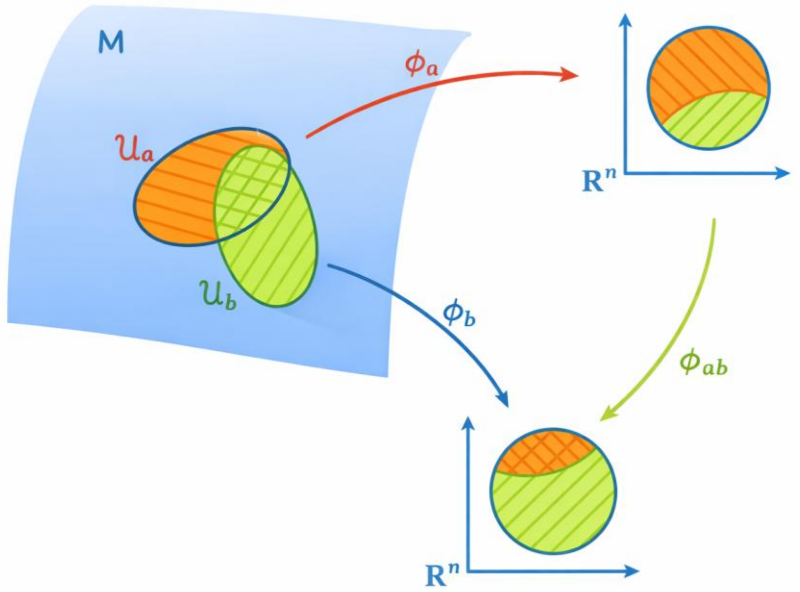
Recap: Charts and Atlas of a manifold

- Let $(M, \mathcal{O})$ be a set equipped with a topology.
- A chart for M is the pair (U, ϕ) with $U \subset M$ an open subset of M and $\phi: U \to \mathbb{R}^n$ with $\phi(U) \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$.
- A C^k-atlas for M is the collection $\mathcal{A} := \{(U_i, \phi_i)\}_{i \in A}$

with the properties that $M = \bigcup_{i \in A} U_i$ and whenever $U_a \cap U_b \neq \emptyset$ we have that the overlap/transition map

$$\phi_{ab} := \phi_b \circ \phi_a^{-1}: \mathbb{R}^n \to \mathbb{R}^n$$

is of class C^k.

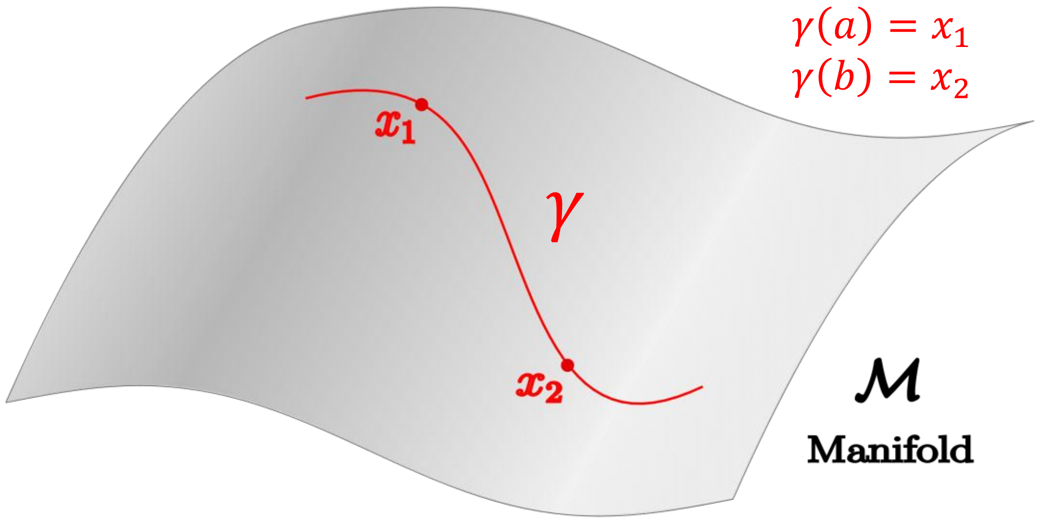


Recap: Curve on a manifold

- A **curve on a manifold** M is simply a map from an interval of real numbers into M.
- A **curve** is a map

