SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 10: Rigid Body Kinematics II

جامعة الملك فهد للبترول والمعادن
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Outline

- Recap Last Lectures
- Lie group structure of $SO(3)$ and $SE(3)$
- Properties of Angular velocities & Twists

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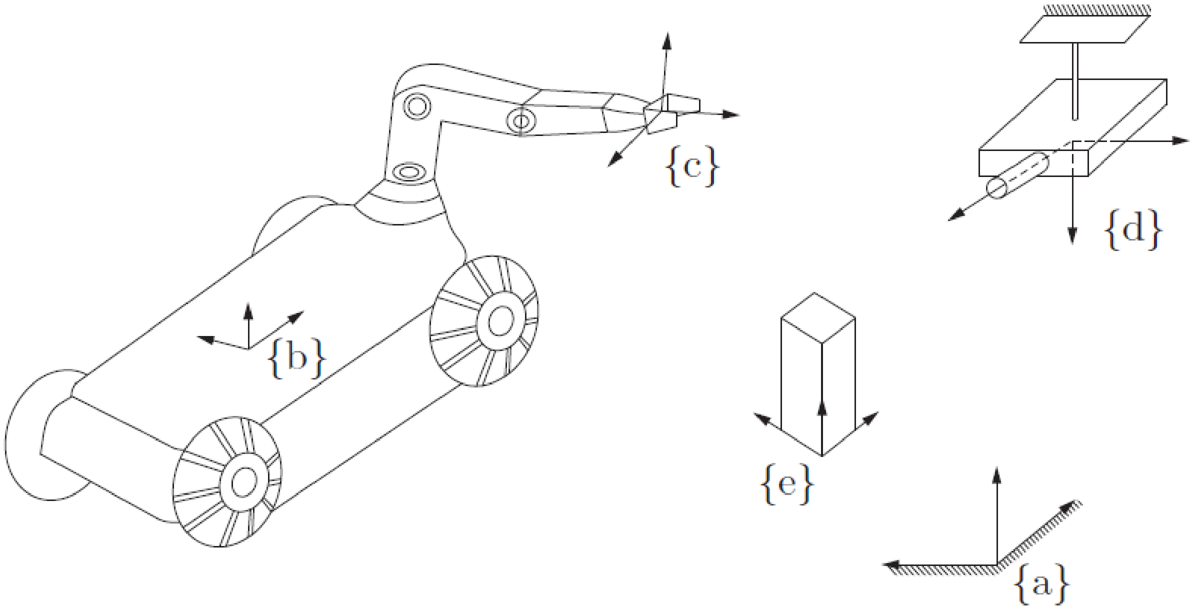
Recap: Kinematic Modeling Notation

- A frame will be denoted by Ψ_i or $\{i\}$.
- The relative pose of $\{i\}$ with respect to $\{k\}$ is described by

$$H_i^k = \begin{pmatrix} R_i^k & \xi_i^k \\ 0 & 1 \end{pmatrix} \in SE(3), \qquad R_i^k \in SO(3), \qquad \xi_i^k \in \mathbb{R}^3$$



