SCE 594: Special Topics in Intelligent Automation & Robotics

Topic 3: Fixed-base manipulator modeling

Lecture 13: Modeling Ideal Joints

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Outline

- Motivation and Terminology
- Displacement Subgroups
- Exponential map of a Lie group

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- Motivation and Terminology
- Displacement Subgroups
- Exponential map of a Lie group

Motivation

- In Topic 3, we will be dealing with the kinematic & dynamic modeling of fixed-base open chain manipulators.

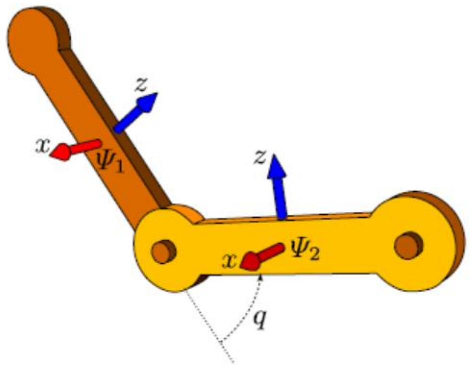
Fixed-base open chain

Fixed-base closed chain

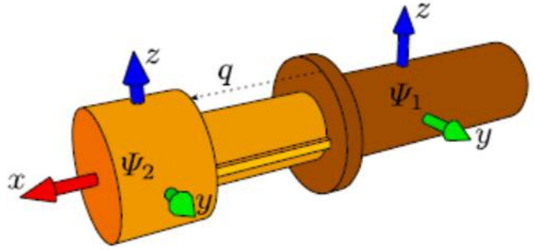
Floating-base open chain

Motivation

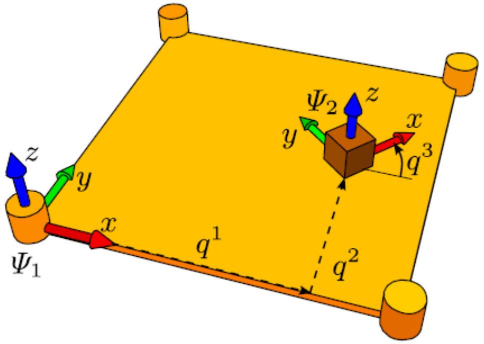
- Majority of robotic systems can be modeled as several rigid bodies connected by ideal joints.
- An ideal joint (aka kinematic pair) is a purely kinematic relation between two rigid bodies restricting the relative twist $\mathcal{V}_1^{*,2}$.



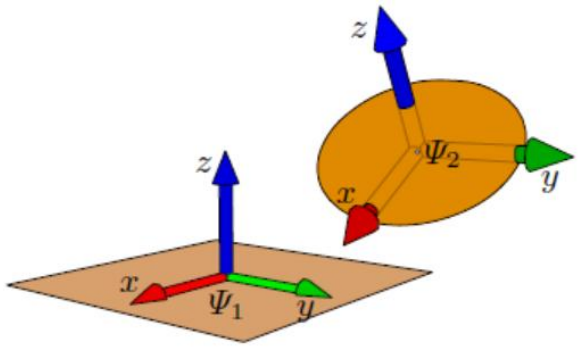
Revolute joint (1 DoF)



Prismatic joint (1 DoF)



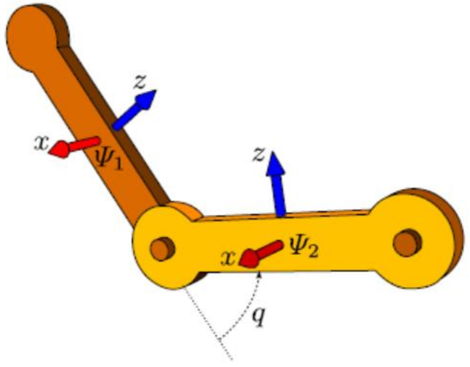
Planar joint (3 DoF)



Free motion (6 DoF)

Motivation

- Majority of robotic systems can be modeled as several rigid bodies connected by ideal joints.
- An ideal joint (aka kinematic pair) is a purely kinematic relation between two rigid bodies restricting the relative twist $\mathcal{V}_1^{*,2}$.
- The degrees of freedom (DoF) of a joint is the number of independent coordinates of $\mathcal{V}_1^{*,2}$.