$$\gamma: I \to M,$$

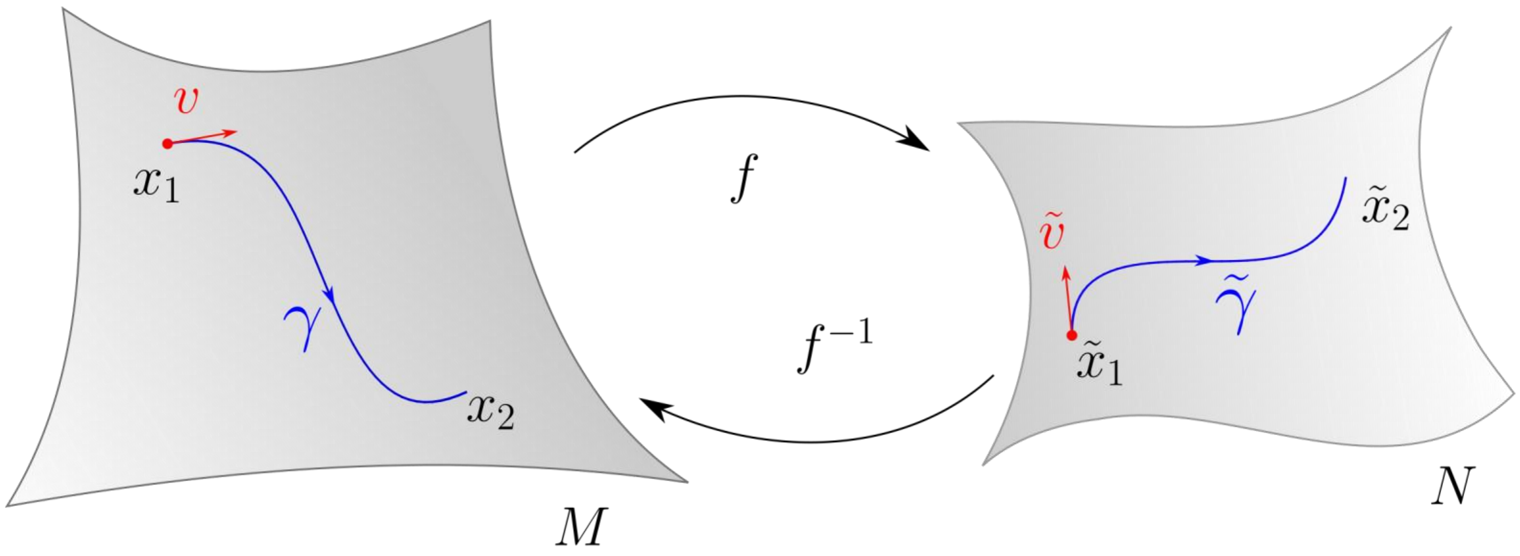
where $I \subset \mathbb{R}$ is an interval (e.g. $[0,1]$ or $[a,b]$).



Recap: Maps between Manifold

- A **map between manifolds** is just a function that takes points of one manifold to points of another