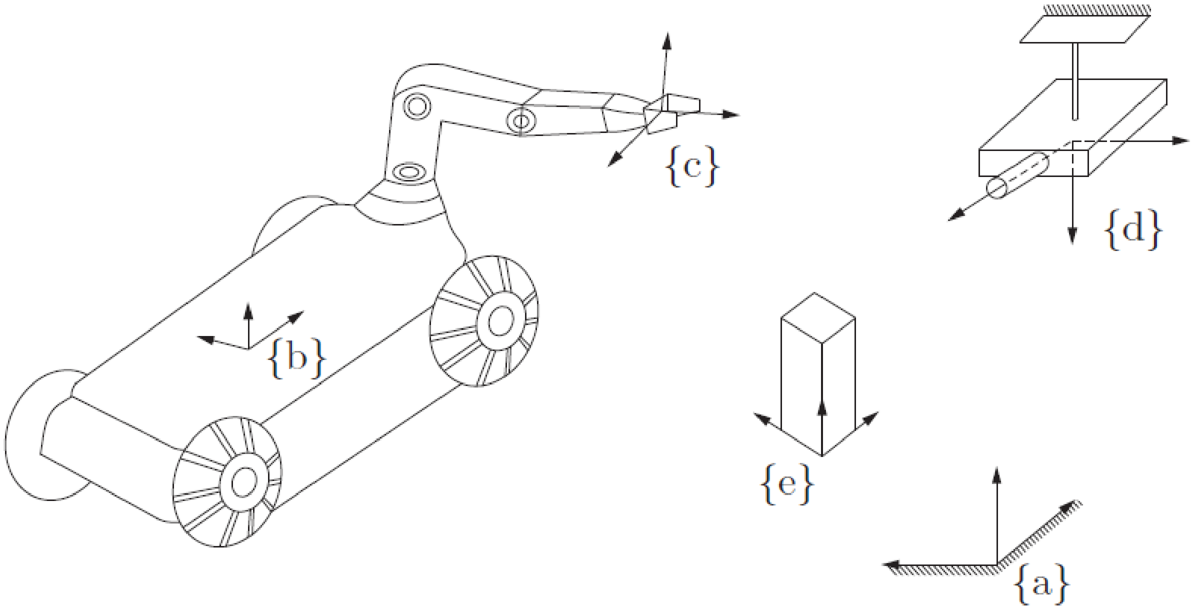
Recap: Kinematic Modeling Notation

- Given the relative pose of $\{i\}$ with respect to $\{j\}$ and the relative pose of $\{j\}$ with respect to $\{k\}$, we have that

$$H_i^k = H_j^k H_i^j \in SE(3)$$

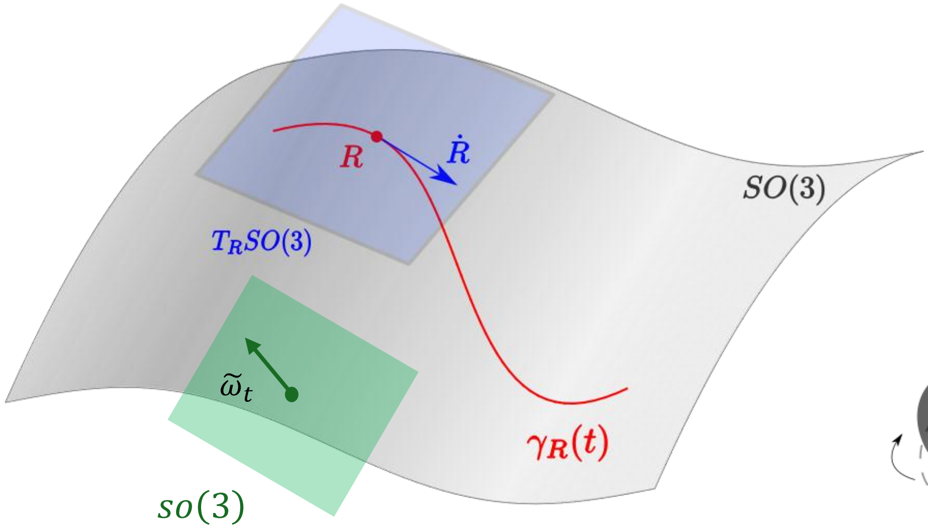
$$H_j^k := \left(H_k^j\right)^{-1} = \begin{pmatrix} R_j^k & -R_j^k \xi_k^j \\ 0 & 1 \end{pmatrix} \in SE(3)$$

$$R_j^k := \left(R_k^j\right)^\top \in SO(3)$$

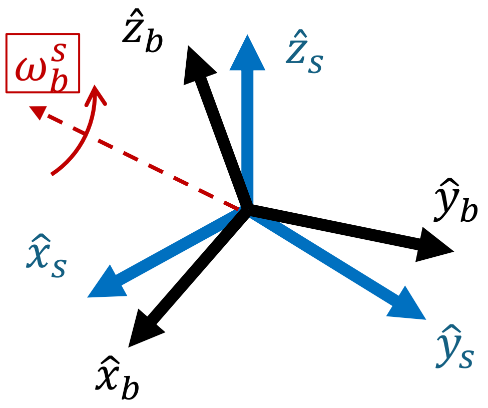


Recap: Lie Group Structure of SO(3)