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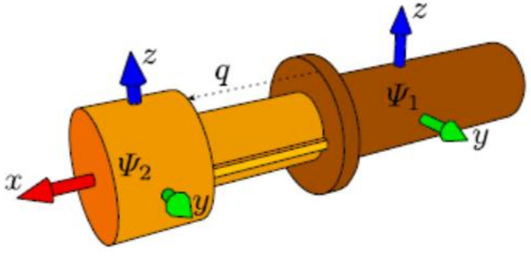

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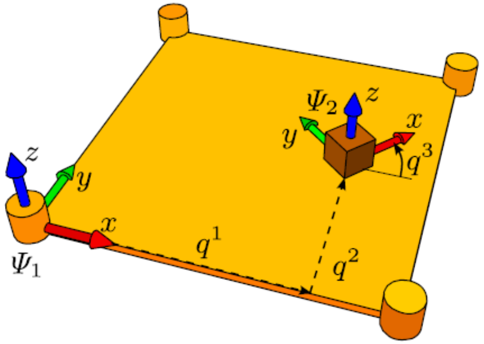

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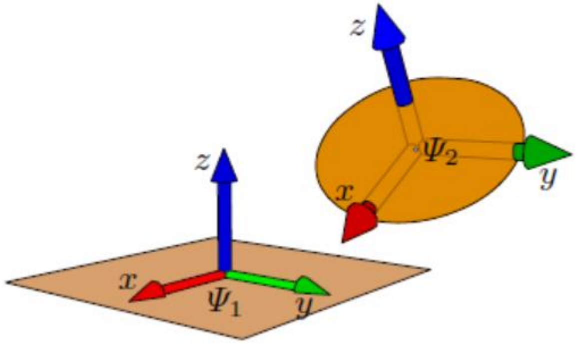
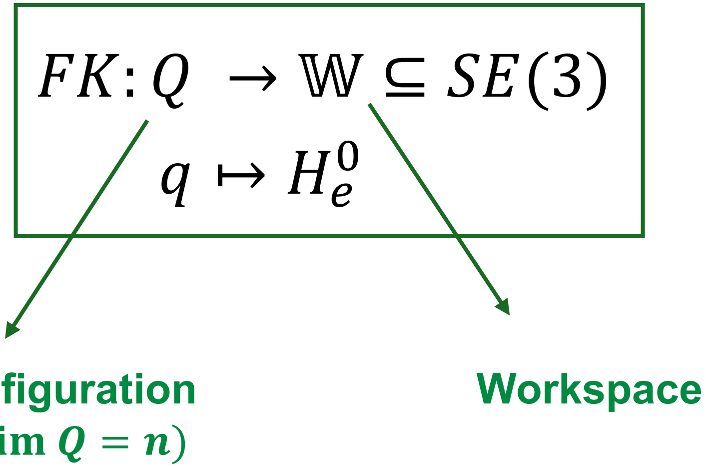


Free motion (6 DoF)

Forward Kinematics

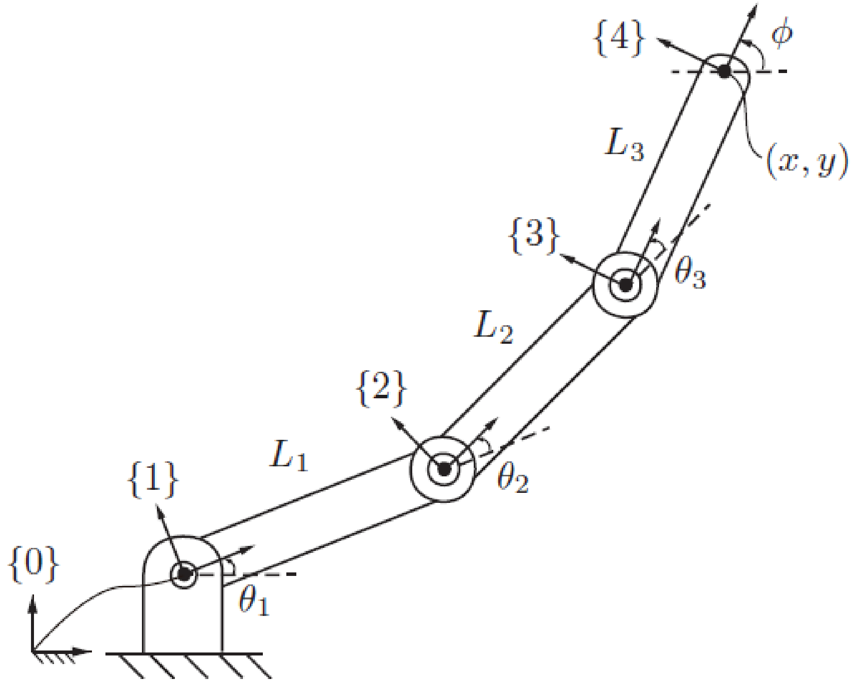
- The forward kinematics of a robot refers to the calculation of the position and orientation of its end-effector frame given:
  - The robot's geometry (link lengths, joint types, etc.)
  - The joint configurations $q \coloneqq (q_1, \dots, q_n) \in Q$
- Mathematically, it is the map