$$f: M \to N$$
$$x \mapsto \tilde{x}$$



Recap: Maps from Manifold to $\mathbb{R}$

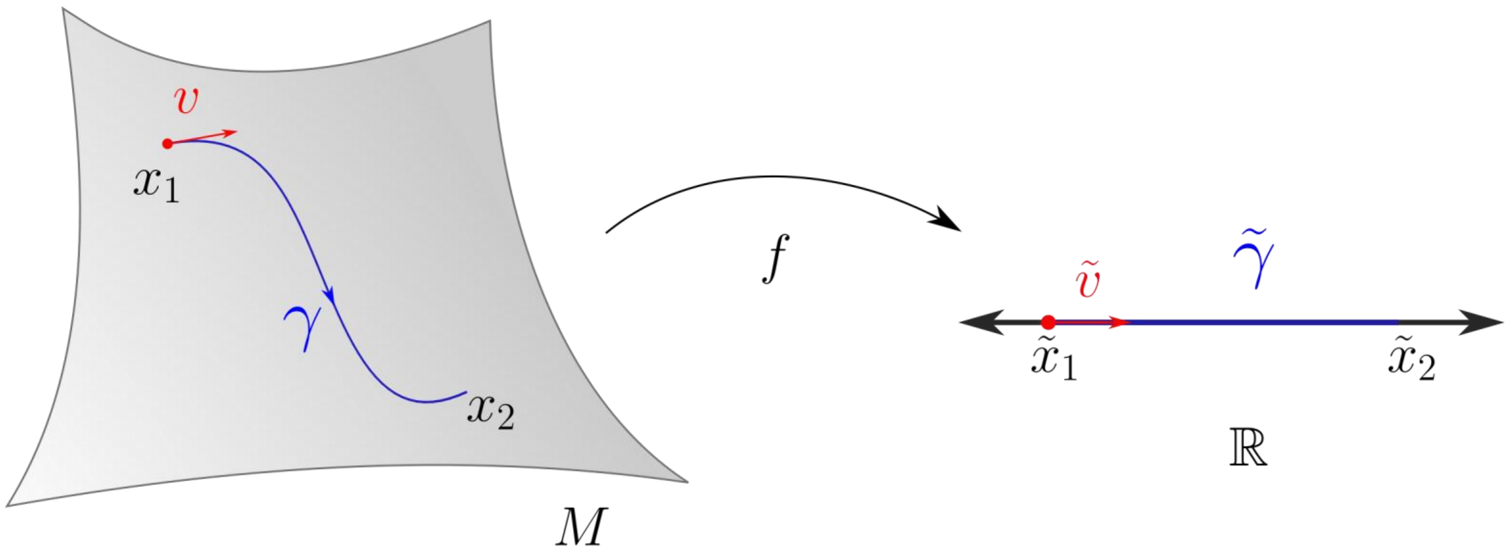
- A **map on the manifold** is a special case where $N = \mathbb{R}$

$$f: M \to \mathbb{R}$$
$$x \mapsto \tilde{x}$$

We say that:

$$f \in C^k(M)$$

if its local representative in the chart is of class C^k



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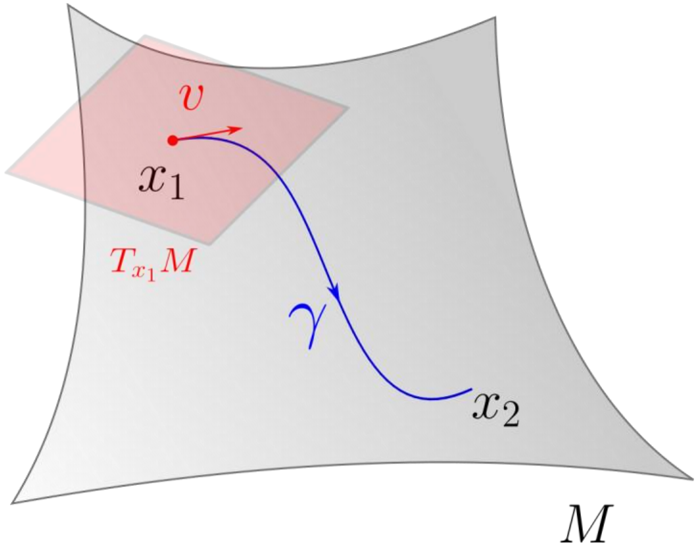
Tangent space of a manifold

- Consider a (smooth) curve $\gamma: I \to M, \quad t \mapsto \gamma(t) = x$
- If we differentiate with respect to the curve parameter t, we get the notion of tangent vector to the curve γ.
- If $\gamma(0) = x_1$, then $v := \gamma'(0)$ is the tangent vector at point $x_1 \in M$ to the curve γ.



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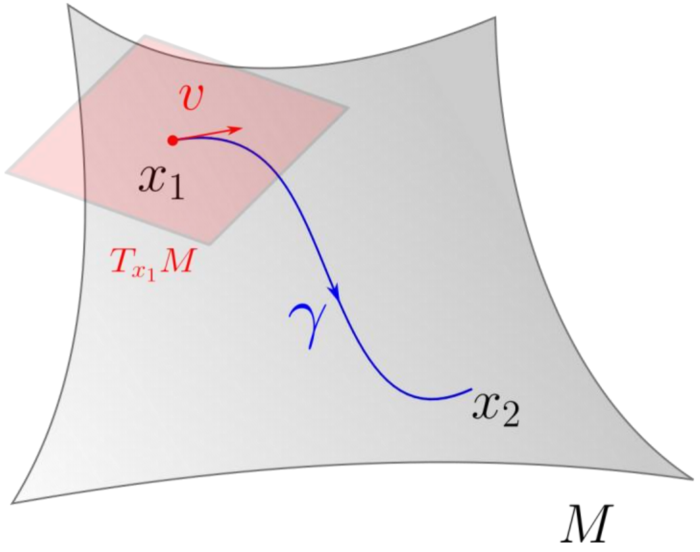

What does $\gamma'(t)$ mean ?