Physical Space	Abstract Manifold	Notation
Relative orientation of 2 frames	Point on a manifold	$R_t \in SO(3)$
A smooth sequence of relative orientations	Smooth curve on a manifold	$\gamma_R(t)$
Rate of change of orientation	Element of the tangent space at $R_t \in SO(3)$ (Tangent vector to a curve)	$\dot{R}_t \in T_{R_t}SO(3)$
Angular Velocity	Element of the tangent space at $I \in SO(3)$ (Vector in the Lie algebra)	$\tilde{\omega}_t \in so(3)$



Abstract Manifold



Physical Space

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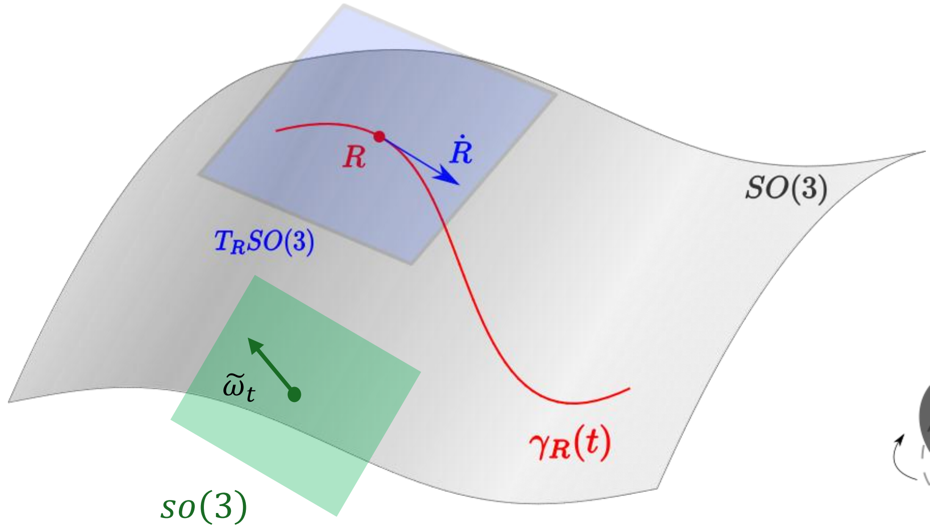
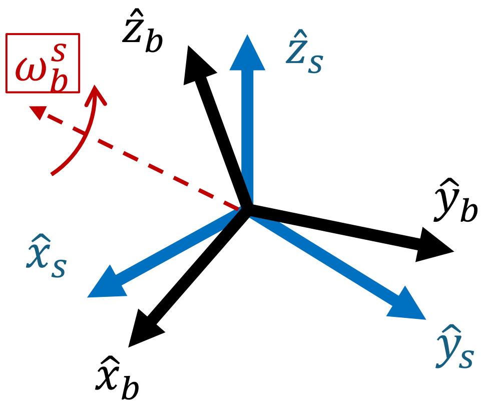
Angular Velocity	Element of the tangent space at $I \in SO(3)$ (Vector in the Lie algebra)	$\tilde{\omega}_t \in so(3)$

$$T_R SO(3) \coloneqq \{\dot{R} \in \mathbb{R}^{3\times 3} \mid \dot{R}^\top R \text{ is skew symmetric}\}$$

$$so(3) \coloneqq T_I SO(3) \coloneqq \{\tilde{\omega} \in \mathbb{R}^{3\times 3} \mid \tilde{\omega} \text{ is skew symmetric}\}$$



