SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 12: Rigid Body Dynamics II

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Outline

- Recap last lectures
- In depth analysis

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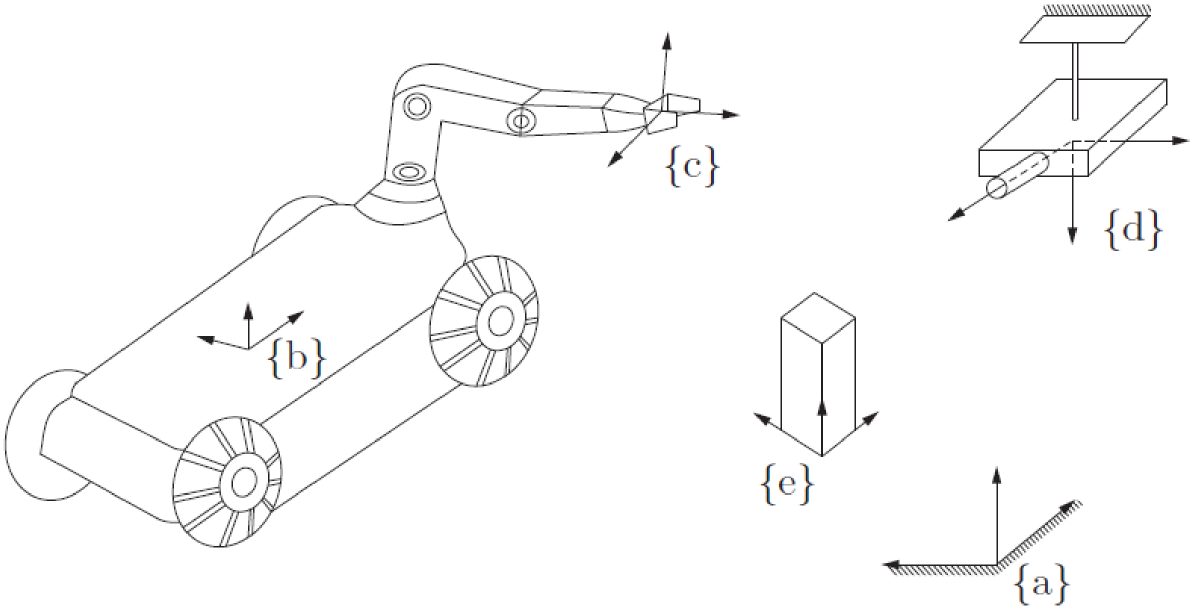
Recap: Kinematic Modeling Notation

- A frame will be denoted by Ψ_i or $\{i\}$.
- The relative pose of $\{i\}$ with respect to $\{k\}$ is described by

$$H_i^k = \begin{pmatrix} R_i^k & \xi_i^k \\ 0 & 1 \end{pmatrix} \in SE(3), \qquad R_i^k \in SO(3), \qquad \xi_i^k \in \mathbb{R}^3$$



Recap: Kinematic Relations

1. Point mass translation :

- Configuration:

$$\xi_b^s \in \mathbb{R}^3$$

- Rate-of-change of configuration:

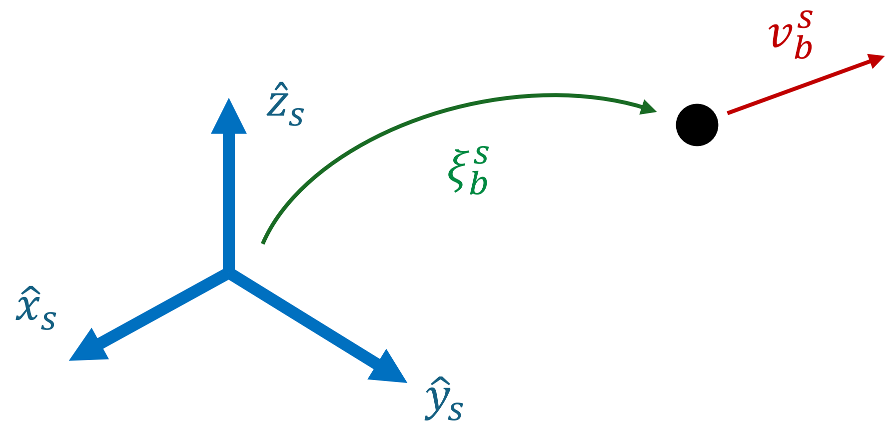
$$\dot{\xi}_b^s \in \mathbb{R}^3$$

- Velocity expressed in $\{s\}$:

$$v_b^{s,s} \in \mathbb{R}^3$$

- Kinematic relation:

$$\dot{\xi}_b^s = v_b^{s,s}$$



Recap: Kinematic Relations

2. Rigid body rotation :

- Configuration:

$$R_b^s \in SO(3)$$

- Rate-of-change of configuration:

$$\dot{R}_b^s \in T_{R_b^s} SO(3)$$

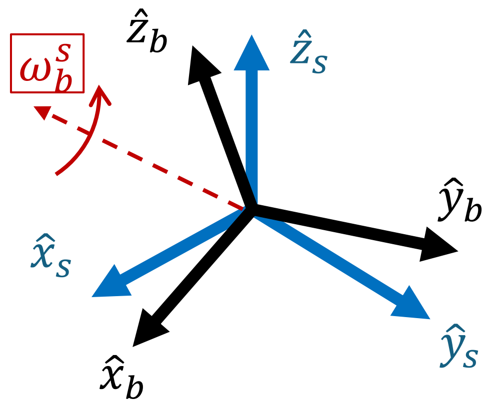
- Angular velocity expressed in $\{*\}$:

$$\tilde{\omega}_b^{*,s} \in T_I SO(3) =: so(3)$$

- Kinematic relation:

$$\dot{R}_b^s = R_b^s \tilde{\omega}_b^{b,s}$$

$$\dot{R}_b^s = \tilde{\omega}_b^{s,s} R_b^s$$



Recap: Kinematic Relations

$$SE(3) \coloneqq SO(3) \ltimes \mathbb{R}^3$$

3. Rigid body motion :

- Configuration:

$$H_b^s \in SE(3)$$

- Rate-of-change of configuration:

$$\dot{H}_b^s \in T_{H_b^s} SE(3)$$

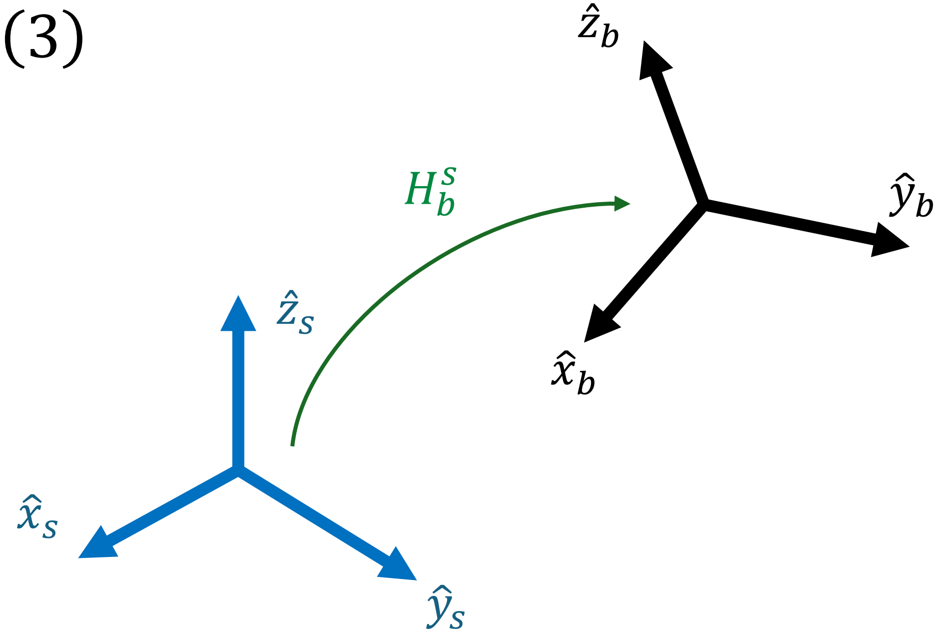
- Twist expressed in $\{*\}$:

$$\tilde{\mathcal{V}}_b^{*,s} \in T_I SE(3) =: se(3)$$

- Kinematic relation:

$$\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$$

$$\dot{H}_b^s = \tilde{\mathcal{V}}_b^{s,s} H_b^s$$



Recap: Matrix form of velocities in 3D

- Angular velocity

$$\omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \in \mathbb{R}^3$$

$$S \downarrow \qquad \uparrow S^{-1}$$

$$\widetilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \in so(3)$$

- Twist

$$\mathcal{V} = \begin{pmatrix} \omega \\ v \end{pmatrix} \in \mathbb{R}^6$$

$$\tilde{S} \downarrow \qquad \uparrow \tilde{S}^{-1}$$

$$\tilde{\mathcal{V}} = \begin{pmatrix} \widetilde{\omega} & v \\ 0_{3\times 1} & 0 \end{pmatrix} \in se(3)$$

Recap: Relation between spatial and body twist

- Spatial Twist

$$\mathcal{V}_b^{s,s} = \begin{pmatrix} \omega_b^{s,s} \\ v_b^{s,s} \end{pmatrix} \in \mathbb{R}^6$$