- Choose a coordinate chart (U, ϕ), $\phi: U \subset M \to \mathbb{R}^n$
- Look at the curve in coordinates $\phi(\gamma(t)) \in \mathbb{R}^n$
- Differentiate the curve $\phi \circ \gamma: \mathbb{R} \to \mathbb{R}^n$

Tangent space of a manifold

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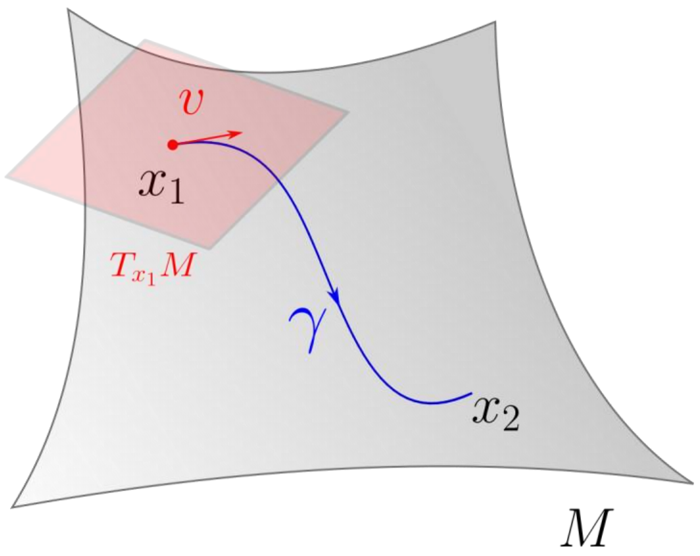
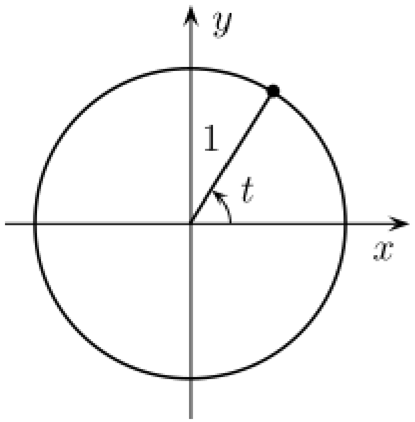
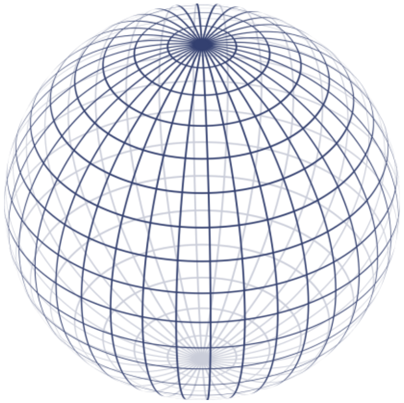
- The tangent space $T_x M$ at a point $x \in M$ is the space of all possible instantaneous directions you can move from x.



