

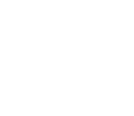
SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 11: Rigid Body Dynamics I



Outline

- Recap last lectures
- Properties of Twists
- Point mass dynamics
- Rigid body dynamics



Outline

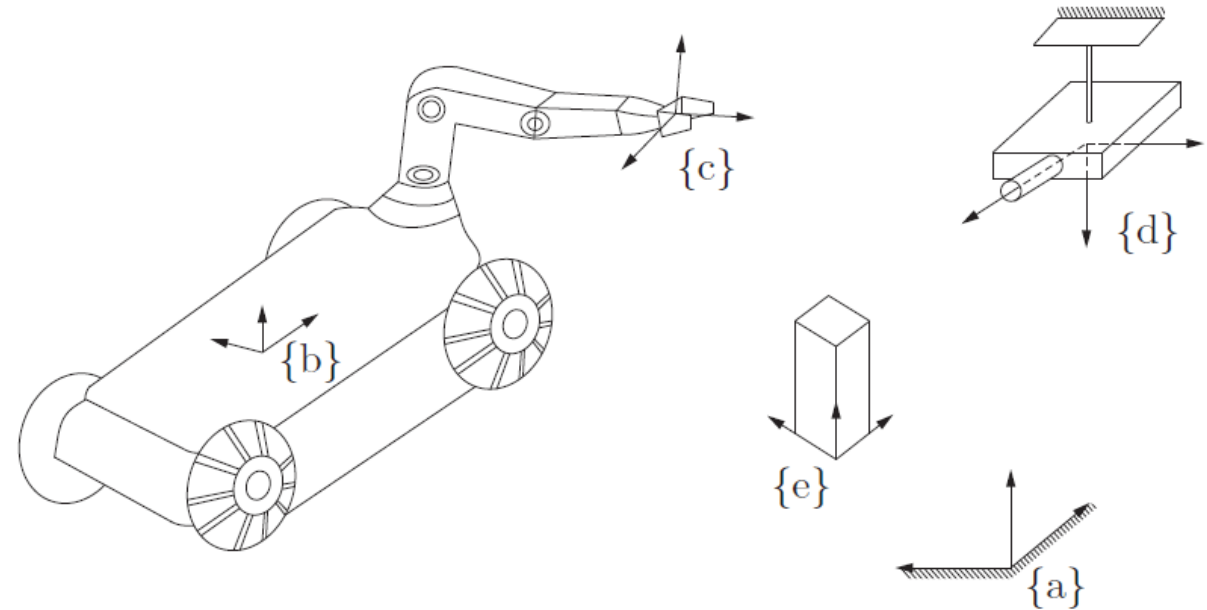
- Recap last lectures
- Properties of Twists
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- Rigid body dynamics



Recap: Kinematic Modeling Notation

- A frame will be denoted by Ψ_i or $\{i\}$.
- The relative pose of $\{i\}$ with respect to $\{k\}$ is described by

$$H_i^k = \begin{pmatrix} R_i^k & \xi_i^k \\ 0 & 1 \end{pmatrix} \in SE(3), \quad R_i^k \in SO(3), \quad \xi_i^k \in \mathbb{R}^3$$



Recap: Kinematic Relations

1. Point mass translation :

- Configuration:

$$\xi_b^s \in \mathbb{R}^3$$

- Rate-of-change of configuration:

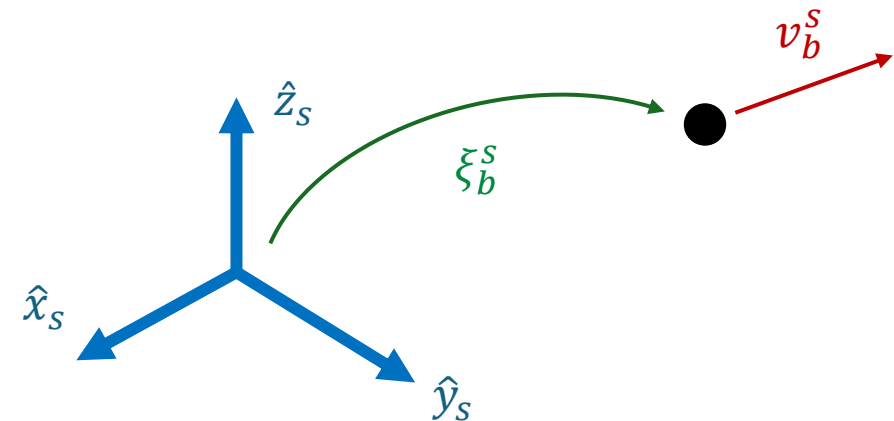
$$\dot{\xi}_b^s \in \mathbb{R}^3$$

- Velocity expressed in $\{s\}$:

$$v_b^{s,s} \in \mathbb{R}^3$$

- Kinematic relation:

$$\dot{\xi}_b^s = v_b^{s,s}$$



Recap: Kinematic Relations

2. Rigid body rotation :

- Configuration:

$$R_b^S \in SO(3)$$

- Rate-of-change of configuration:

$$\dot{R}_b^S \in T_{R_b^S} SO(3)$$

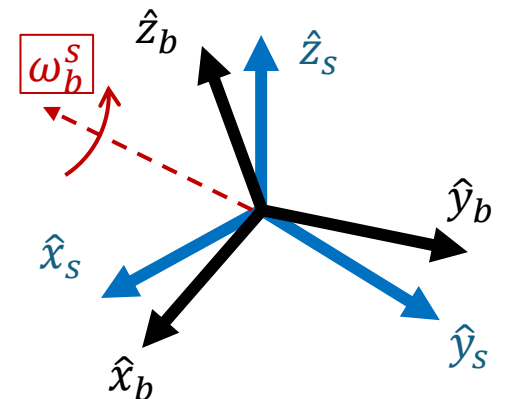
- Angular velocity expressed in $\{*\}$:

$$\tilde{\omega}_b^{*,S} \in T_I SO(3) =: so(3)$$

- Kinematic relation:

$$\dot{R}_b^S = R_b^S \tilde{\omega}_b^{b,S}$$

$$\dot{R}_b^S = \tilde{\omega}_b^{S,S} R_b^S$$



Recap: Kinematic Relations

3. Rigid body motion :

- Configuration:

$$H_b^S \in SE(3)$$

- Rate-of-change of configuration:

$$\dot{H}_b^S \in T_{H_b^S} SE(3)$$

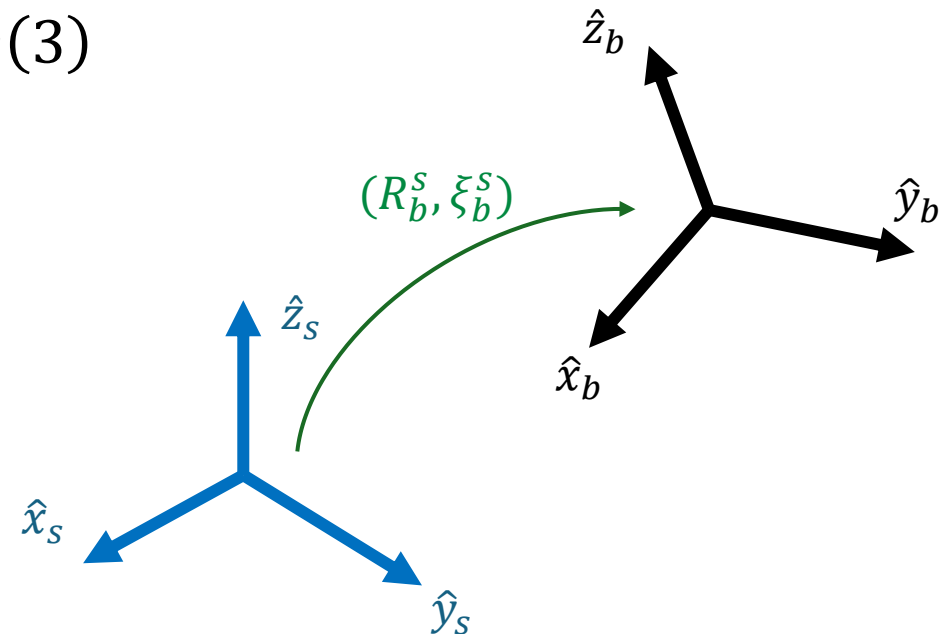
- Twist expressed in $\{*\}$:

$$\tilde{\mathcal{V}}_b^{*,S} \in T_I SE(3) =: se(3)$$

- Kinematic relation:

$$\dot{H}_b^S = H_b^S \tilde{\mathcal{V}}_b^{b,S}$$

$$\dot{H}_b^S = \tilde{\mathcal{V}}_b^{S,S} H_b^S$$



Recap: Matrix form of velocities in 3D

- Angular velocity

$$\omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \in \mathbb{R}^3$$

$$s \downarrow \quad \uparrow s^{-1}$$

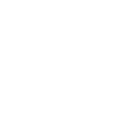
$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \in so(3)$$