Abstract Manifold



Physical Space

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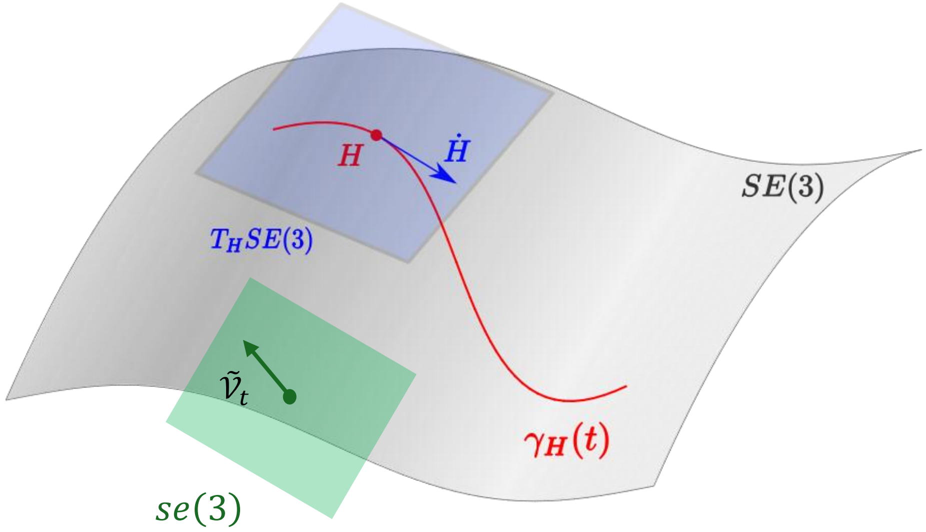
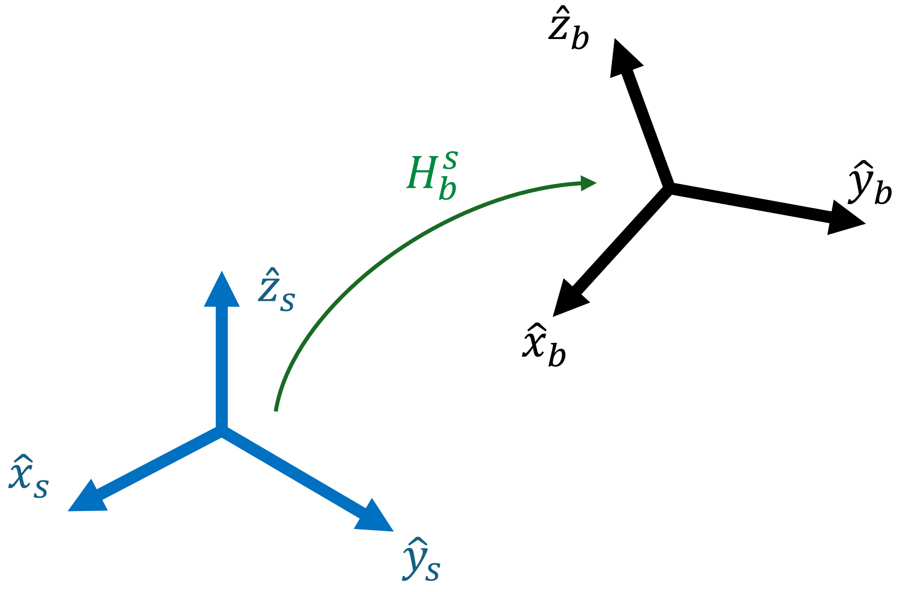
Lie Group Structure of SE(3)

Physical Space	Abstract Manifold	Notation
Relative pose of 2 frames	Point on a manifold	$H_t \in SE(3)$
A smooth sequence of relative poses	Smooth curve on a manifold	$\gamma_H(t)$
Rate of change of pose	Element of the tangent space at $H_t \in SE(3)$ (Tangent vector to a curve)	$\dot{H}_t \in T_{H_t}SE(3)$
Generalized Velocity (**Twist**)	Element of the tangent space at $I \in SE(3)$ (Vector in the Lie algebra)	$\tilde{\mathcal{V}}_t \in se(3)$



Abstract Manifold



Physical Space

Lie group

- How to relate a tangent vector $\dot{h} \in T_h G$ to an element of the Lie algebra $g := T_e G$ for a generic Lie group G?
  - Using pushforward of the Left translation map
  - Using pushforward of the Right translation map

Left and Right translation maps

- Consider the Lie group $(G, \blacksquare)$
- For a given element $h \in G$, the left translation is the map

$$\begin{aligned} \mathcal{L}_h: G &\to G \\ q &\mapsto h \blacksquare q \end{aligned}$$

- For a given element $h \in G$, the right translation is the map

$$\begin{aligned} \mathcal{R}_h: G &\to G \\ q &\mapsto q \blacksquare h \end{aligned}$$

Left (Right) translations encode how the group acts on itself from the left (right).