$$FK: Q \to \mathbb{W} \subseteq SE(3)$$
$$q \mapsto H_e^0$$

Joint/Configuration Space $(\dim Q = n)$

Workspace

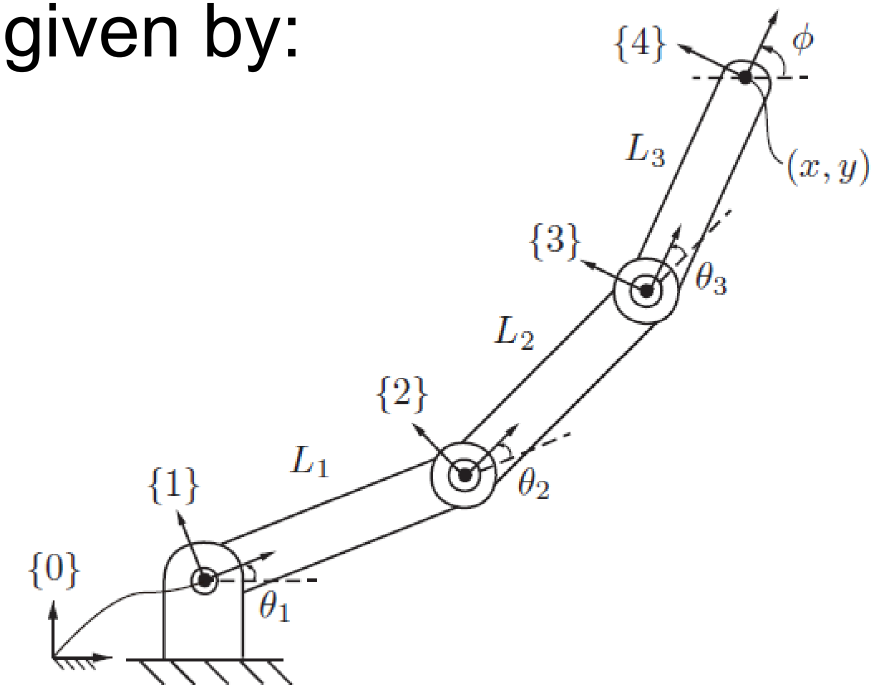


Forward Kinematics

- The forward kinematics of a robot refers to the calculation of the position and orientation of its end-effector frame given:
  - The robot’s geometry (link lengths, joint types, etc.)
  - The joint configurations $q \coloneqq (q_1, \dots, q_n) \in Q$
- For example, consider the 3R planar open chain below.
- The pose of the end-effector’s frame $\{4\}$ is given by:

$$H_4^0 = \begin{pmatrix} c_\phi & -s_\phi & 0 & x \\ s_\phi & c_\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3), \\ y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3), \\ \phi &= \theta_1 + \theta_2 + \theta_3. \end{aligned}$$



Differential/Velocity Kinematics

- The forward kinematics defines a map from the joint configuration $q \in Q$ to the end effector pose $H_e^0 \in \mathbb{W} \subseteq SE(3)$.
- The differential/velocity kinematics is the differential of this map which is a mapping from $\dot{q} \in T_q Q$ to $\dot{H}_e^0 \in T_{H_e^0} SE(3)$.

Differential/Velocity Kinematics

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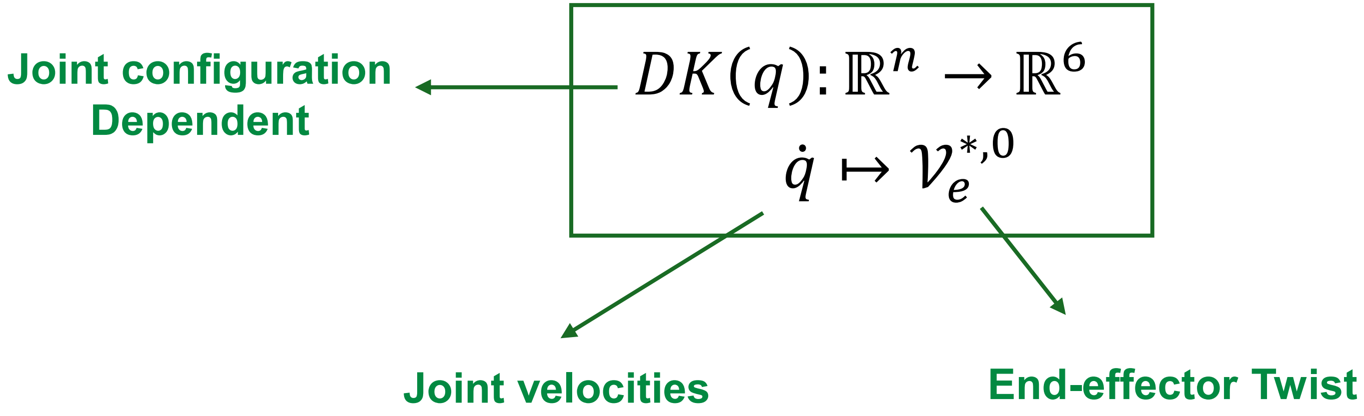
- Geometrically, we can equivalently represent it as a map from $\dot{q} \in T_q Q \cong \mathbb{R}^n$ to the end effector's twist $\mathcal{V}_e^{*,0} \in \mathbb{R}^6$



$$\boxed{\begin{aligned} DK(q)&: \mathbb{R}^n \to \mathbb{R}^6 \\ \dot{q} &\mapsto \mathcal{V}_e^{*,0} \end{aligned}}$$