- Twist

$$\mathcal{V} = \begin{pmatrix} \omega \\ v \end{pmatrix} \in \mathbb{R}^6$$

$$\tilde{s} \downarrow \quad \uparrow \tilde{s}^{-1}$$

$$\tilde{\mathcal{V}} = \begin{pmatrix} \tilde{\omega} & v \\ 0_{3 \times 1} & 0 \end{pmatrix} \in se(3)$$



Outline

- Recap last lectures
- **Properties of Twists**
- Point mass dynamics
- Rigid body dynamics



Properties

• Angular velocities

- $\tilde{\omega}_i^{*,k} = -\tilde{\omega}_k^{*,i}$
- $\tilde{\omega}_i^{*,k} = \tilde{\omega}_i^{*,m} + \tilde{\omega}_m^{*,k}$

$$\boxed{so(3)}$$

- $\omega_i^{*,k} = -\omega_k^{*,i}$
- $\omega_i^{*,k} = \omega_i^{*,m} + \omega_m^{*,k}$

$$\boxed{\mathbb{R}^3}$$

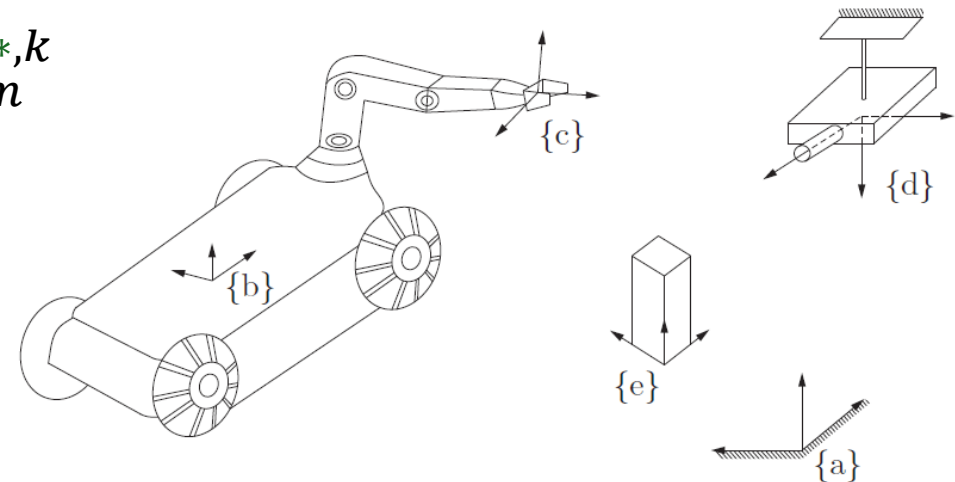
• Twists

- $\tilde{\mathcal{V}}_i^{*,k} = -\tilde{\mathcal{V}}_k^{*,i}$
- $\tilde{\mathcal{V}}_i^{*,k} = \tilde{\mathcal{V}}_i^{*,m} + \tilde{\mathcal{V}}_m^{*,k}$

$$\boxed{se(3)}$$

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$$\boxed{\mathbb{R}^6}$$



Relation between spatial and body angular velocity

- Spatial Angular Velocity

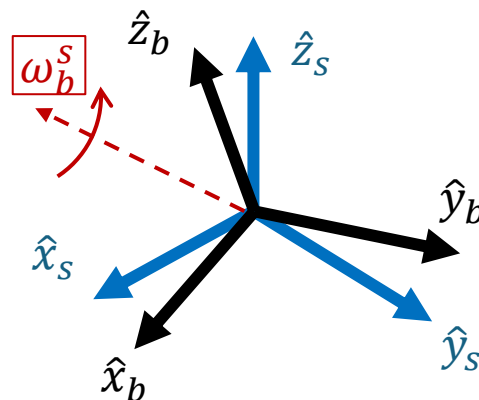
$$\omega_b^{s,s} \in \mathbb{R}^3$$

- Body Angular Velocity

$$\omega_b^{b,s} \in \mathbb{R}^3$$

$$\dot{R}_b^s = R_b^s \tilde{\omega}_b^{b,s} = \tilde{\omega}_b^{s,s} R_b^s$$

$$\tilde{\omega}_b^{s,s} = R_b^s \tilde{\omega}_b^{b,s} R_b^b$$



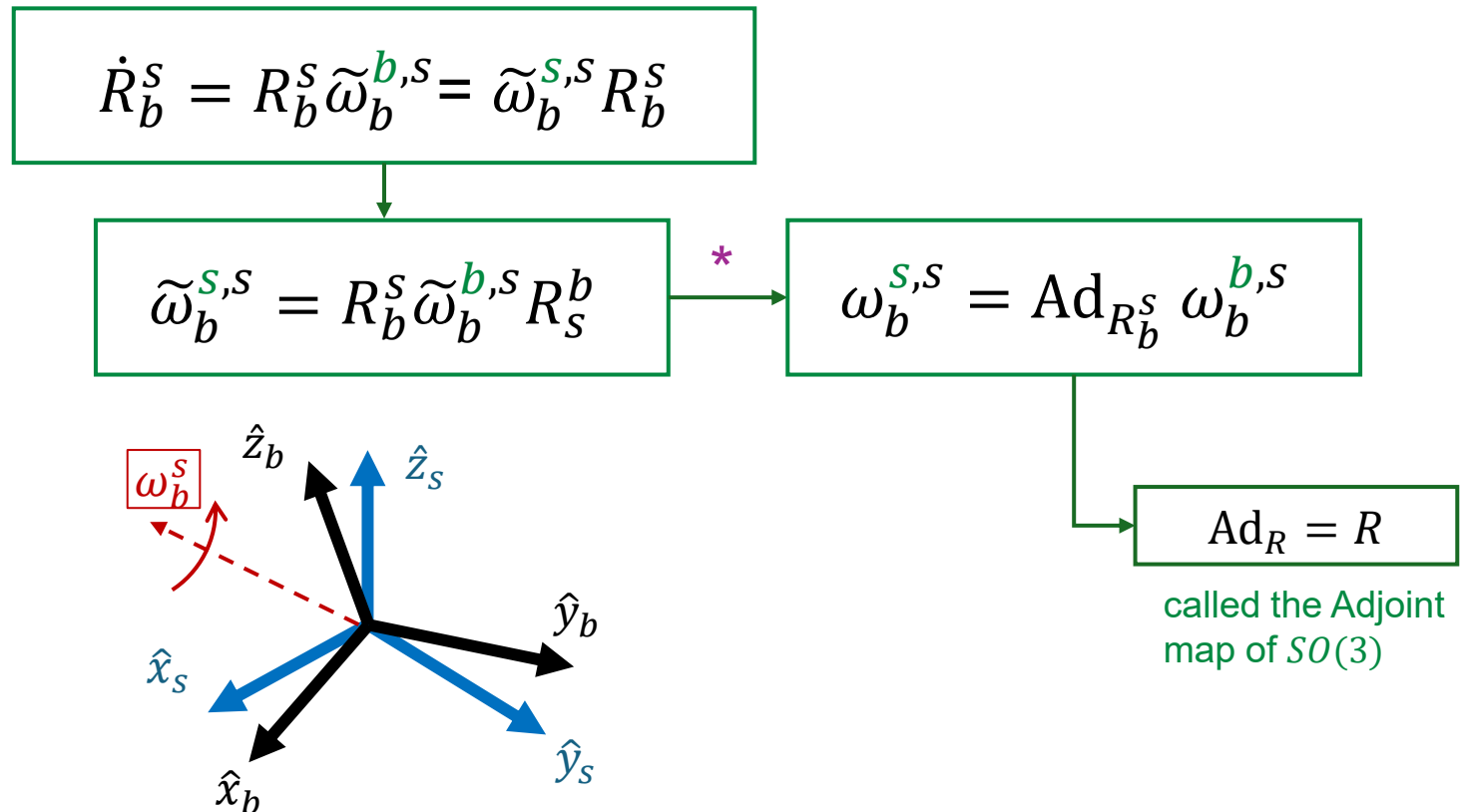
Relation between spatial and body angular velocity