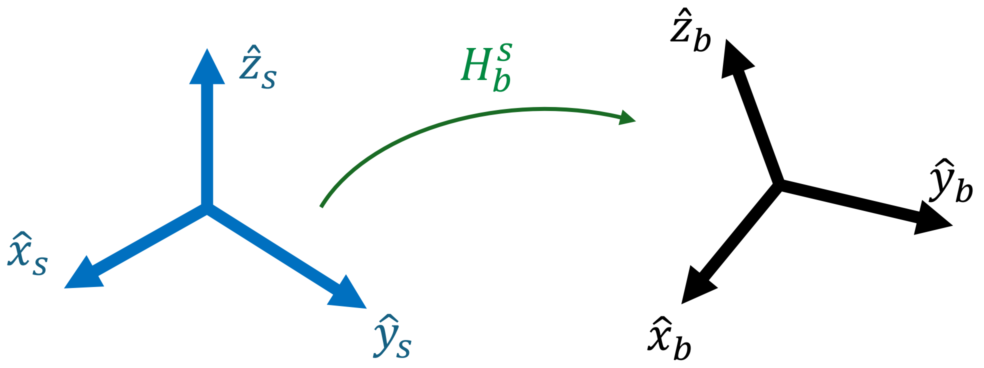
- Body Twist

$$\mathcal{V}_b^{b,s} = \begin{pmatrix} \omega_b^{b,s} \\ v_b^{b,s} \end{pmatrix} \in \mathbb{R}^6$$

$$\mathcal{V}_b^{s,s} = \mathrm{Ad}_{H_b^s} \mathcal{V}_b^{b,s}$$

$$\mathrm{Ad}_H = \begin{pmatrix} R & 0 \\ \tilde{\xi} R & R \end{pmatrix}$$

called the Adjoint map of $SE(3)$



Recap: Properties of Adjoint Map of SE(3)

- Adjoint map of SE(3)

$$\mathrm{Ad}_H : \mathbb{R}^6 \to \mathbb{R}^6$$

- Closed form expressions:
  - Composition

$$\mathrm{Ad}_{H_1}\mathrm{Ad}_{H_2} = \mathrm{Ad}_{H_1 H_2}$$

  - Inverse

$$(\mathrm{Ad}_H)^{-1} = \mathrm{Ad}_{H^{-1}}$$

  - Time derivative

$$\frac{d}{dt}\left(\mathrm{Ad}_{H_i^k}\right) = \mathrm{Ad}_{H_i^k}\,\mathrm{ad}_{\mathcal{V}_i^{i,k}}$$

$$\mathcal{V}_b^{s,s} = \mathrm{Ad}_{H_b^s}\,\mathcal{V}_b^{b,s}$$

$$\mathrm{Ad}_H = \begin{pmatrix} R & 0 \\ \tilde{\xi}R & R \end{pmatrix}$$

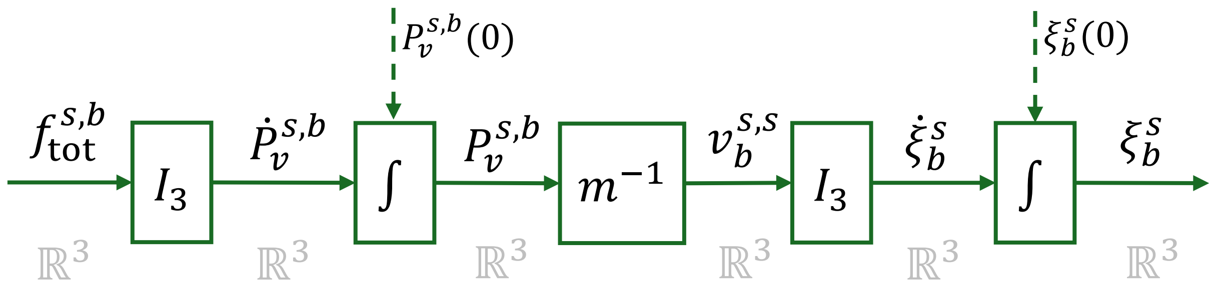
6 × 6 matrix

$$\mathrm{ad}_{\mathcal{V}} = \begin{pmatrix} \tilde{\omega} & 0_{3\times 3} \\ \tilde{v} & \tilde{\omega} \end{pmatrix}$$