Topic 3 plan

- Lecture 13: Modeling ideal joints
- Lecture 14: Forward and Differential Kinematics
- Lecture 15 & 16: Dynamics

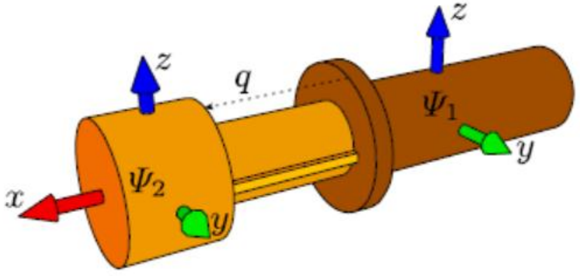
Outline

- Motivation and Terminology
- Displacement Subgroups
- Exponential map of a Lie group

Displacement Subgroup

- Subgroups of $SE(3)$ represent constrained displacements of a rigid body that arise due to the presence of joints.
- Consider a joint connecting two rigid bodies with frames Ψ_i and Ψ_j.
- The displacement subgroup of $SE(3)$ associated with the joint connecting body i to body j is denoted by

$$G_i^j \subseteq SE(3)$$



Ψ_i : Child frame
Ψ_j: Parent frame

b DoF joint constraints $n - b$ DoFs

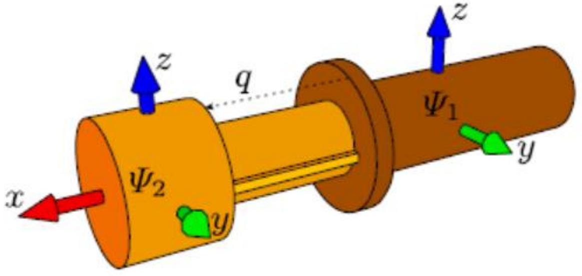
Displacement Subgroup

- Subgroups of $SE(3)$ represent constrained displacements of a rigid body that arise due to the presence of joints.
- Consider a joint connecting two rigid bodies with frames Ψ_i and Ψ_j.
- The displacement subgroup of $SE(3)$ associated with the joint connecting body i to body j is denoted by

$$G_i^j \subseteq SE(3)$$

Represents the set of all possible relative poses of body i with respect to body j that are allowed by the joint.

G_i^j is a b-dimensional Lie subgroup of $SE(3)$



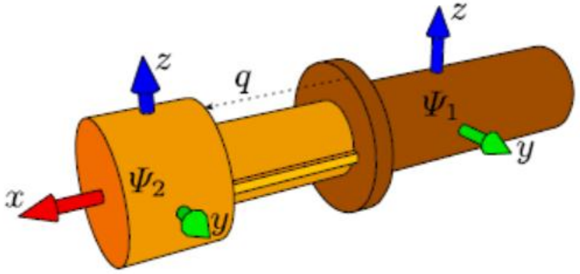
b DoF joint constraints $n - b$ DoFs

Configuration of Joint

- The joint configuration of joint i connecting body i to body j is denoted by

$$q_i \in G_i$$

- G_i : The configuration manifold of joint i
- G_i is a b-dimensional Lie group isomorphic to G_i^j



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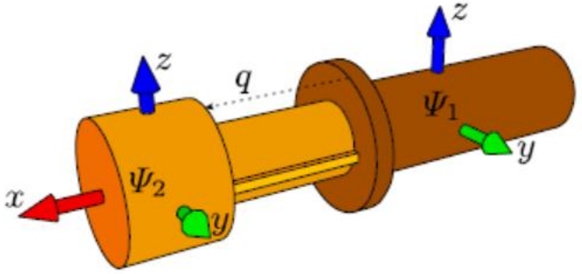
$$\boxed{\begin{aligned} \varphi_i : G_i \ &\rightarrow G_i^j \\ q_i &\mapsto H_i^j \end{aligned}}$$