- Spatial Angular Velocity

$$\omega_b^{s,s} \in \mathbb{R}^3$$

- Body Angular Velocity

$$\omega_b^{b,s} \in \mathbb{R}^3$$



*See Reading material



Relation between spatial and body twist

- Spatial Twist

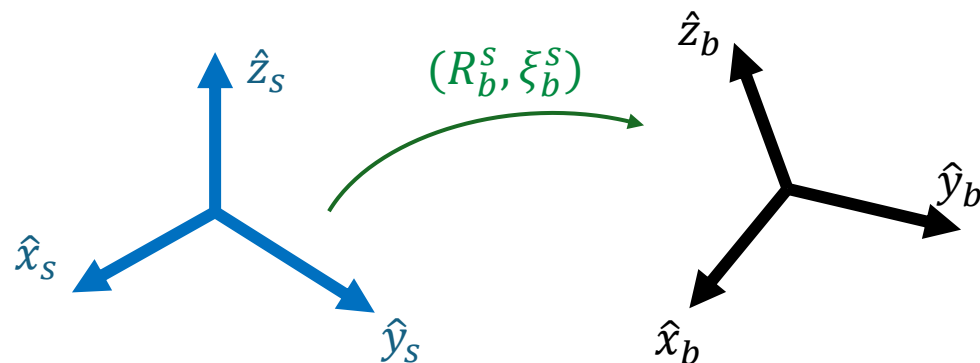
$$\mathcal{V}_b^{s,s} = \begin{pmatrix} \omega_b^{s,s} \\ v_b^{s,s} \end{pmatrix} \in \mathbb{R}^6$$

- Body Twist

$$\mathcal{V}_b^{b,s} = \begin{pmatrix} \omega_b^{b,s} \\ v_b^{b,s} \end{pmatrix} \in \mathbb{R}^6$$

$$\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s} = \tilde{\mathcal{V}}_b^{s,s} H_b^s$$

$$\tilde{\mathcal{V}}_b^{s,s} = H_b^s \tilde{\mathcal{V}}_b^{b,s} H_s^b$$



Relation between spatial and body twist

- Spatial Twist

$$\mathcal{V}_b^{s,s} = \begin{pmatrix} \omega_b^{s,s} \\ v_b^{s,s} \end{pmatrix} \in \mathbb{R}^6$$

- Body Twist

$$\mathcal{V}_b^{b,s} = \begin{pmatrix} \omega_b^{b,s} \\ v_b^{b,s} \end{pmatrix} \in \mathbb{R}^6$$

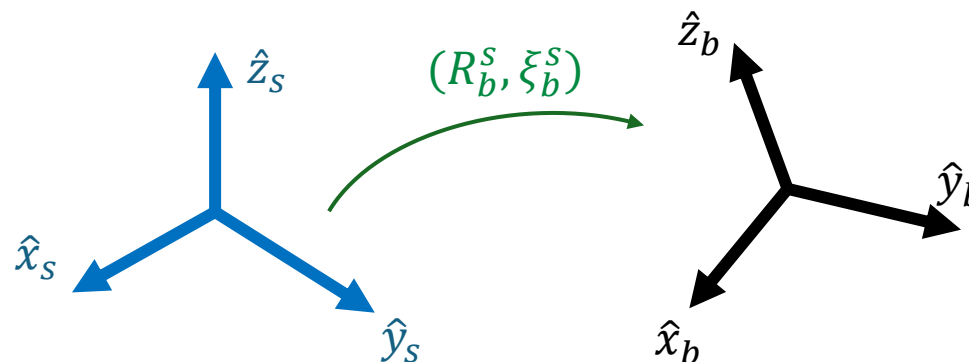
$$\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s} = \tilde{\mathcal{V}}_b^{s,s} H_b^s$$

$$\tilde{\mathcal{V}}_b^{s,s} = H_b^s \tilde{\mathcal{V}}_b^{b,s} H_b^b$$

$$\mathcal{V}_b^{s,s} = \text{Ad}_{H_b^s} \mathcal{V}_b^{b,s}$$

$$\text{Ad}_H = \begin{pmatrix} R & 0 \\ \tilde{\xi}R & R \end{pmatrix}$$

called the Adjoint map of $SE(3)$



* Homework Problem



Properties of Adjoint Map of SE(3)

- Adjoint map of SE(3)

$$\text{Ad}_H: \mathbb{R}^6 \rightarrow \mathbb{R}^6$$

- Closed form expressions:

- Composition

$$\text{Ad}_{H_1} \text{Ad}_{H_2} = \text{Ad}_{H_1 H_2}$$

- Inverse

$$(\text{Ad}_H)^{-1} = \text{Ad}_{H^{-1}}$$

- Time derivative

$$\frac{d}{dt} \left(\text{Ad}_{H_i^k} \right) = \text{Ad}_{H_i^k} \text{ad}_{\mathcal{V}_i^{i,k}}$$

$$\mathcal{V}_b^{s,s} = \text{Ad}_{H_b^s} \mathcal{V}_b^{b,s}$$

$$\text{Ad}_H = \begin{pmatrix} R & 0 \\ \tilde{\xi}R & R \end{pmatrix}$$

6 × 6
matrix



Properties of Adjoint Map of SE(3)

- Adjoint map of SE(3)

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- Closed form expressions:

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- Time derivative

$$\frac{d}{dt} \left(\text{Ad}_{H_i^k} \right) = \text{Ad}_{H_i^k} \text{ad}_{\mathcal{V}_i^{i,k}}$$

- Adjoint operator of Se(3)

$$\text{ad}_{\mathcal{V}}: \mathbb{R}^6 \rightarrow \mathbb{R}^6$$

$$\mathcal{V}_b^{s,s} = \text{Ad}_{H_b^s} \mathcal{V}_b^{b,s}$$

$$\text{Ad}_H = \begin{pmatrix} R & 0 \\ \tilde{\xi}R & R \end{pmatrix}$$

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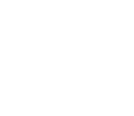
$$\text{ad}_{\mathcal{V}} = \begin{pmatrix} \tilde{\omega} & 0_{3 \times 3} \\ \tilde{v} & \tilde{\omega} \end{pmatrix}$$

6 × 6
matrix



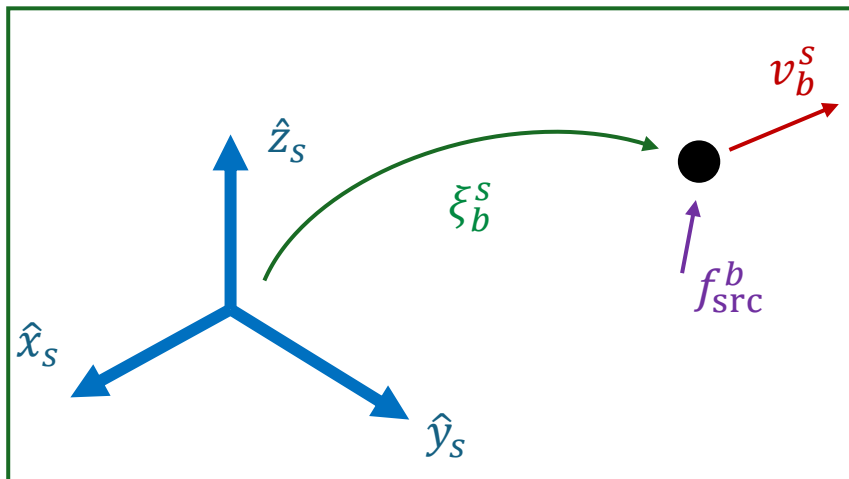
Outline

- Recap last lectures
- Properties of Twists
- **Point mass dynamics**
- Rigid body dynamics