Examples



Circle $\mathbb{S}^1$

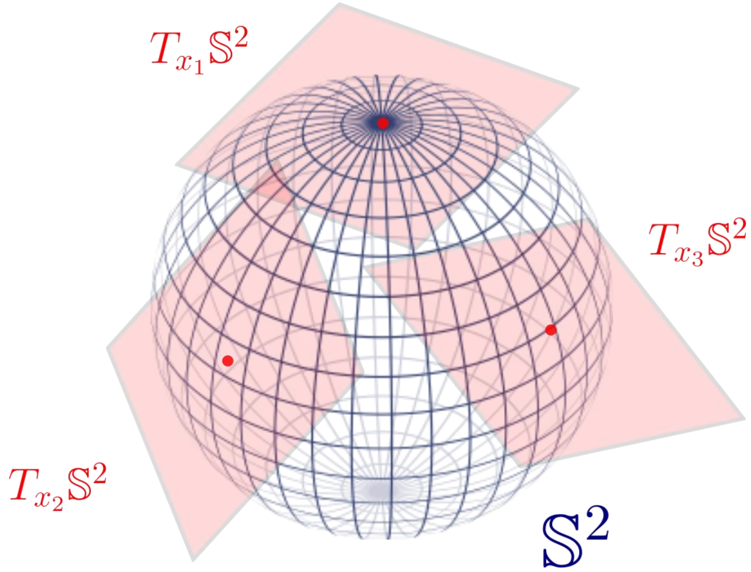


Sphere $\mathbb{S}^2$

Tangent bundle

- The tangent bundle is what you get when you take all tangent spaces of a manifold and “stack them together” into one geometric object.
- The tangent bundle is the disjoint union of all tangent spaces:

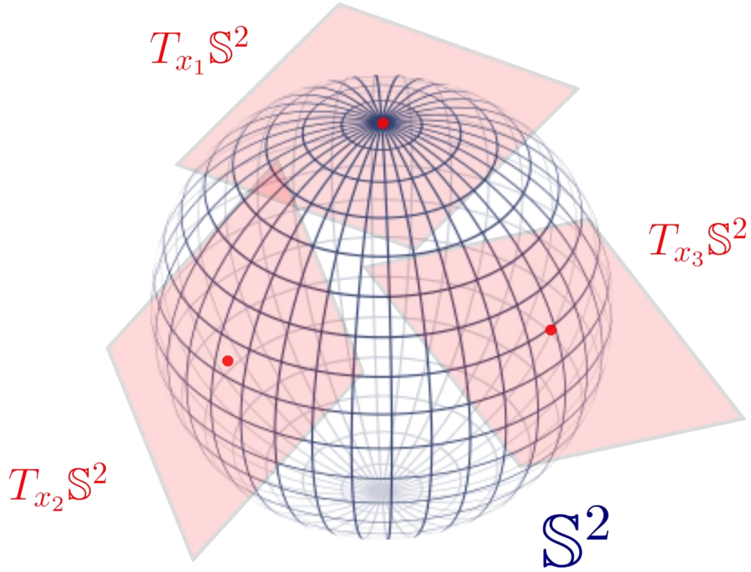
$$TM := \mathop{\dot{\bigcup}}_{x \in M} T_x M$$



Tangent bundle

$$TM := \mathring{\bigcup_{x \in M}} T_x M$$

- A point of TM is a pair (x, v) where $x \in M$ and $v \in T_x M$.
- TM itself is a smooth manifold
- If $\dim M = n$, then $\dim TM = 2n$
- Locally, it TM "looks like" $\mathbb{R}^n \times \mathbb{R}^n$



TM is fundamental in Lagrangian mechanics !!

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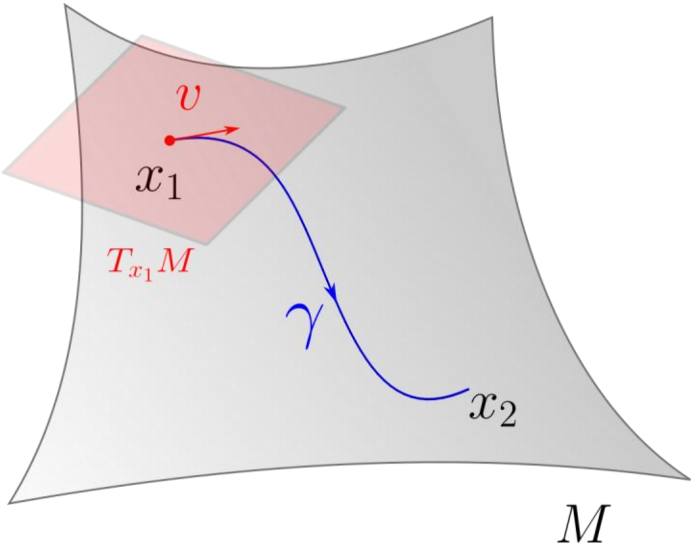
Cotangent space of a manifold

- The tangent space $T_x M$ at a point $x \in M$ is the space of all possible instantaneous directions (velocities) you can move from x.
- $T_x M$ is an n-dimensional vector space.
- The cotangent space $T_x^* M$ is the space of all linear maps on $T_x M$.
- An element $\alpha \in T_x^* M$ is the linear map