Joint configuration

Relative pose of body i w.r.t. body j

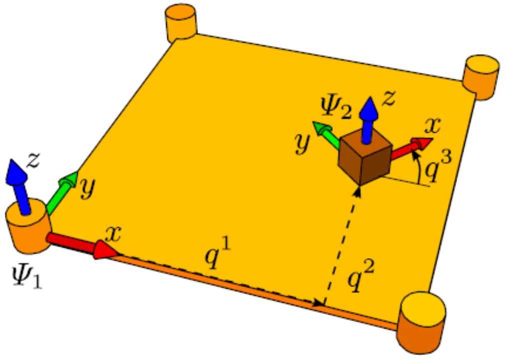


b DoF joint constraints $n - b$ DoFs

Example: Planar Joint

- A planar joint allows for 2 translational DoFs and 1 rotational DoF about an axis $\hat{n}_i \in \mathbb{S}^2$ normal to the plane of motion.
- In this case,
  - $b = 3$
  - The relative pose $\mathrm{H}_i^j \in G_i^j = SE(2)$
  - The joint configuration $q_i = (q^1, q^2, q^3) \in \mathbb{R}^2 \times (-\pi, \pi] =: G_i$
  - The map $\varphi_i : q_i \mapsto \mathrm{H}_i^j$ is given by

$$H_i^j := \varphi_i(q_i) = \begin{pmatrix} c_{q^3} & -s_{q^3} & 0 & q^1 \\ s_{q^3} & c_{q^3} & 0 & q^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Planar joint (3 DoF)
$\mathbf{\Psi_1 = \Psi_j\,,\ \ \Psi_2 = \Psi_i}$

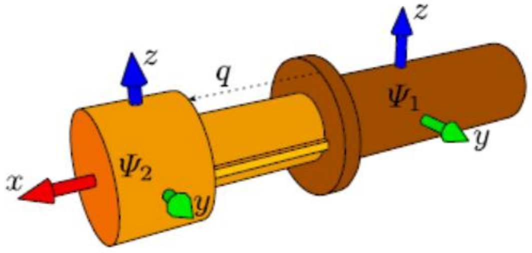
$\hat{n}_i = \hat{z}_2$ in this example

Example: Prismatic Joint

- A planar joint allows for 1 translational DoF along an axis $\hat{n}_i \in \mathbb{S}^2$

- In this case,
  - $b = 1$
  - The relative pose $\mathrm{H}_i^j \in G_i^j = \mathrm{Tr}(3)$
  - The joint configuration $q_i \in \mathbb{R} =: G_i$ is the linear displacement
  - The map $\varphi_i : q_i \mapsto \mathrm{H}_i^j$ is given by

$$H_i^j \coloneqq \varphi_i(q_i) = \begin{pmatrix} 1 & 0 & 0 & q_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



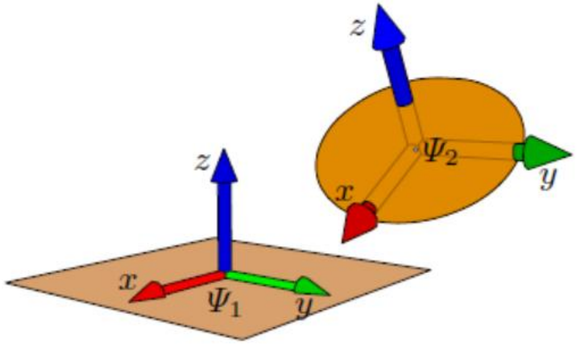
Prismatic joint (1 DoF)
$\boldsymbol{\Psi_1 = \Psi_j}$, $\boldsymbol{\Psi_2 = \Psi_i}$

$\hat{n}_i = \hat{x}_2$ in this example

Example: Floating Joint

- A floating joint is a degenerate case.

- In this case,
  - $b = 6$
  - The relative pose $\mathrm{H}_i^j \in G_i^j = SE(3)$
  - The joint configuration $q_i \in SE(3) =: G_i$ is full pose
  - The map $\varphi_i : q_i \mapsto \mathrm{H}_i^j$ is given by the identity map.



Floating Joint (6 DoF)

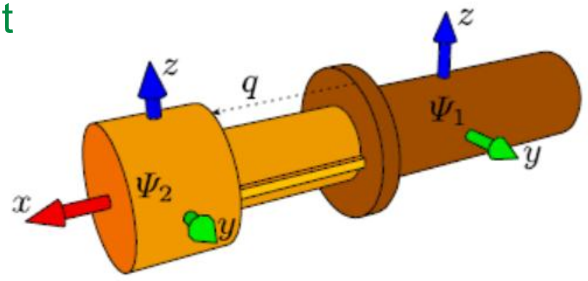
Velocity of Joint

- The joint configuration is mapped to the relative pose by

$$\begin{aligned} \varphi_i : G_i &\to G_i^j \\ q_i &\mapsto H_i^j \end{aligned}$$