Left and Right translation maps

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$$\begin{aligned} \mathcal{R}_h : G &\to G \\ q &\mapsto q \blacksquare h \end{aligned}$$

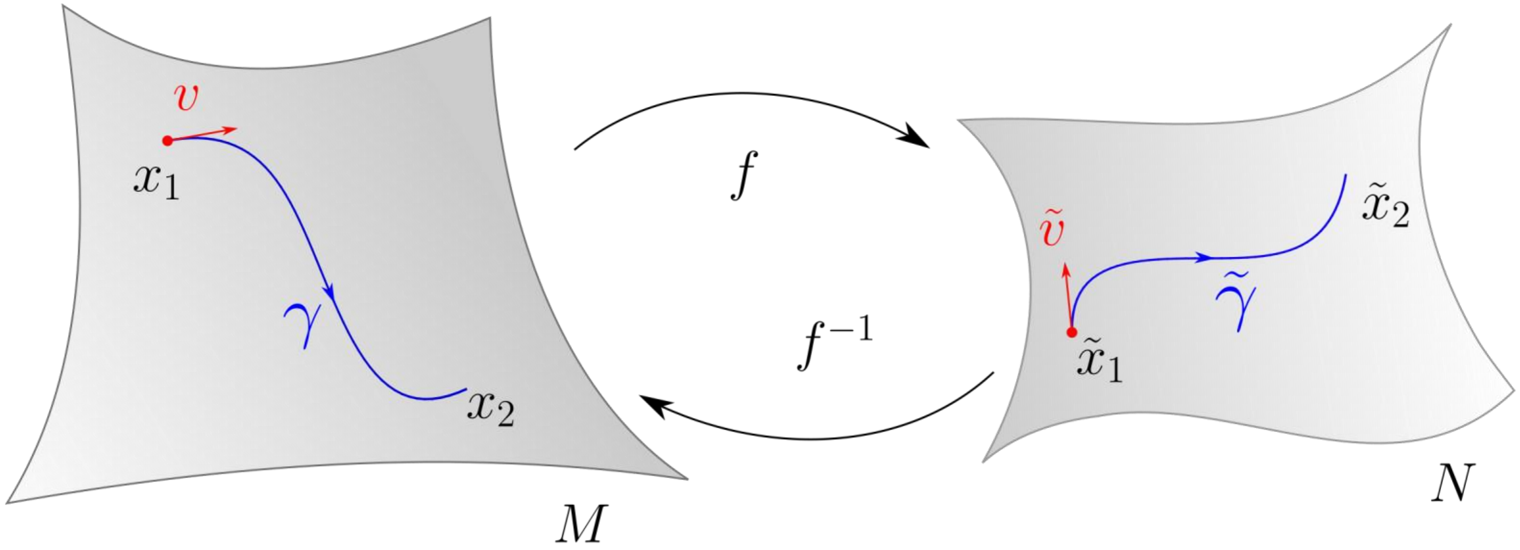
For a matrix Lie group G (e.g., $SO(n), SE(n)$) -> $\blacksquare$ is matrix multiplication.

Pushforward of a map between manifolds

- Consider the map $f : M \to N, \;\; x \mapsto \tilde{x}$
- The pushforward of the map f is the map

$$f_{*,x}\colon T_x M \;\to T_{f(x)} N$$

$$v \mapsto \tilde{v}$$



Pushforward of Left and Right translation maps

- Consider the Lie group $(G, \blacksquare)$
- The pushforward of the left translation is the map

$$(\mathcal{L}_h)_{*,q} : T_q G \ \rightarrow T_{h \blacksquare q} G$$
$$\dot{q} \mapsto v_{\mathcal{L}}$$

- The pushforward of the right translation is the map

$$(R_h)_{*,q} : T_q G \ \rightarrow T_{q \blacksquare h} G$$
$$\dot{q} \mapsto v_{\mathcal{R}}$$

These maps let you compare/relate vectors living at different points of the Lie group.