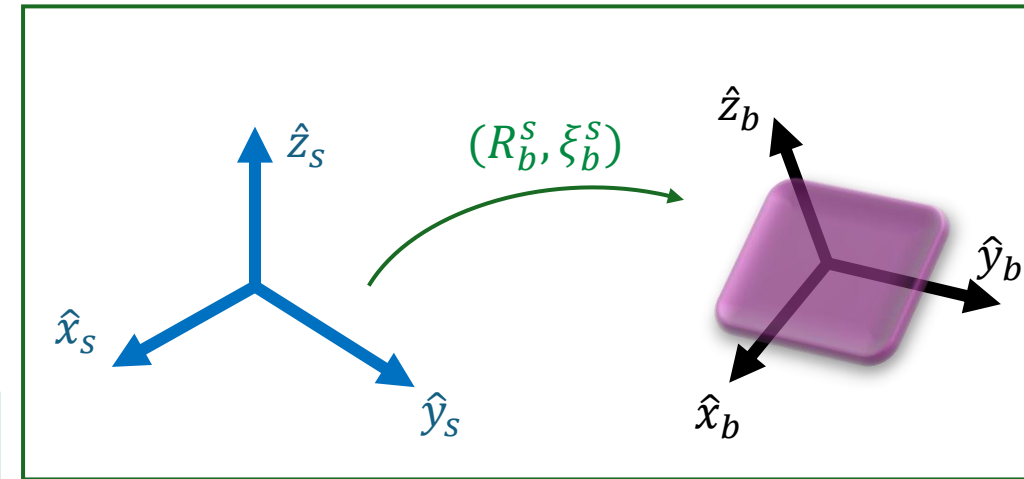
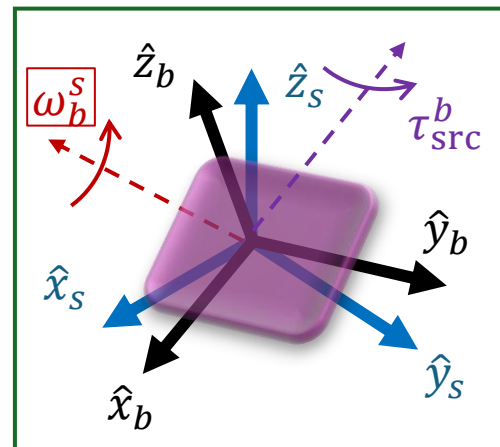
Dynamic modeling

- A dynamic model describes the motion of a system while considering the forces and torques that cause the motion.
- It includes both kinematics and conservation laws



Forces on translating point mass

Torques on rotating rigid body

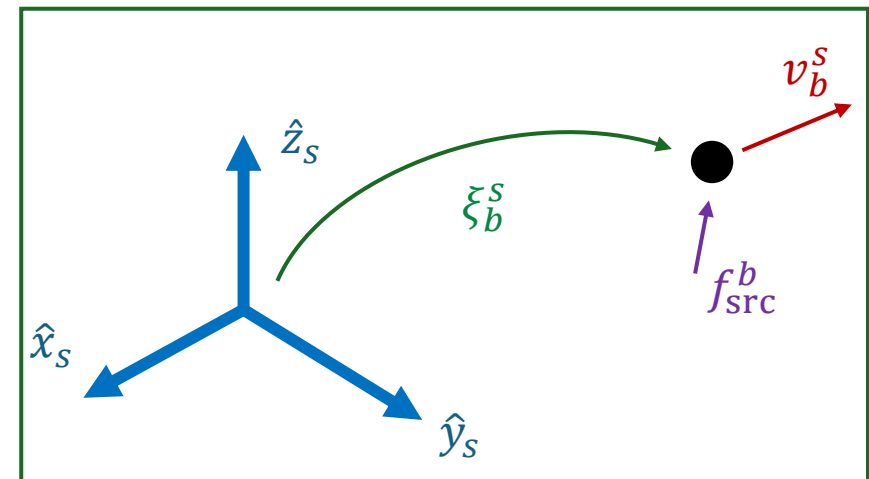


Wrenches on moving rigid body



Point mass dynamics

- We will denote by:
 - $f_{\text{src}}^b \in \mathbb{R}^3$: the abstract force from source src acting on point mass b
 - $f_{\text{src}}^{s,b} \in \mathbb{R}^3$: the force from source src acting on point mass b , expressed in $\{s\}$.

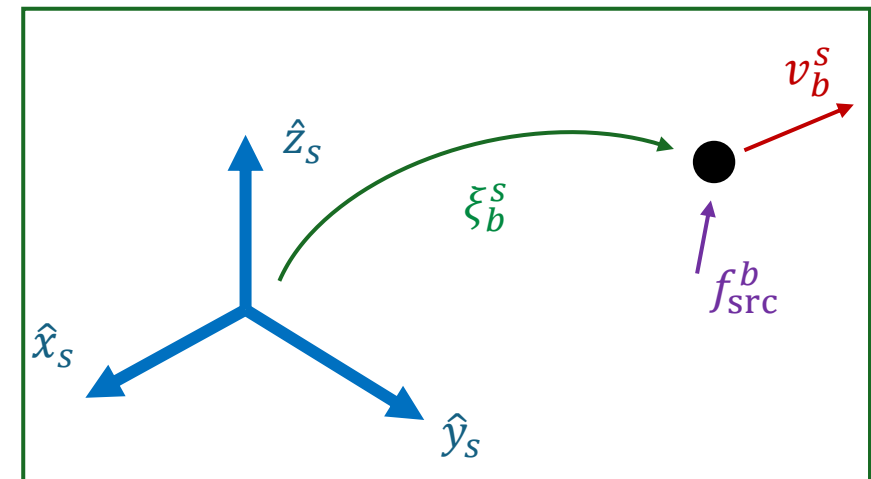


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 - $E_k(v_b^{s,s}) = \frac{1}{2} m(v_b^{s,s})^\top v_b^{s,s}$

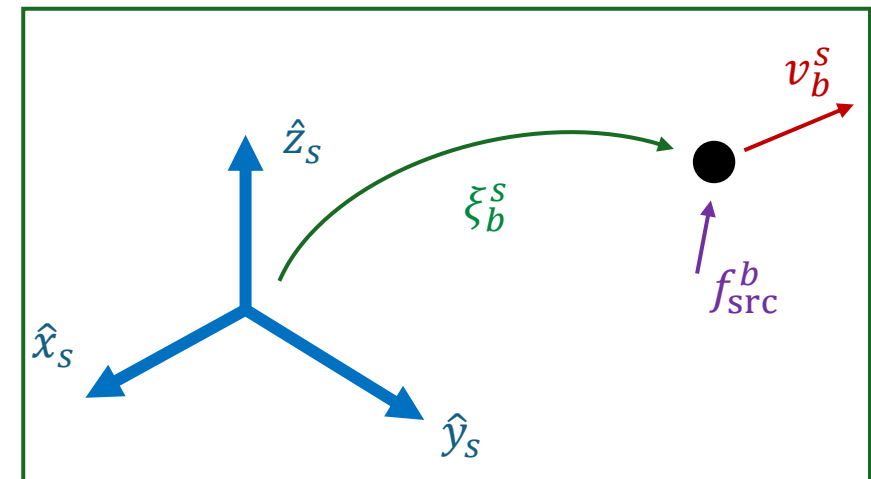


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 - $P_v^{s,b} := \frac{\partial E_k}{\partial v_b^{s,s}}(v_b^{s,s}) = m v_b^{s,s}$

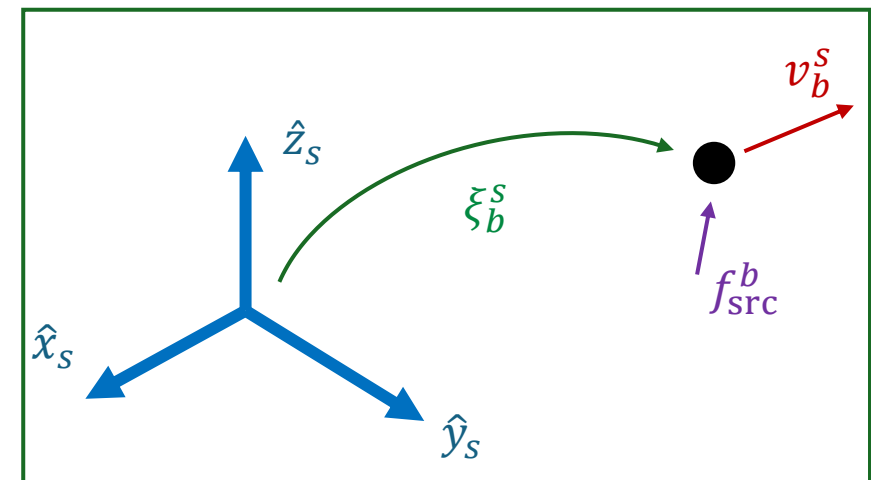


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 - $P_v^{s,b} := \frac{\partial E_k}{\partial v_b^{s,s}}(v_b^{s,s}) = m v_b^{s,s}$
- Newton's law:
 - $\dot{P}_v^{s,b} = f_{\text{tot}}^{s,b}$
 - or
 - $\dot{v}_b^{s,s} = \frac{1}{m} f_{\text{tot}}^{s,b}$