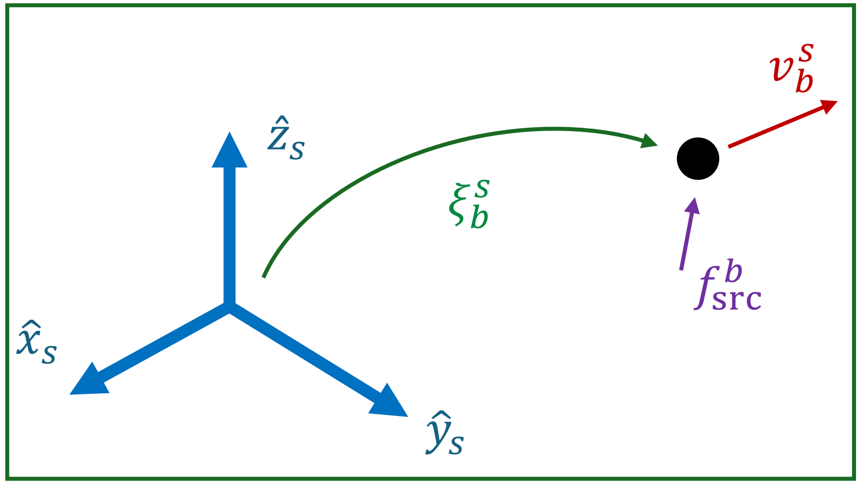
6 × 6 matrix

Adjoint operator of se(3)

Recap: Point mass dynamics

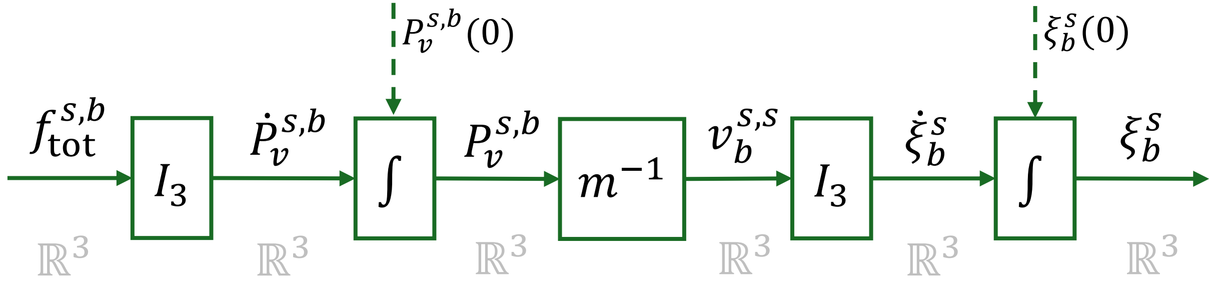


- $\dot{\xi}_b^s = v_b^{s,s}$ — Kinematic relation
- $\dot{P}_v^{s,b} = f_{\text{tot}}^{s,b}$ — Momentum balance
- $v_b^{s,s} = m^{-1} P_v^{s,b}$ — Constitutive relation



Forces on translating point mass

Recap: Point mass dynamics



- $\dot{\xi}_b^s = v_b^{s,s}$ — Kinematic relation
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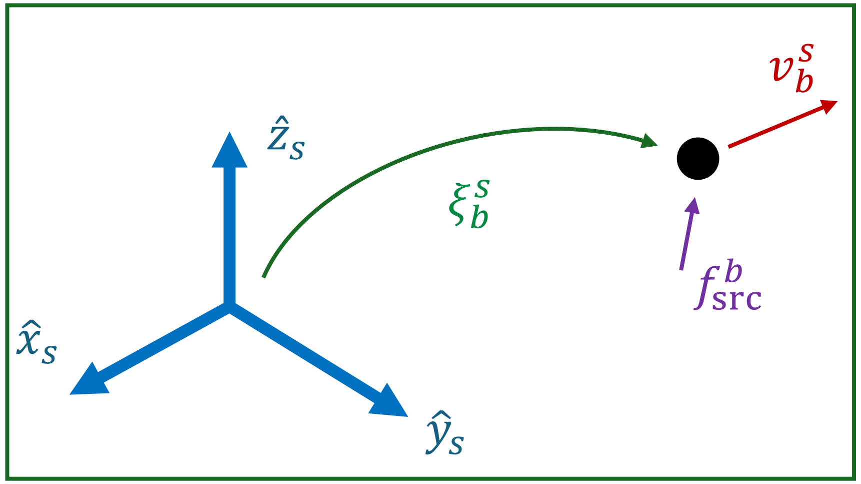
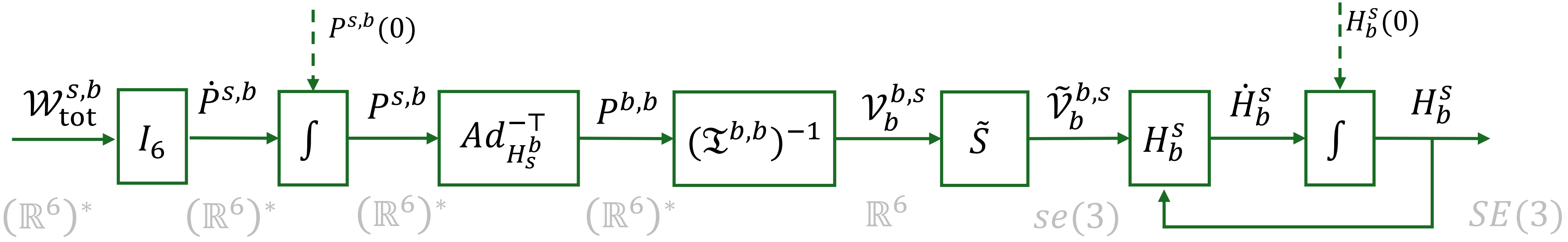
- $m\, \ddot{\xi}_b^s = f_{tot}^{s,b}$ — Familiar form