Pushforward of Left and Right translation maps

- What happens when $h = q^{-1}$?
- The pushforward of the left translation becomes

$$\left(\mathcal{L}_{q^{-1}}\right)_{*,q} : T_q G \rightarrow T_e G =: g$$

- The pushforward of the right translation becomes

$$\left(\mathcal{R}_{q^{-1}}\right)_{*,q} : T_q G \rightarrow T_e G =: g$$

Any map "to the Lie algebra" will take a tangent vector somewhere on the group and bringing it back to the identity.

Special Case: SO(3)

- Consider the Lie group $(SO(3),\cdot)$
- For a given element $R \in SO(3)$, we have that
  - $\mathcal{L}_{R^\top}: SO(3) \to SO(3)$ is matrix multiplication from the left
  - $(\mathcal{L}_{R^\top})_{*,R}: T_R SO(3) \to so(3)$ is matrix multiplication from the left

  - $\mathcal{R}_{R^\top}: SO(3) \to SO(3)$ is matrix multiplication from the right
  - $(\mathcal{R}_{R^\top})_{*,R}: T_R SO(3) \to so(3)$ is matrix multiplication from the right

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  - $\mathcal{R}_{R^\top}: SO(3) \to SO(3)$ is matrix multiplication from the right
  - $(\mathcal{R}_{R^\top})_{*,R}: T_R SO(3) \to so(3)$ is matrix multiplication from the right

$$\mathcal{L}_{R^\top}(\bar{R}) = R^\top \cdot \bar{R} \in SO(3)$$

$$\mathcal{R}_{R^\top}(\bar{R}) = \bar{R} \cdot R^\top \in SO(3)$$

$$(\mathcal{L}_{R^\top})_{*,R}(\dot{R}) = R^\top \cdot \dot{R} \in so(3)$$

$$(\mathcal{R}_{R^\top})_{*,R}(\dot{R}) = \dot{R} \cdot R^\top \in so(3)$$

2 different representations of angular velocity

Special Case: SE(3)

- Consider the Lie group $(SE(3),\cdot)$
- For a given element $H \in SE(3)$, we have that
  - $\mathcal{L}_{H^{-1}}: SE(3) \rightarrow SE(3)$ is matrix multiplication from the left
  - $(\mathcal{L}_{H^{-1}})_{*,H}: T_H SE(3) \rightarrow se(3)$ is matrix multiplication from the left

  - $\mathcal{R}_{H^{-1}}: SE(3) \rightarrow SE(3)$ is matrix multiplication from the right
  - $(\mathcal{R}_{H^{-1}})_{*,H}: T_H SE(3) \rightarrow se(3)$ is matrix multiplication from the right

Special Case: SE(3)

- Consider the Lie group $(SE(3),\cdot)$
- For a given element $H \in SE(3)$, we have that
  - $\mathcal{L}_{H^{-1}}: SE(3) \rightarrow SE(3)$ is matrix multiplication from the left
  - $(\mathcal{L}_{H^{-1}})_{*,H}: T_H SE(3) \rightarrow se(3)$ is matrix multiplication from the left

  - $\mathcal{R}_{H^{-1}}: SE(3) \rightarrow SE(3)$ is matrix multiplication from the right
  - $(\mathcal{R}_{H^{-1}})_{*,H}: T_H SE(3) \rightarrow se(3)$ is matrix multiplication from the right

$$\mathcal{L}_{H^{-1}}(\bar{H}) = H^{-1} \cdot \bar{H} \in SE(3)$$

$$\mathcal{R}_{H^{-1}}(\bar{H}) = \bar{H} \cdot H^{-1} \in SE(3)$$

$$(\mathcal{L}_{H^{-1}})_{*,H}(\dot{H}) = H^{-1} \cdot \dot{H} \in se(3)$$

$$(\mathcal{R}_{H^{-1}})_{*,H}(\dot{H}) = \dot{H} \cdot H^{-1} \in se(3)$$

2 different representations of Twist

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Angular Velocity

- Configuration:

$$R_b^s \in SO(3)$$

$$R_s^b := (R_b^s)^\top$$

- Rate-of-change of configuration:

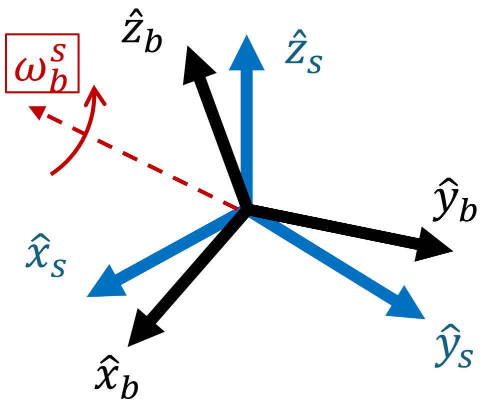
$$\dot{R}_b^s \in T_{R_b^s} SO(3)$$

- Body angular velocity expressed in $\{b\}$:

$$\tilde{\omega}_b^{b,s} := R_s^b \, \dot{R}_b^s \in so(3)$$

- Spatial angular velocity expressed in $\{s\}$:

$$\tilde{\omega}_b^{s,s} := \dot{R}_b^s \, R_s^b \in so(3)$$



Twist

- Configuration:

$$H_b^s \in SE(3)$$

$$\boxed{H_s^b := (H_b^s)^{-1}}$$

- Rate-of-change of configuration:

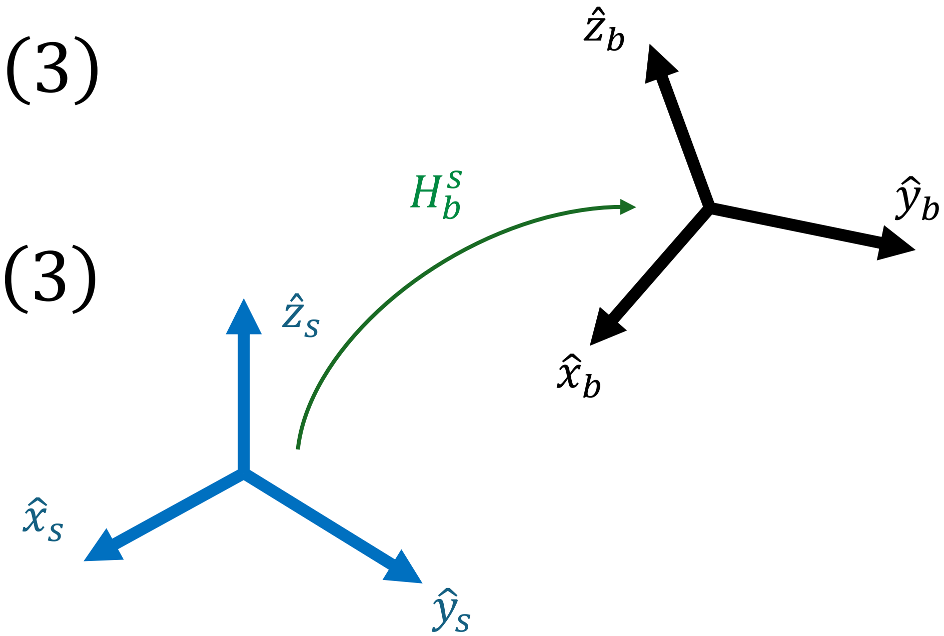
$$H_b^s \in T_{H_b^s} SE(3)$$

- Body twist expressed in $\{b\}$:

$$\tilde{\mathcal{V}}_b^{b,s} := H_s^b \, \dot{H}_b^s \in se(3)$$

- Spatial twist expressed in $\{s\}$:

$$\tilde{\mathcal{V}}_b^{s,s} := \dot{H}_b^s \, H_s^b \in se(3)$$



Properties

- Angular velocities
  - $\tilde{\omega}_i^{*,k} = -\tilde{\omega}_k^{*,i}$
  - $\tilde{\omega}_i^{*,k} = \tilde{\omega}_i^{*,m} + \tilde{\omega}_m^{*,k}$

- Twists
  - $\tilde{\mathcal{V}}_i^{*,k} = -\tilde{\mathcal{V}}_k^{*,i}$
  - $\tilde{\mathcal{V}}_i^{*,k} = \tilde{\mathcal{V}}_i^{*,m} + \tilde{\mathcal{V}}_m^{*,k}$