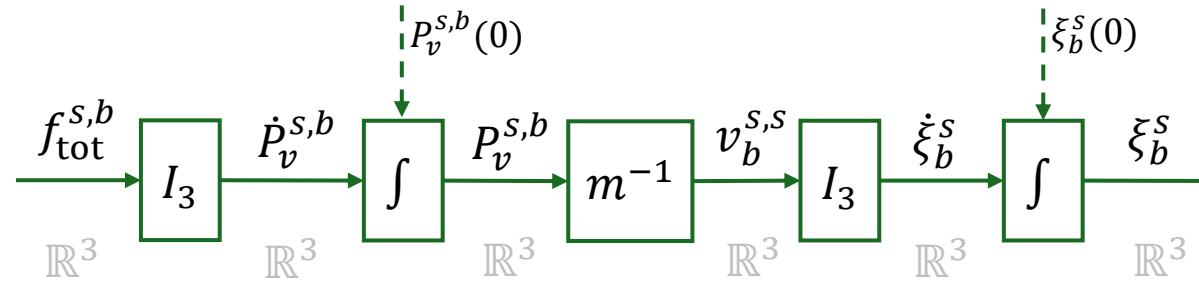


Forces on translating point mass

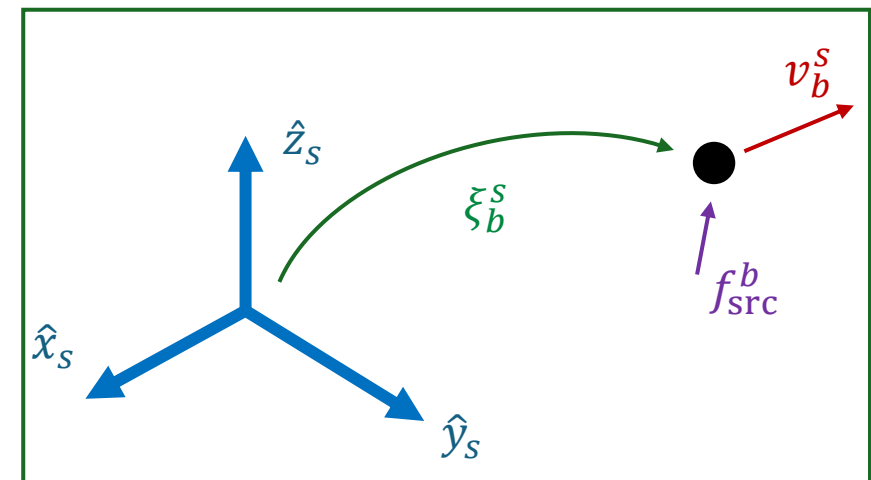


Point mass dynamics

- In summary,



- | | |
|---|-----------------------|
| <ul style="list-style-type: none"> $\dot{\xi}_b^s = v_b^{s,s}$ | Kinematic relation |
| <ul style="list-style-type: none"> $\dot{P}_v^{s,b} = f_{\text{tot}}^{s,b}$ | Momentum balance |
| <ul style="list-style-type: none"> $v_b^{s,s} = m^{-1} P_v^{s,b}$ | Constitutive relation |

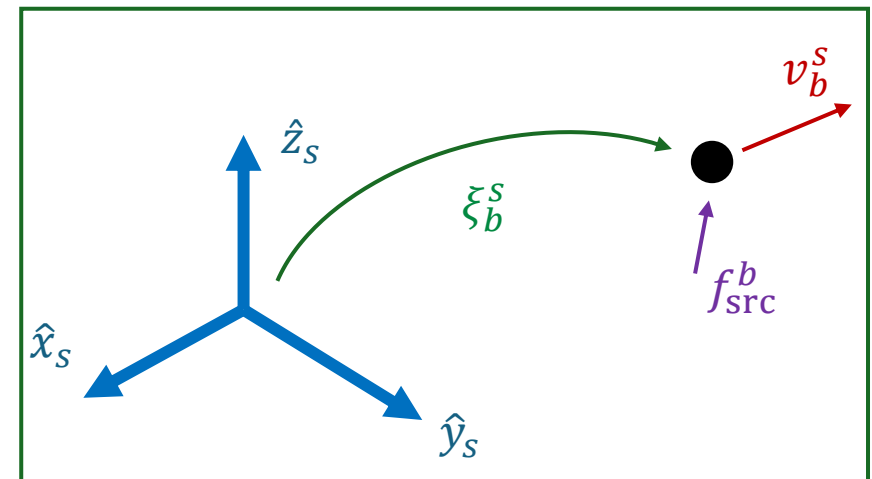
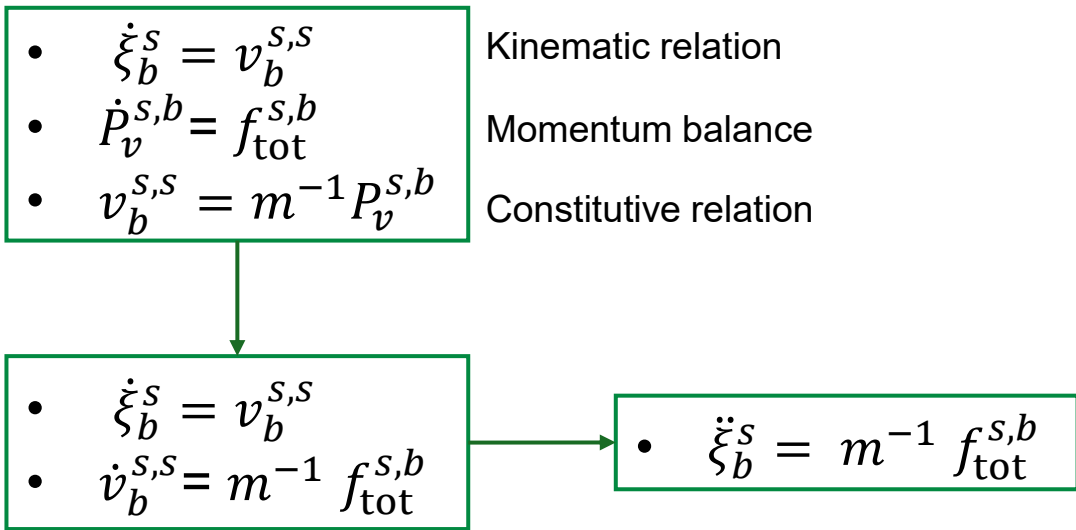
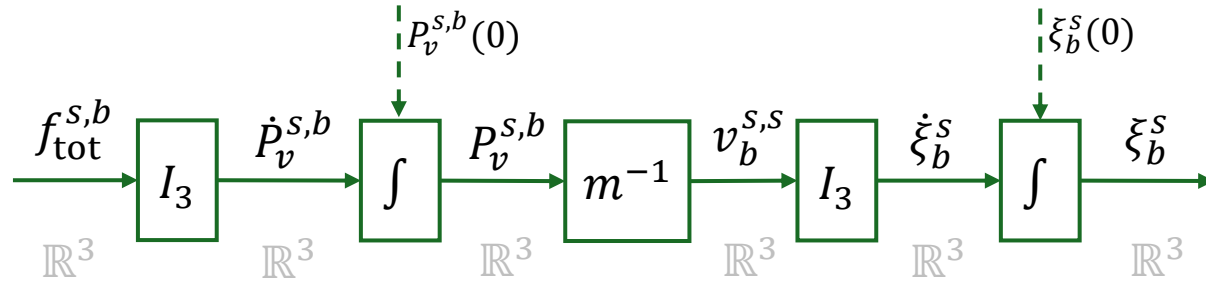


Forces on translating point mass



Point mass dynamics

- In summary,



Forces on translating point mass



Outline

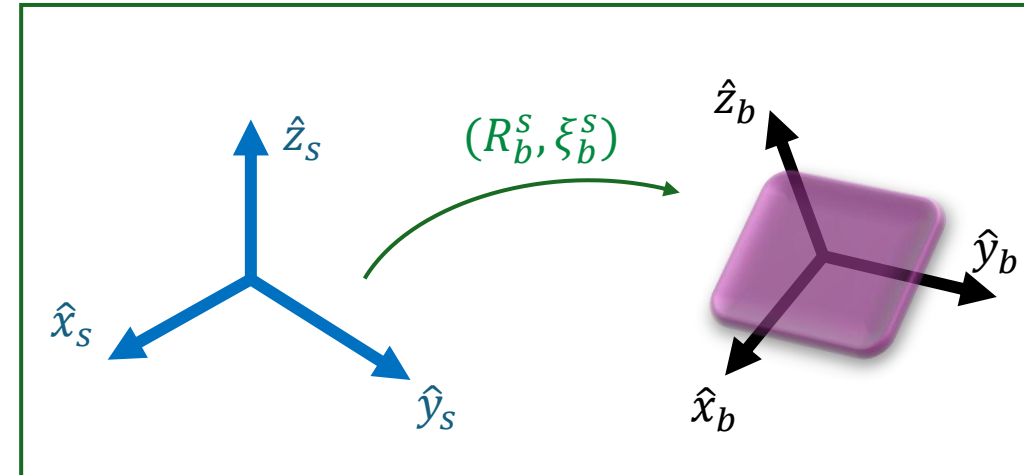
- Recap last lectures
- Properties of Twists
- Point mass dynamics
- **Rigid body dynamics**



Rigid body motion dynamics

• We will denote by:

- $\mathcal{W}_{\text{src}}^b \in (\mathbb{R}^6)^*$: the abstract wrench from source src acting on body attached to $\{b\}$
- $\mathcal{W}_{\text{src}}^{*,b} \in (\mathbb{R}^6)^*$: the wrench from source src acting on body attached to $\{b\}$, expressed in $\{*\}$.