- The induced map on the velocities is given by

$$\begin{aligned} \iota_i : \overbrace{\mathfrak{g}_i}^{\text{Lie algebra of } G_i} &\to \overbrace{\mathfrak{g}_i^j}^{\text{Lie algebra of } G_i^j} \\ \underbrace{\mathrm{v}_i}_{\text{Joint velocity}} &\mapsto \underbrace{\mathcal{V}_i^{i,j}}_{\text{Relative twist allowed by the joint}} \end{aligned}$$



b DoF joint constraints $n-b$ DoFs

Velocity of Joint

- The joint configuration is mapped to the relative pose by

$$\begin{aligned}\varphi_i : G_i &\to G_i^j \\ q_i &\mapsto H_i^j\end{aligned}$$

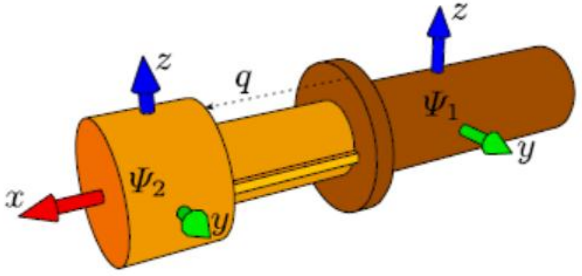
- The induced map on the velocities is given by

$$\begin{aligned}\iota_i : \mathfrak{g}_i &\to \mathfrak{g}_i^j \\ \mathrm{v}_i &\mapsto \mathcal{V}_i^{i,j}\end{aligned}$$

$\mathcal{S}_i^{i,j} \in \mathbb{R}^{6\times b}$

$$\boxed{\mathcal{V}_i^{i,j} = \mathcal{S}_i^{i,j}\mathrm{v}_i}$$

$\mathfrak{g}_i^j \cong \mathbb{R}^6$ $\mathfrak{g}_i \cong \mathbb{R}^b$



b DoF joint constraints $n - b$ DoFs

Velocity of Joint

- The joint configuration is mapped to the relative pose by

$$\varphi_i : G_i \to G_i^j$$

$$q_i \mapsto H_i^j$$

- The induced map on the velocities is given by

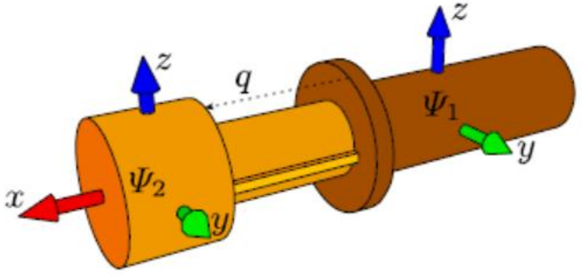
$$\iota_i : \mathfrak{g}_i \to \mathfrak{g}_i^j$$

$$\mathrm{v}_i \mapsto \mathcal{V}_i^{i,j}$$

- The joint velocity is related to its configuration by

$$\chi_{q_i} : \mathfrak{g}_i \to T_{q_i} G_i$$

$$\mathrm{v}_i \mapsto \dot{q}_i$$



b DoF joint constraints $n - b$ DoFs

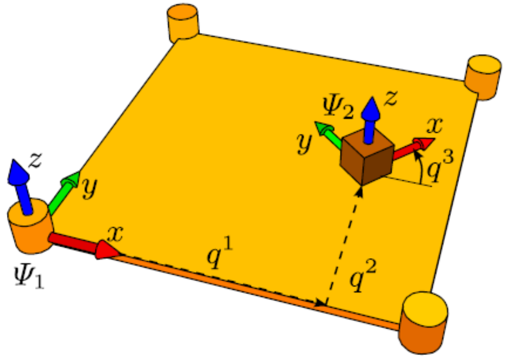
Example: Planar Joint

- In this case,
  - The joint configuration $q_i = (q^1, q^2, q^3) \in \mathbb{R}^2 \times (-\pi, \pi] =: G_i$
  - The joint velocity $\mathrm{v}_i = (v^x, v^y, \omega^z) \in \mathbb{R}^3 \cong \mathfrak{g}_i$
  - The map $\iota_i : \mathrm{v}_i \mapsto \mathcal{V}_i^{i,j}$ is given by

$$\mathcal{V}_i^{i,j} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v^x \\ v^y \\ \omega^z \end{pmatrix}$$

  - The map $\chi_{q_i} : \mathrm{v}_i \mapsto \dot{q}_i$ is given by

$$\dot{q}_i = \begin{pmatrix} c_{q^3} & -s_{q^3} & 0 \\ s_{q^3} & c_{q^3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^x \\ v^y \\ \omega^z \end{pmatrix}$$

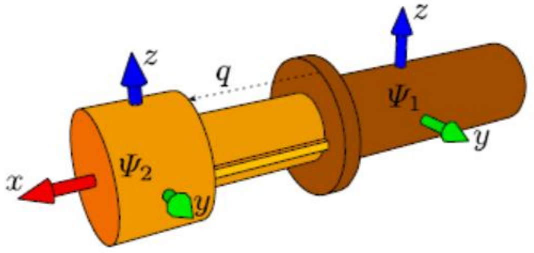


Planar joint (3 DoF)
$\mathbf{\Psi_1 = \Psi_j}$, $\mathbf{\Psi_2 = \Psi_i}$

Example: Prismatic Joint

- In this case,
  - The joint configuration $q_i \in \mathbb{R} =: G_i$ is the linear displacement
  - The joint velocity $\mathrm{v}_i = \dot{q}_i \in \mathbb{R}^1 \cong \mathfrak{g}_i$. Therefore, χ_{q_i} is identity.
  - The map $\iota_i : \mathrm{v}_i \mapsto \mathcal{V}_i^{i,j}$ is given by

$$\mathcal{V}_i^{i,j} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mathrm{v}_i$$

