1) Momentum Balance in $\{b\}$:

$$\Rightarrow \boxed{\dot{P}^{s,b} = W^{s,b}} \quad (0)$$

$$\Rightarrow W^{s,b} = Ad^{T}_{H^b_s} W^{b,b} \quad (1)$$

$$P^{s,b} = Ad^{T}_{H^b_s} P^{b,b} \quad (2)$$

(2)&(1) $\Rightarrow$ (0)

$$\frac{d}{dt}\left(Ad^{T}_{H^b_s} P^{b,b}\right) = Ad^{T}_{H^b_s} W^{b,b}$$

$$\parallel$$

$$\frac{d}{dt}\left(Ad^{T}_{H^b_s}\right) P^{b,b} + Ad^{T}_{H^b_s} \dot{P}^{b,b}$$

$$\Downarrow \circledast Ad^{-T}_{H^b_s}$$

$$\boxed{\dot{P}^{b,b} = W^{b,b} - Ad^{-T}_{H^b_s} \frac{d}{dt}\left(Ad^{T}_{H^b_s}\right) P^{b,b}} \quad (3)$$

$$\Rightarrow \frac{d}{dt} Ad_{H_s^b} = Ad_{H_s^b}\, ad_{v_s^{s,b}} = Ad_{H_s^b}\, ad_{-v_b^{s,s}} = -Ad_{H_s^b}\, ad_{v_b^{s,s}} \quad (4)$$

$$\Rightarrow ad_{v_*^{m,o}} = Ad_{H_k^m}\, ad_{v_*^{k,o}}\, Ad_{H_m^k} \Rightarrow Ad_{H_m^k}\, ad_{v_*^{m,o}} = ad_{v_*^{k,o}}\, Ad_{H_m^k}$$

$* = k = b \,,\; o = m = s$

$$\boxed{Ad_{H_s^b}\, ad_{v_b^{s,s}} = ad_{v_b^{b,s}}\, Ad_{H_s^b}} \quad (5)$$

$\Rightarrow$ (5) in (4) :

$$\frac{d}{dt}\left(Ad_{H_s^b}\right) = -ad_{v_b^{b,s}}\, Ad_{H_s^b} \quad (6)$$

$\Rightarrow$ (6) in (3):

$$\dot{p}^{b,b} = W^{b,b} - Ad^{-T}_{H^b_s} \frac{d}{dt}\left(Ad^T_{H^b_s}\right) p^{b,b}$$

$$= W^{b,b} + \underbrace{Ad^{-T}_{H^b_s}\, Ad^T_{H^b_s}}_{I_6}\, ad^T_{v^{b,s}_b}\, p^{b,b}$$