Wrenches on moving rigid body

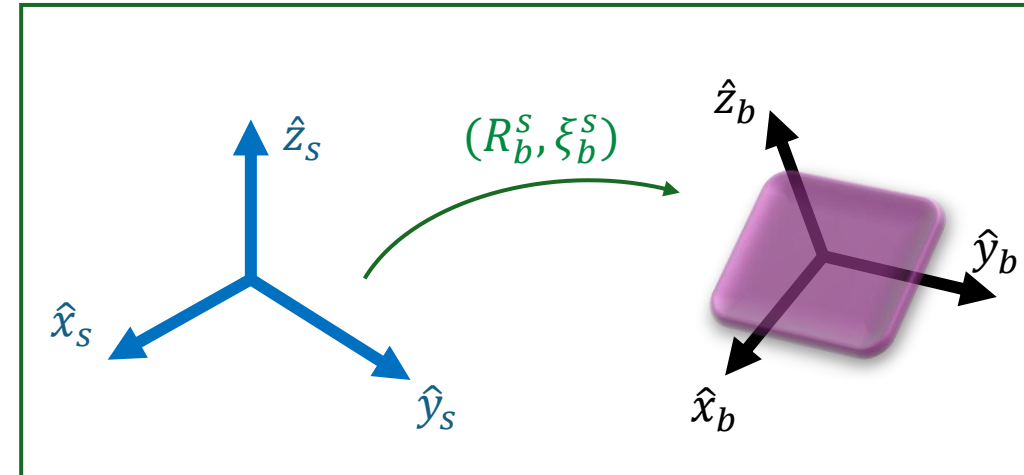


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- The kinetic energy of the rigid body, $E_k: SE(3) \times \mathbb{R}^6 \rightarrow \mathbb{R}$ is given by:
 - $E_k(H_b^S, \mathcal{V}_b^{*,S}) = \frac{1}{2} (\mathcal{V}_b^{*,S})^\top \mathfrak{I}^{*,b}(H_b^S) \mathcal{V}_b^{*,S}$

Generalized Inertia of the body attached to $\{b\}$ expressed in $\{*\}$

$$\mathfrak{I}^{*,b}(H_b^S): \mathbb{R}^6 \rightarrow (\mathbb{R}^6)^*$$

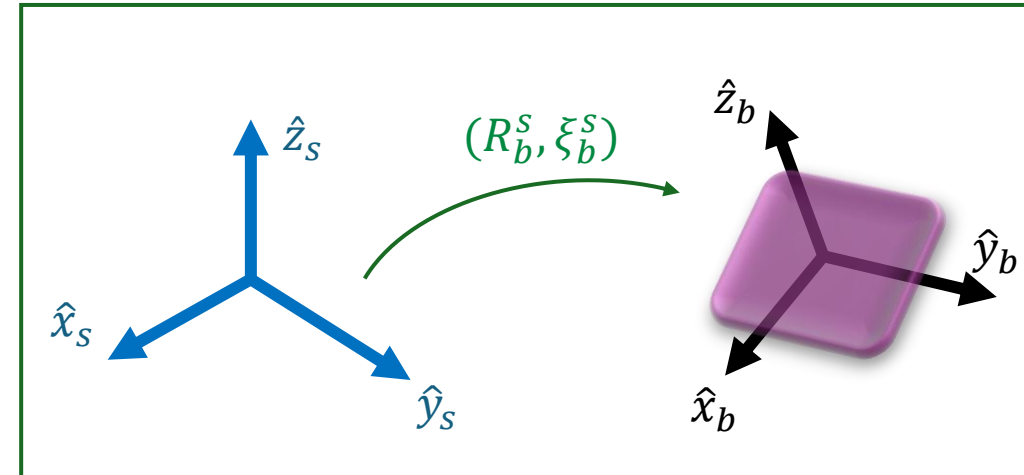


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 - $P^{*,b} := \frac{\partial E_k}{\partial \mathcal{V}_b^{*,S}}(H_b^S, \mathcal{V}_b^{*,S}) = \mathfrak{T}^{*,b}(H_b^S) \mathcal{V}_b^{*,S}$



Wrenches and generalized momenta are covectors !!

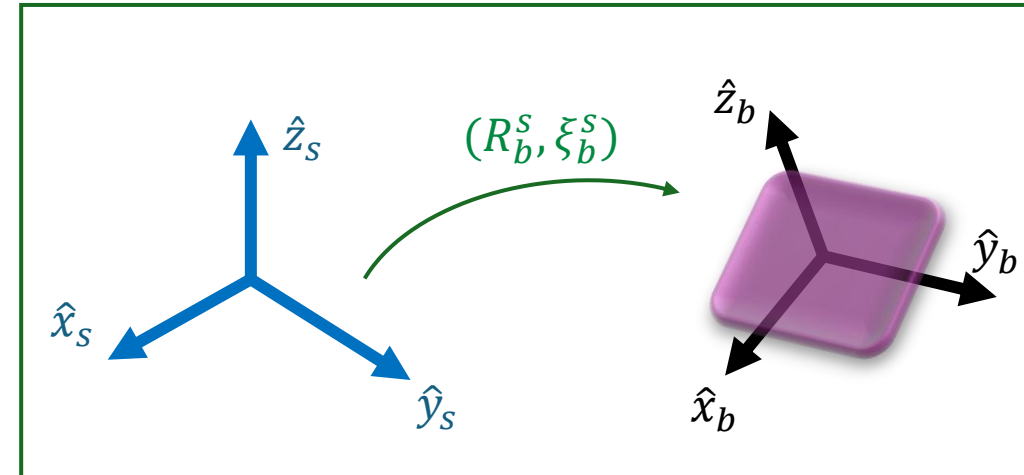
$$\mathcal{W}^{*,b}, P^{*,b} \in (\mathbb{R}^6)^*$$

Wrenches on moving rigid body



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- Newton-Euler's law in $\{s\}$:
 - $\dot{p}^{s,b} = \mathcal{W}_{\text{tot}}^{s,b}$



Wrenches and generalized momenta are covectors !!

$$\mathcal{W}^{*,b}, P^{*,b} \in (\mathbb{R}^6)^*$$

Wrenches on moving rigid body



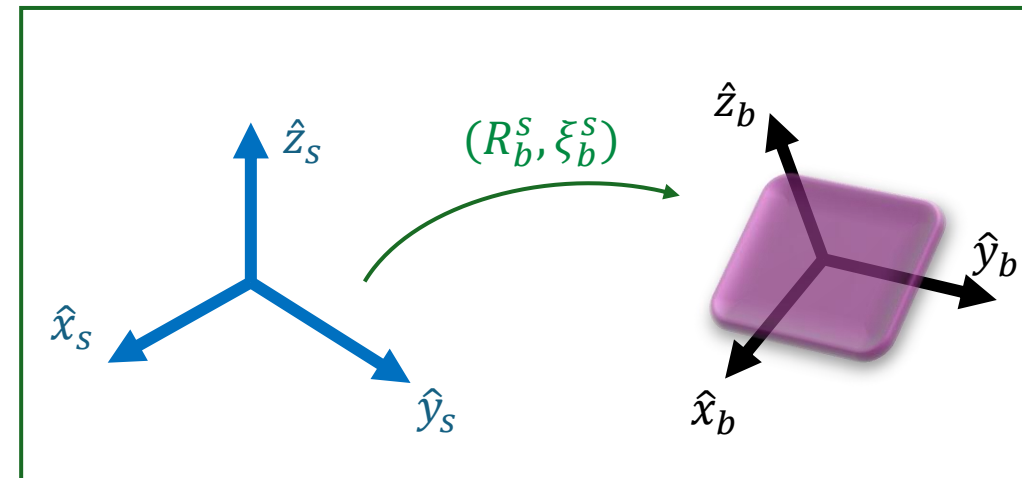
Generalized Inertia

- Symmetric & positive definite
- Constant in any body-fixed frame

$$\mathcal{I}^{b,b} = \begin{pmatrix} J^{b,b} & m \tilde{\xi}_{\text{cm}}^b \\ -m \tilde{\xi}_{\text{cm}}^b & m I_3 \end{pmatrix}$$