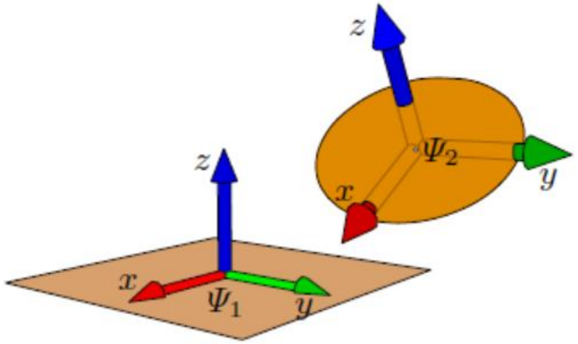


Prismatic joint (1 DoF)
$\boldsymbol{\Psi_1 = \Psi_j\,,\ \Psi_2 = \Psi_i}$

Example: Floating Joint

- In this case,
  - The joint configuration $q_i = H_i^j \in SE(3) =: G_i$ is full pose
  - The joint velocity $\mathrm{v}_i = \mathcal{V}_i^{i,j} \in \mathbb{R}^6 \cong \mathfrak{g}_i$. Therefore, $\iota_i$is identity.
  - The map $\chi_{q_i}: \mathcal{V}_i^{i,j} \mapsto \dot{H}_i^j$is given by

$$\dot{H}_i^j = \left(H_i^j\right)^{-1} \tilde{S}\left(\mathcal{V}_i^{i,j}\right) = H_j^i \tilde{\mathcal{V}}_i^{i,j}$$



Floating Joint (6 DoF)

Up Next: Exponential Map

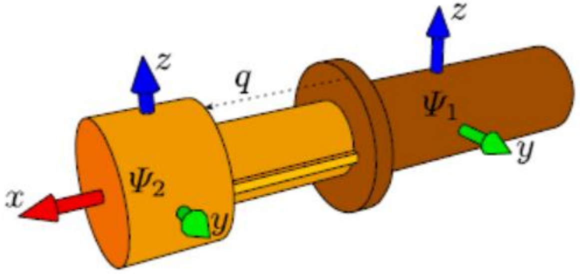
- The maps

$$\boxed{\begin{aligned}\varphi_i : G_i &\to G_i^j \\ q_i &\mapsto H_i^j\end{aligned}}$$

$$\boxed{\begin{aligned}\iota_i : \mathfrak{g}_i &\to \mathfrak{g}_i^j \\ \mathrm{v}_i &\mapsto \mathcal{V}_i^{i,j}\end{aligned}}$$

can be geometrically (and elegantly) represented using the exponential map of a Lie group.

- **Advantages:**
  - Unifies revolute and prismatic joints
  - Avoids any pre-defined choices of body-fixed frames
  - Makes forward kinematics clean and modular
  - Connects naturally to differential kinematics



b DoF joint constraints $n - b$ DoFs

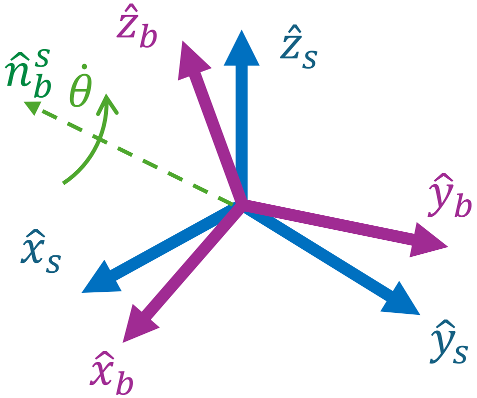
Outline

- Motivation and Terminology
- Displacement Subgroups
- Exponential map of a Lie group
    - SO(3)
    - SE(3)

Exponential coordinates of rotations

- Suppose that a frame $\{\mathrm{b}\}$ with unit axes $\{\hat{x}_b, \hat{y}_b, \hat{z}_b\}$ is attached to a rotating body.
- The angular velocity vector describing the instantaneous rotation of $\{\mathrm{b}\}$ relative to $\{\mathrm{s}\}$ is given by $\omega_b^s := \hat{n}_b^s \dot{\theta}$.
- The instantaneous rate of change of $R_b^s \in SO(3)$ is given by