$$\Downarrow$$

$$= W^{b,b} + ad^T_{v^{b,s}_b}\, p^{b,b}$$

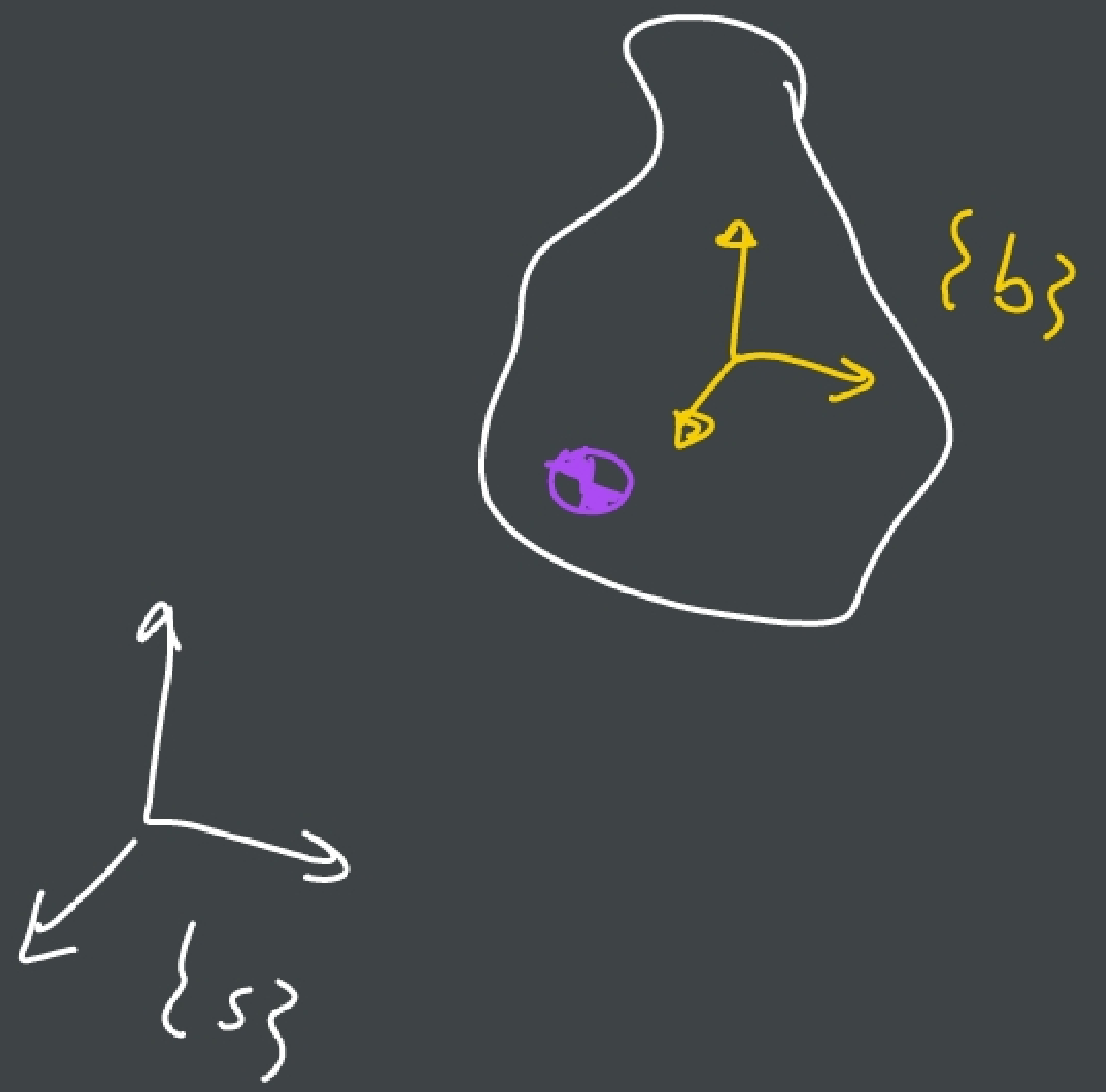
$\#$

* Summary :

* Rigid body dynamics in $\{b\}$

where $\{b\}$ is any arbitrary body-fixed frame

$$\boxed{\begin{aligned} \dot{H}_b^s &= H_b^s \tilde{\mathcal{V}}_b^{b,s} \\ P^{b,b} &= \underset{\sim}{I}^{b,b} \mathcal{V}_b^{b,s} \\ \dot{P}^{b,b} &= ad^T_{\mathcal{V}_b^{b,s}} P^{b,b} + \mathcal{W}^{b,b} \end{aligned}}$$



* Eliminate $p^{b,b}$

$$\boxed{\begin{aligned} \dot{H}_b^s &= H_b^s \tilde{\mathcal{V}}_b^{b,s} \\ \tilde{I}^{b,b} \dot{\mathcal{V}}_b^{b,s} &= \underbrace{ad^T_{\mathcal{V}_b^{b,s}} \tilde{I}^{b,b} \mathcal{V}_b^{b,s}}_{\text{bec. \{b\} is a moving frame}} + \mathcal{W}^{b,b} \end{aligned}}$$

Euler-Poincare Eqns.

* Split Rot. & Trans. $\{b\}$ @ CoM

$$\mathcal{I} = \begin{pmatrix} J & 0 \\ 0 & m I_3 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} \omega \\ v \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} \tau \\ f \end{pmatrix}, \quad H = \begin{pmatrix} R & \xi \\ 0_{1\times 3} & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \dot{R}^s_b &= R^s_b \, \tilde{\omega}^{b,s}_b \\ \dot{\xi}^s_b &= R^s_b \, v^{b,s}_b \\ J^{b,b} \dot{\omega}^{b,s}_b &= \overbrace{-\omega^{b,s}_b \wedge \left(J^{b,b} \omega^{b,s}_b\right)}^{\text{Gyroscopic torques}} + \tau^{b,b} \\ m \, \dot{v}^{b,s}_b &= \underbrace{-\omega^{b,s}_b \wedge \left(m \, v^{b,s}_b\right)}_{\text{Coriolis forces}} + f^{b,b} \end{aligned}}$$

$*$ let $\boxed{\bar{v} := R_b^s v_b^{b,s}}$

$$\dot{R}_b^s = R_b^s \tilde{\omega}_b^{b,s}$$

$$J^{b,b} \dot{\omega}_b^{b,s} = -\omega_b^{b,s} \wedge \left(J^{b,b} \omega_b^{b,s}\right) + \tau^{b,b}$$

$$\dot{\xi}_b^s = \bar{v}$$

$$m \dot{\bar{v}} = R_b^s f^{b,b}$$

$$\boxed{\dot{\xi}_b^s = \bar{v} \neq v_b^{s,s}}$$

$$\begin{pmatrix} \omega_b^{b,s} \\ \dot{\xi}_b^s \end{pmatrix} \neq$$

not a twist !!

$$\mathcal{V}_b^{s,s} = Ad_{H_b^s} \mathcal{V}_b^{b,s}$$

$$\begin{pmatrix} \omega_b^{s,s} \\ v_b^{s,s} \end{pmatrix} = \begin{pmatrix} R & 0 \\ \tilde{\xi} R & R \end{pmatrix} \begin{pmatrix} \omega_b^{b,s} \\ v_b^{b,s} \end{pmatrix} \Longrightarrow v_b^{s,s} = R_b^s v_b^{b,s} + \tilde{\xi}_b^s R_b^s \omega_b^{b,s}$$

$$= \bar{v} + \xi_b^s \wedge \omega_b^{s,s}$$

$$\boxed{v_b^{s,s} = \dot{\xi}_b^s + \xi_b^s \wedge \omega_b^{s,s}}$$

$$\boxed{v_b^{s,s} \neq \bar{v}}$$