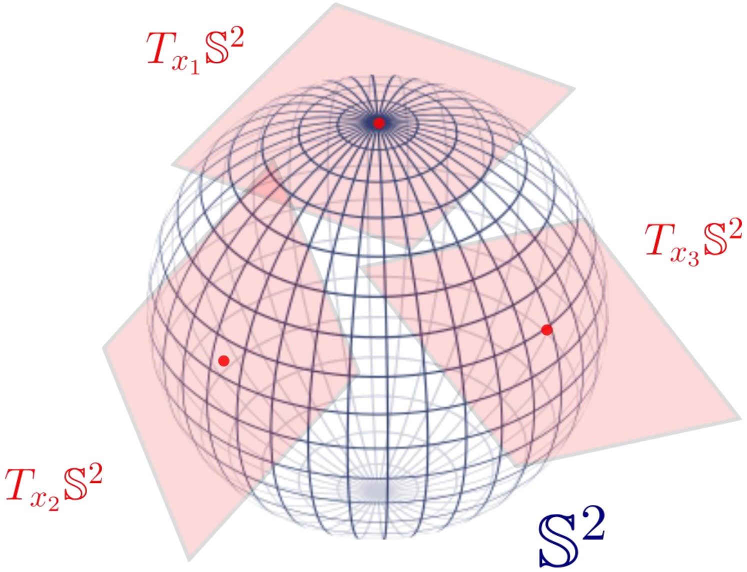
$$\alpha: T_x M \to \mathbb{R}$$
$$v \mapsto \alpha(v)$$



Cotangent bundle

$$T^*M := \mathring{\bigcup_{x \in M}} T_x^*M$$

- A point of T^*M is a pair (x, α) where $x \in M$ and $\alpha \in T_x^*M$.
- T^*M itself is a smooth manifold
- If $\dim M = n$, then $\dim T^*M = 2n$
- Locally, it T^*M "looks like" $\mathbb{R}^n \times (\mathbb{R}^n)^*$



T^*M is fundamental in Hamiltonian mechanics !!

Differential of a function on M

- A very important example of a co-vector on T_x^*M is the so called differential of a function.
- Let $f: M \to \mathbb{R}$ be a smooth function, then at each point $x \in M$ it has the differential

$$df_x \in T_x^*M$$

defined such that for any $v \in T_xM$

$$df_x(v) = \text{is the directional derivative of } f \text{ at } x \text{ along } v.$$

How you know it in $\mathbb{R}^n$:

$$df_x(v) = \left(\nabla f(x)\right)^\top v$$

Gradient of a function

Teaser !

Abstract / physical concept	Manifold / geometric object	Notation	Intuition
Configuration space of a robot	Smooth manifold	Q	All possible configurations
A specific configuration	Point on the manifold	$q \in Q$	One robot pose/joint state
Local coordinates	Chart / coordinate map	$(U, \varphi), q \mapsto (q^1, \dots, q^n)$	Numbers describing $q$locally (generalized coordinates)
Degrees of freedom	Manifold dimension	$\dim Q = n$	# independent coordinates
Motion/trajectory	Smooth curve on Q	$q(t)$	Time-parameterized configuration
Instantaneous velocity	Tangent vector at q	$\dot{q} \in T_q Q$	"Direction of motion" at that point
Generalized force	Covector at q	$\tau \in T_q^* Q$	maps a velocity/variation to power/work rate
Power	Pairing of covector and vector	$\tau(\dot{q})$	Generalized force × velocity

Teaser !

Abstract / physical concept	Manifold / geometric object	Notation	Intuition
Configuration space of a robot	Smooth manifold	Q	All possible configurations
Velocity space (Lagrangian approach)	Tangent bundle	TQ	All $(q, \dot{q})$ pairs
Phase space (Hamiltonian approach)	Cotangent bundle	T^*Q	All (q, p) pairs
Lyapunov/Potential function	Smooth map	$V: Q \rightarrow \mathbb{R}$	Functions of configuration q
Gradient of a potential	Differential of a function on M	$dV_q \in T_q^*Q$	Direction of steepest increase
Rate of change of Lyapunov function	Pairing of covector and vector	$\dot{V} \coloneqq dV_q(\dot{q})$	Rate of change of V along trajectories

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