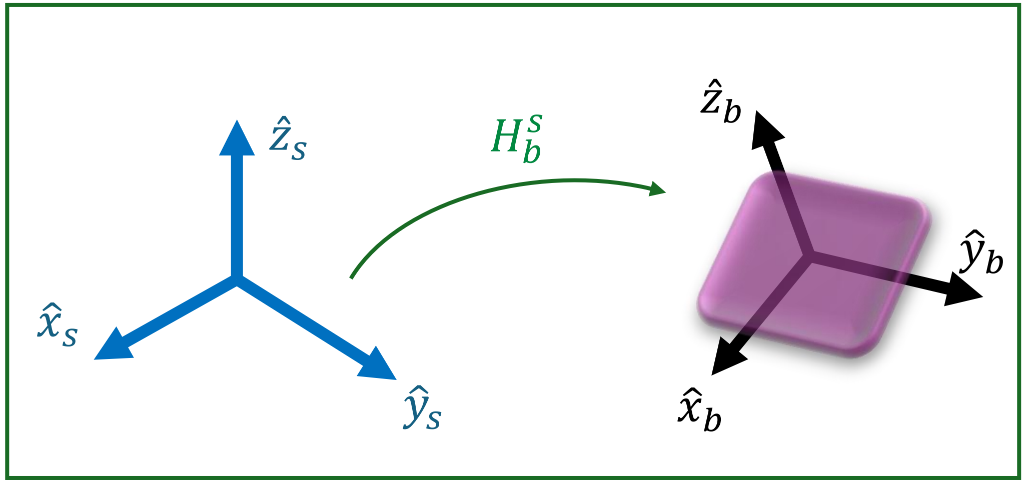


Forces on translating point mass

Recap: Rigid body dynamics



- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$ — Kinematic relation
- $\dot{P}^{s,b} = \mathcal{W}_{\text{tot}}^{s,b}$ — Momentum balance
- $\mathcal{V}_b^{b,s} = (\mathfrak{I}^{b,b})^{-1} P^{b,b}$ — Constitutive relation



$P^{*,b}$: generalized momentum of the body expressed in $\{*\}$
$\mathcal{W}_{\text{tot}}^{*,b}$: resultant wrench on body expressed in $\{*\}$
$\mathfrak{I}^{*,b}$: generalized inertia of the body expressed in $\{*\}$

Wrenches on moving rigid body

Outline

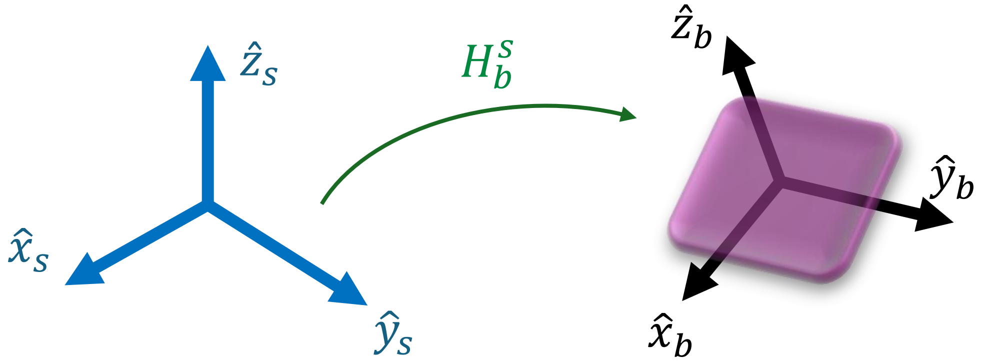
- Recap last lectures
- In depth analysis

Abstract concepts vs. numerical representations

- Abstract:
  - $\mathcal{V}_b^s$: generalized velocity of the body with respect to fixed world
  - P^b: generalized momentum of the body
  - $\mathcal{W}_{\text{tot}}^b$: resultant wrench on body
  - $\mathfrak{I}^b$: generalized inertia of the body

$$\mathcal{V}_b^s \in \mathbb{V}, \qquad P^b, \mathcal{W}_{\text{tot}}^b \in \mathbb{V}^*, \qquad \mathfrak{I}^b : \mathbb{V} \to \mathbb{V}^*$$

- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$
- $\dot{P}^{s,b} = \mathcal{W}_{\text{tot}}^{s,b}$
- $\mathcal{V}_b^{b,s} = (\mathfrak{I}^{b,b})^{-1} P^{b,b}$