- Changes coordinates as

$$\mathcal{I}^{s,b} = Ad_{H_s^b}^\top \mathcal{I}^{b,b} Ad_{H_s^b}$$



Recall how bilinear forms change coordinates

Wrenches on moving rigid body



Kinetic energy invariance

- Kinetic energy is a scalar, so it is a **coordinate-free** concept.

$$E_k(H_b^S, \mathcal{V}_b^{b,S}) = E_k(H_b^S, \mathcal{V}_b^{S,S}) = E_k(H_b^S, \mathcal{V}_b^{*,S})$$

- Let's examine the kinetic energy of the twist in $\{b\}$:

$$E_k(H_b^S, \mathcal{V}_b^{b,S}) = E_k(\mathcal{V}_b^{b,S}) = \frac{1}{2} (\mathcal{V}_b^{b,S})^\top \mathfrak{I}^{b,b} \mathcal{V}_b^{b,S}$$



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$$E_k(H_b^s, \mathcal{V}_b^{b,s}) = E_k(H_b^s, \mathcal{V}_b^{s,s}) = E_k(H_b^s, \mathcal{V}_b^{*,s})$$

- Let's examine the kinetic energy of the twist in $\{b\}$:

$$E_k(H_b^s, \mathcal{V}_b^{b,s}) = E_k(\mathcal{V}_b^{b,s}) = \frac{1}{2} (\mathcal{V}_b^{b,s})^\top \mathfrak{I}^{b,b} \mathcal{V}_b^{b,s}$$

- Using the definition of the generalized momentum $P^{b,b} = \mathfrak{I}^{b,b} \mathcal{V}_b^{b,s}$,
we have that

$$2E_k = (\mathcal{V}_b^{b,s})^\top P^{b,b} = (P^{b,b})^\top \mathcal{V}_b^{b,s}$$

The duality pairing between twist $\mathcal{V}_b^{*,s}$ and generalized momentum $P^{*,b}$ is twice the kinetic energy $2E_k$



Generalized momentum

- Due to the invariance of kinetic energy, if we express the momentum and twist in the $\{s\}$ frame we have that:

$$2E_k = (P^{b,b})^\top \mathcal{V}_b^{b,s} = (P^{b,b})^\top Ad_{H_S^b} \mathcal{V}_b^{s,s} = (Ad_{H_S^b}^\top P^{b,b})^\top \mathcal{V}_b^{s,s} = (P^{s,b})^\top \mathcal{V}_b^{s,s}$$



Generalized momentum

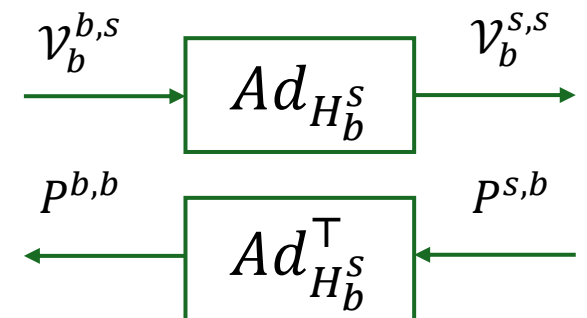
- Due to the invariance of kinetic energy, if we express the momentum and twist in the $\{s\}$ frame we have that:

$$2E_k = (P^{b,b})^\top \mathcal{V}_b^{b,s} = (P^{b,b})^\top Ad_{H_b^s} \mathcal{V}_b^{s,s} = \left(Ad_{H_b^s}^\top P^{b,b} \right)^\top \mathcal{V}_b^{s,s} = (P^{s,b})^\top \mathcal{V}_b^{s,s}$$

- Therefore,

Vector $\mathcal{V}_b^{s,s} = Ad_{H_b^s} \mathcal{V}_b^{b,s} \in \mathbb{R}^6$

$P^{s,b} = Ad_{H_b^s}^{-\top} P^{b,b} \in (\mathbb{R}^6)^*$ Covector



A vector and a covector change coordinates differently !!



Wrench

- Just as with twists, we can merge torques and forces into a six-dimensional object we shall call a wrench.

$$\mathcal{W}^{*,b} = \begin{pmatrix} \tau^{*,b} \\ f^{*,b} \end{pmatrix} \in (\mathbb{R}^6)^*$$

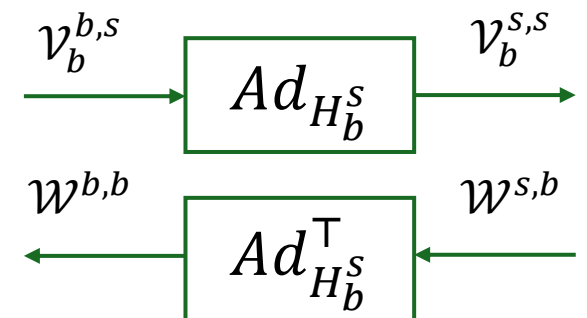
- The duality pairing of a wrench and a twist gives mechanical power:

$$\text{Power} = (\mathcal{W}^{*,b})^\top \mathcal{V}_b^{*,s}$$

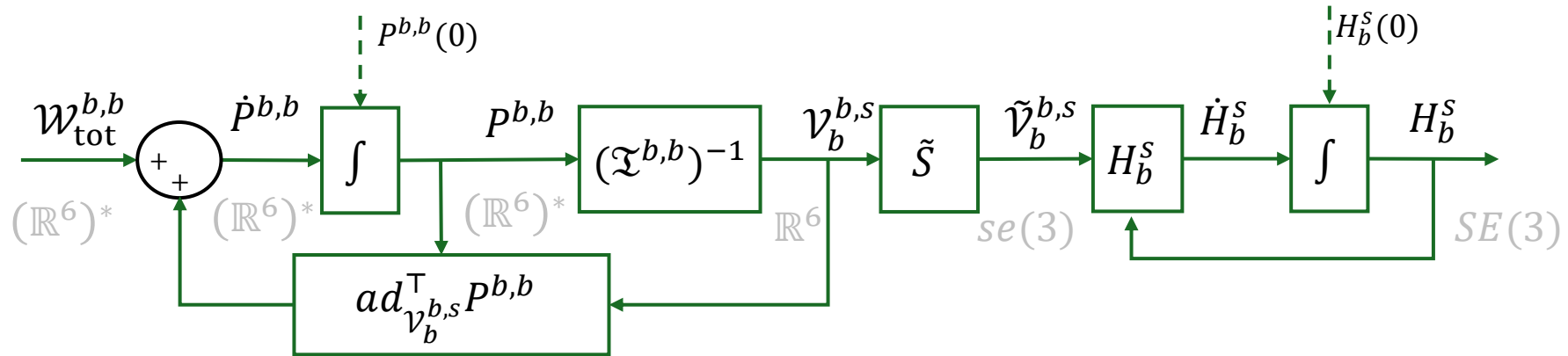
- Since power is also a coordinate-free concept, we have that

Vector $\mathcal{V}_b^{s,s} = Ad_{H_b^s} \mathcal{V}_b^{b,s} \in \mathbb{R}^6$

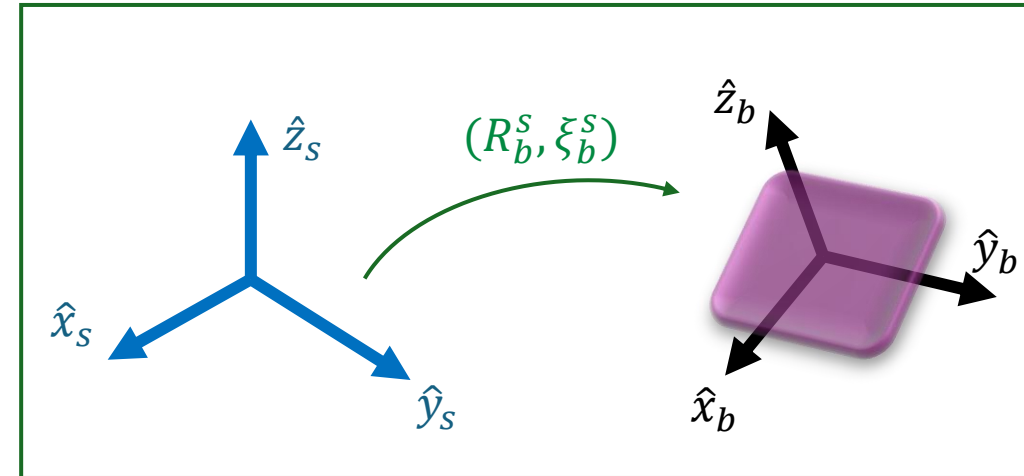
$\mathcal{W}^{s,b} = Ad_{H_b^s}^{-\top} \mathcal{W}^{b,b} \in (\mathbb{R}^6)^*$ Covector



Rigid body dynamics



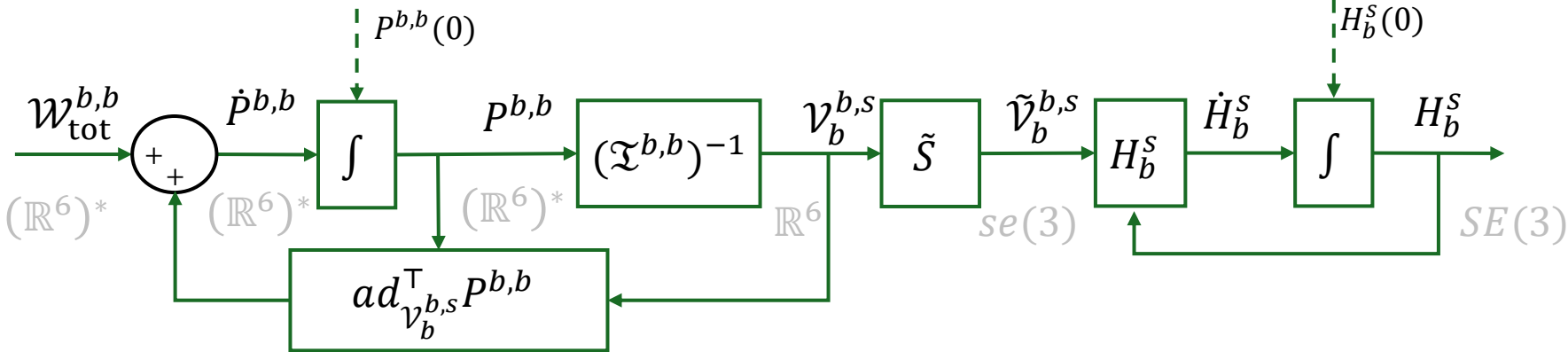
- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$ Kinematic relation
- $\dot{p}^{s,b} = \mathcal{W}_{tot}^{s,b}$ Momentum balance
- $\mathcal{V}_b^{b,s} = (\mathcal{I}^{b,b})^{-1} p^{b,b}$ Constitutive relation



Wrenches on moving rigid body



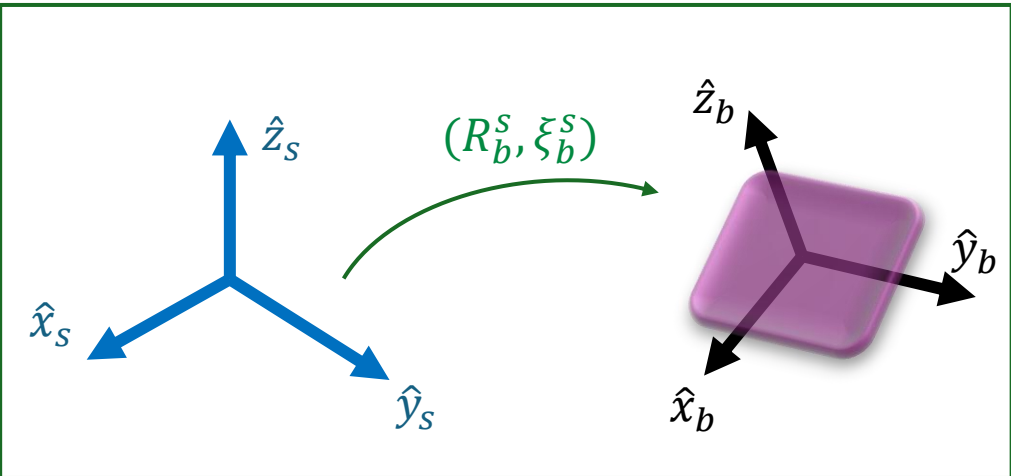
Rigid body dynamics



- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$ Kinematic relation
- $\dot{p}^{s,b} = \mathcal{W}_{tot}^{s,b}$ Momentum balance
- $\mathcal{V}_b^{b,s} = (\mathcal{I}^{b,b})^{-1} p^{b,b}$ Constitutive relation

Rewritten as

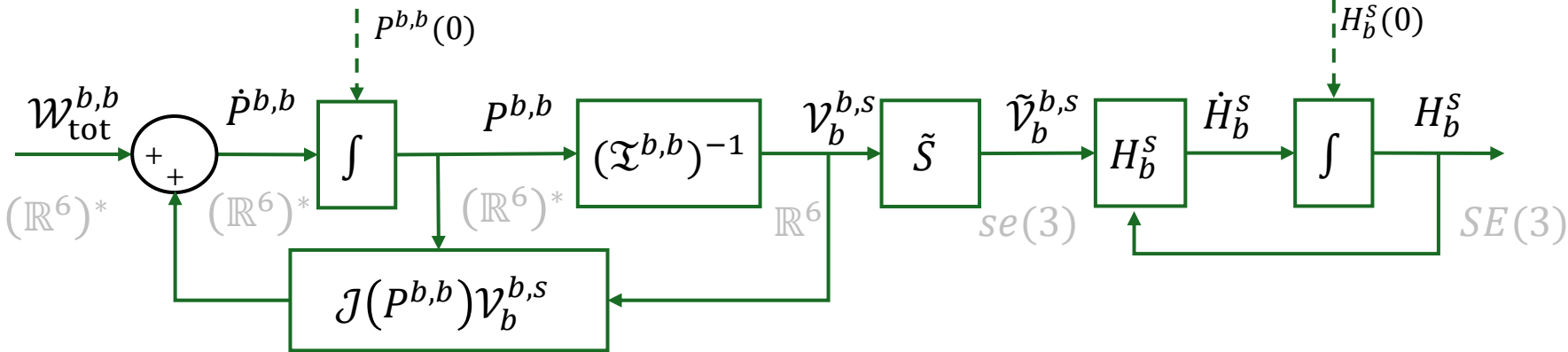
1.
 - $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$
 - $\dot{p}^{b,b} = \mathcal{W}_{tot}^{b,b} + ad_{\mathcal{V}_b^{b,s}}^\top P^{b,b}$
 - $\mathcal{V}_b^{b,s} = (\mathcal{I}^{b,b})^{-1} p^{b,b}$



Wrenches on moving rigid body



Rigid body dynamics

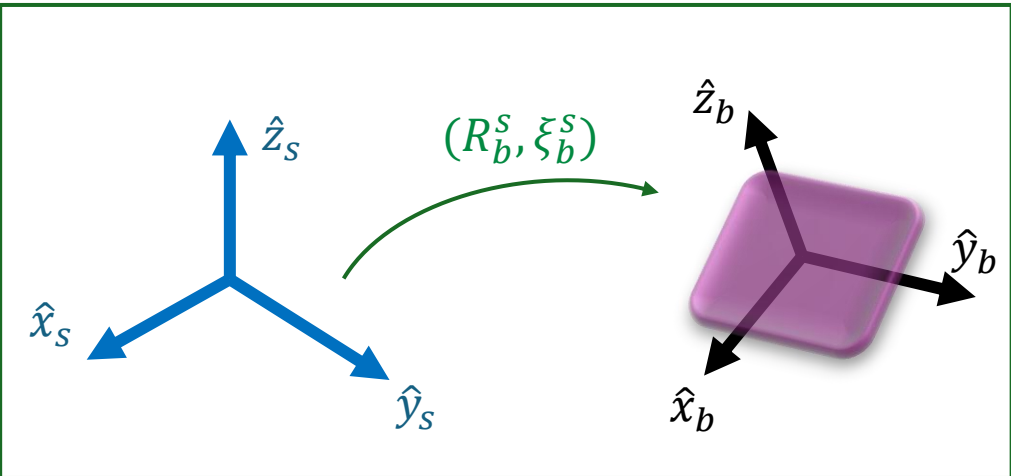


- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$ Kinematic relation
- $\dot{p}^{s,b} = \mathcal{W}_{tot}^{s,b}$ Momentum balance
- $\mathcal{V}_b^{b,s} = (\mathcal{I}^{b,b})^{-1} p^{b,b}$ Constitutive relation

Or rewritten as

- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$
- $\dot{p}^{b,b} = \mathcal{W}_{tot}^{b,b} + \mathcal{J}(P^{b,b})\mathcal{V}_b^{b,s}$
- $\mathcal{V}_b^{b,s} = (\mathcal{I}^{b,b})^{-1} p^{b,b}$

2



Wrenches on moving rigid body