Wrenches on moving rigid body

$\mathbb{V}$ as a vector space:
- $0 \in \mathbb{V}$
- $\mathcal{V}_1 + \mathcal{V}_2 \in \mathbb{V}$

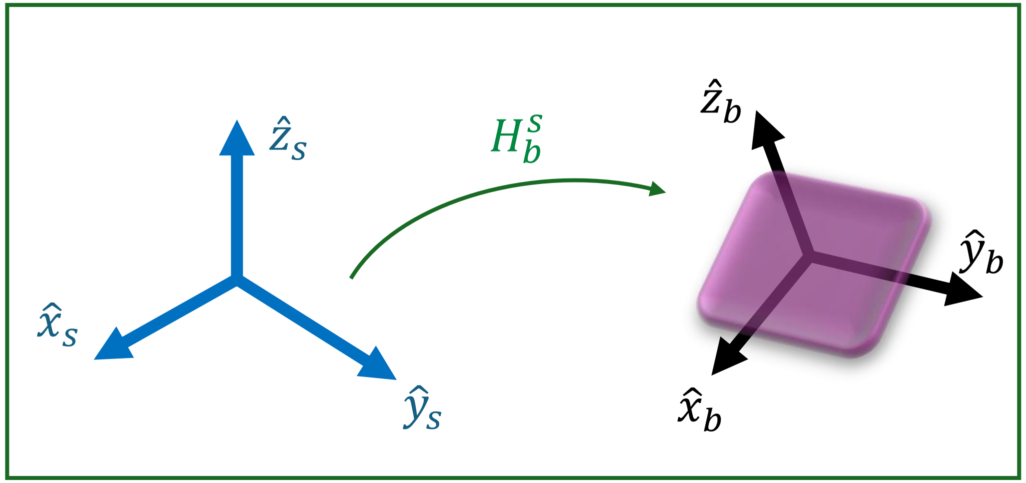
Abstract concepts vs. numerical representations

- Abstract:
  - $\mathcal{V}_b^s$: generalized velocity of the body with respect to fixed world
  - P^b: generalized momentum of the body
  - $\mathcal{W}_{\text{tot}}^b$: resultant wrench on body
  - $\mathfrak{I}^b$: generalized inertia of the body

$$\mathcal{V}_b^s \in \mathbb{V}, \qquad P^b, \mathcal{W}_{\text{tot}}^b \in \mathbb{V}^*, \qquad \mathfrak{I}^b : \mathbb{V} \to \mathbb{V}^*$$

  - $P^b := \mathfrak{I}^b(\mathcal{V}_b^s)$: definition of the momentum of the body
  - $\langle P^b | \mathcal{V}_b^s \rangle$: twice the kinetic energy of the body
  - $\langle \mathcal{W}_{\text{tot}}^b | \mathcal{V}_b^s \rangle$: the mechanical power due to wrench

- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$
- $\dot{P}^{s,b} = \mathcal{W}_{\text{tot}}^{s,b}$
- $\mathcal{V}_b^{b,s} = (\mathfrak{I}^{b,b})^{-1} P^{b,b}$



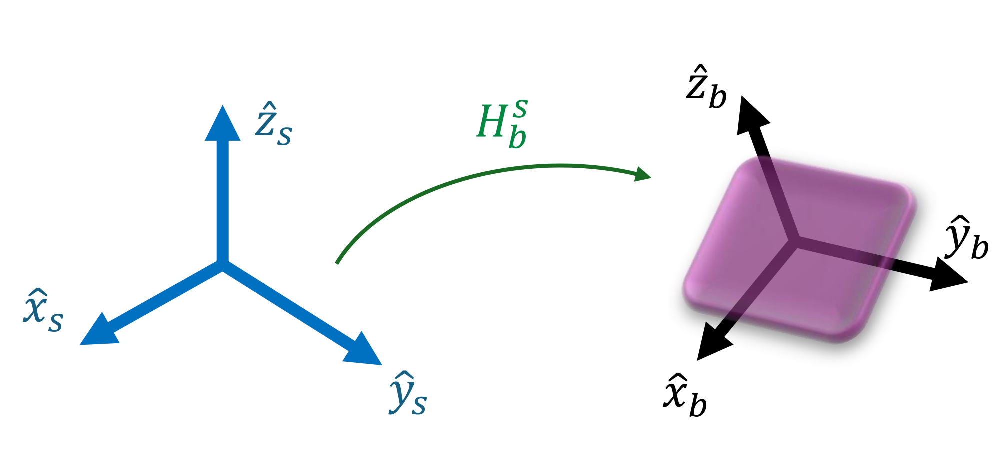
Wrenches on moving rigid body

$\langle \cdot | \cdot \rangle : \mathbb{V}^* \times \mathbb{V} \to \mathbb{R}$ Duality product/pairing

Abstract concepts vs. numerical representations

- Components in $\{m\}$:
  - $\mathcal{V}_b^{m,s} := [\mathcal{V}_b^s]_m \in \mathbb{R}^6$
  - $P^{m,b} := [P^b]_m \in (\mathbb{R}^6)^*$
  - $\mathcal{W}_{\text{tot}}^{m,b} := \left[\mathcal{W}_{\text{tot}}^b\right]_m \in (\mathbb{R}^6)^*$
  - $\mathfrak{I}^{m,b} := [\mathfrak{I}^b]_m : \mathbb{R}^6 \to (\mathbb{R}^6)^*$
- Change of basis from $\{m\}$ to $\{k\}$
  - $\mathcal{V}_b^{k,s} = \text{Ad}_{H_m^k} \mathcal{V}_b^{m,s}$
  - $P^{k,b} = \text{Ad}_{H_m^k}^{-\top} P^{m,b}$
  - $\mathcal{W}_{\text{tot}}^{k,b} = \text{Ad}_{H_m^k}^{-\top} \mathcal{W}_{\text{tot}}^{m,b}$
  - $\mathfrak{I}^{k,b} = \text{Ad}_{H_m^k}^{-\top} \mathfrak{I}^{m,b} \text{Ad}_{H_m^k}^{-1}$

- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$
- $\dot{P}^{s,b} = \mathcal{W}_{\text{tot}}^{s,b}$
- $\mathcal{V}_b^{b,s} = (\mathfrak{I}^{b,b})^{-1} P^{b,b}$



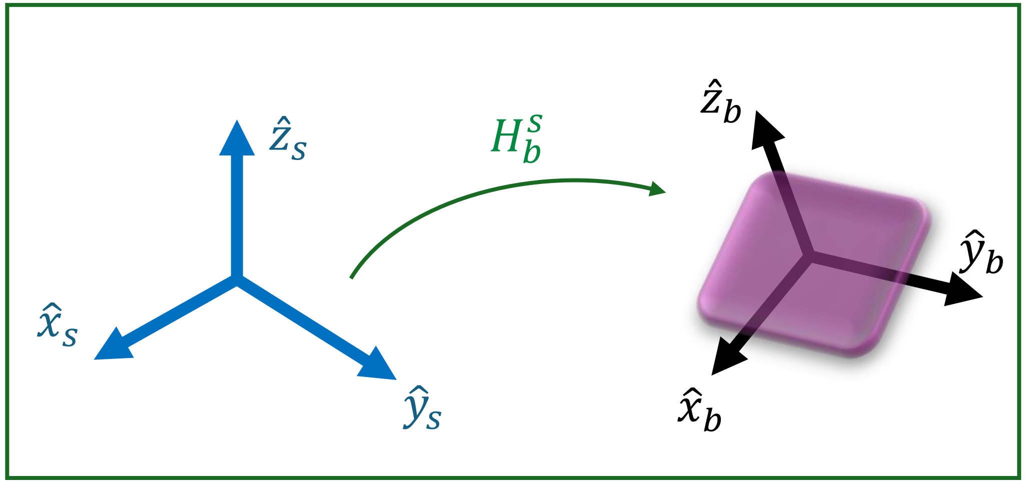
Wrenches on moving rigid body

Abstract concepts vs. numerical representations

- Components in $\{m\}$:
  - $\mathcal{V}_b^{m,s} := [\mathcal{V}_b^s]_m \in \mathbb{R}^6$
  - $P^{m,b} := [P^b]_m \in (\mathbb{R}^6)^*$
  - $\mathcal{W}_{\text{tot}}^{m,b} := [\mathcal{W}_{\text{tot}}^b]_m \in (\mathbb{R}^6)^*$
  - $\mathfrak{T}^{m,b} := [\mathfrak{T}^b]_m : \mathbb{R}^6 \to (\mathbb{R}^6)^*$

- Change of basis from $\{m\}$ to $\{k\}$
  - $\mathcal{V}_b^{k,s} = \text{Ad}_{H_m^k} \mathcal{V}_b^{m,s} = \text{Ad}_{H_k^m}^{-1} \mathcal{V}_b^{m,s}$
  - $P^{k,b} = \text{Ad}_{H_m^k}^{-\top} P^{m,b} = \text{Ad}_{H_k^m}^{\top} P^{m,b}$
  - $\mathcal{W}_{\text{tot}}^{k,b} = \text{Ad}_{H_m^k}^{-\top} \mathcal{W}_{\text{tot}}^{m,b} = \text{Ad}_{H_k^m}^{\top} \mathcal{W}_{\text{tot}}^{m,b}$
  - $\mathfrak{T}^{k,b} = \text{Ad}_{H_m^k}^{-\top} \mathfrak{T}^{m,b} \text{Ad}_{H_m^k}^{-1} = \text{Ad}_{H_k^m}^{\top} \mathfrak{T}^{m,b} \text{ Ad}_{H_k^m}$

- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$
- $\dot{P}^{s,b} = \mathcal{W}_{\text{tot}}^{s,b}$
- $\mathcal{V}_b^{b,s} = (\mathfrak{T}^{b,b})^{-1} P^{b,b}$



Wrenches on moving rigid body

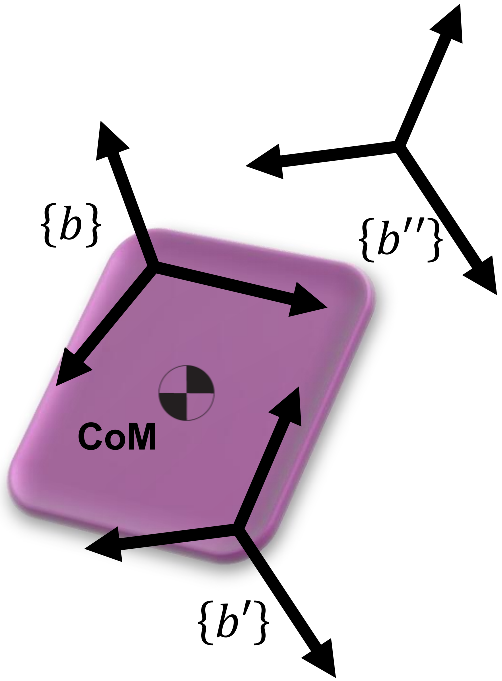
Generalized Inertia

- $\mathfrak{I}^{k,b}$ is always symmetric & positive definite
- $\mathfrak{I}^{k,b}$ is constant if $\{k\}$ is **any** body-fixed frame

$$\mathfrak{I}^{k,b} = \begin{pmatrix} J^{k,b} & m\,\tilde{\xi}^{k}_{\mathrm{cm}} \\ -m\tilde{\xi}^{k}_{\mathrm{cm}} & m\,I_3 \end{pmatrix}$$

- $\mathfrak{I}^{k,b}$ is block-diagonal if $\{k\}$ is at CoM

$$\mathfrak{I}^{k,b} = \begin{pmatrix} J^{k,b} & 0_{3\times 3} \\ 0_{3\times 3} & m\,I_3 \end{pmatrix}$$

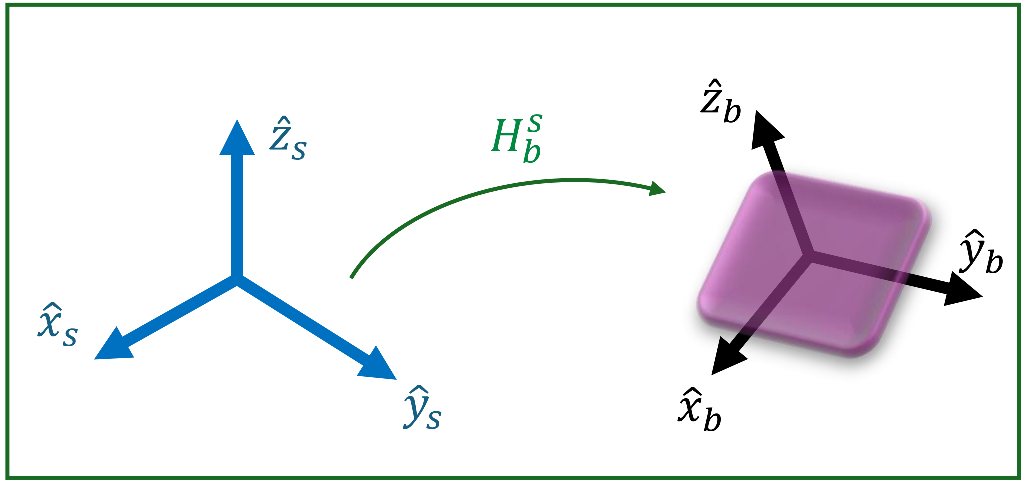


$$J^{k,b} = \begin{pmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{xy} & J_{yy} & J_{yz} \\ J_{xz} & J_{yz} & J_{zz} \end{pmatrix}$$: moment of inertia matrix

Up Next !

- How does the momentum balance equation look like in terms of $P^{b,b}$?
- Can we split rotational and translational dynamics ?
- How is this related to more common forms in the literature ?
- What is the twist ?

- $\dot{H}_b^s = H_b^s \tilde{\mathcal{V}}_b^{b,s}$
- $\dot{P}^{s,b} = \mathcal{W}_{\text{tot}}^{s,b}$
- $\mathcal{V}_b^{b,s} = (\mathfrak{I}^{b,b})^{-1} P^{b,b}$



Wrenches on moving rigid body