$$\boxed{\dot{R}_b^s = \tilde{\omega}_b^{s,s} R_b^s}$$



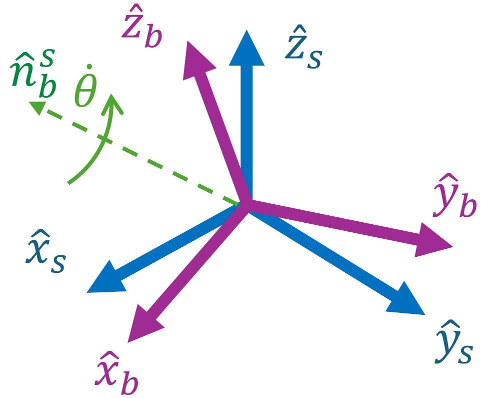
Exponential coordinates of rotations

- Assume that $\omega_b^{s,s} = \hat{n}_b^{s,s}$ is a constant unit vector with $\dot{\theta} = 1$ and thus $\widetilde{\omega}_b^{s,s}$ is also constant.
- The solution to $\dot{R}_b^s = \widetilde{\omega}_b^{s,s} R_b^s$ has a closed-form:

$$\boxed{R_b^s(t) = e^{\tilde{n}_b^{s,s} t} R_b^s(0)}$$

$$\tilde{n}_b^{s,s} \coloneqq S\left(\hat{n}_b^{s,s}\right) \in so(3)$$

- Intuitively, this is the rotation matrix reached from $R_b^s(0)$ by rotating it around $\widehat{\omega}$ at a constant 1 rad/s for t seconds.



Exponential coordinates of rotations

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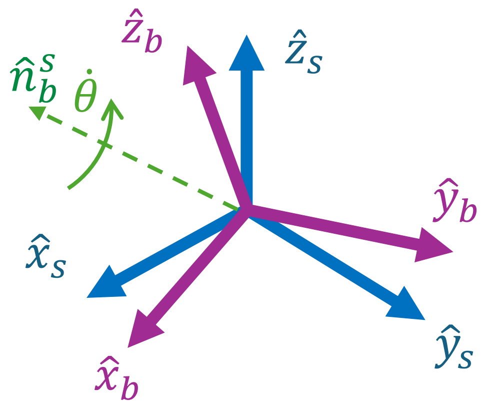


Note that since $\dot{\theta} = 1$, we have that

$$\theta(t) = \int_0^t \dot{\theta}\ dt = \int_0^t dt = t$$

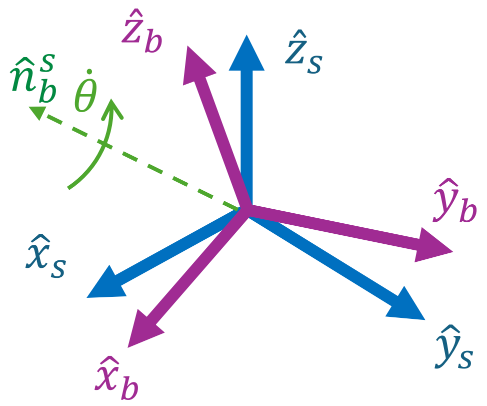
Exponential coordinates of rotations

- By replacing t with θ, we have that

$$\boxed{R_b^s(\theta) = e^{\tilde{n}_b^{s,s}\theta} R_b^s(0)}$$

which is intuitively the rotation matrix achieved by rotating $R_b^s(0)$ around the axis $\hat{n}_b^{s,s}$ by an angle θ.

The pair $(\hat{n}_b^s, \theta) \in \mathbb{S}^2 \times [-\pi, \pi]$ is called the exponential coordinates or axis-angle representation of a rotation matrix $R \in SO(3)$



Exponential coordinates of rotations

- Alternatively, we can either use

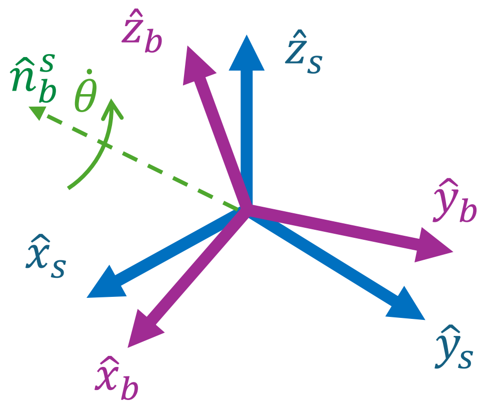
$$\boxed{R_b^s(\theta) = e^{\tilde{n}_b^{s,s}\theta} R_b^s(0)}$$

$$\tilde{n}_b^{s,s} := S\left(\hat{n}_b^{s,s}\right)$$

or

$$\boxed{R_b^s(\theta) = R_b^s(0) e^{\tilde{n}_b^{b,s}\theta}}$$

$$\tilde{n}_b^{b,s} := S\left(\hat{n}_b^{b,s}\right)$$



Exponential map of SO(3)

- The exponential map

$$\exp\colon so(3) \ \to SO(3)$$
$$\tilde{n}\theta \mapsto e^{\tilde{n}\theta}$$

maps elements of the Lie algebra to elements of the Lie group.

- It is surjective, but not injective
  - $\forall\ R \in SO(3),\ \ \exists \tilde{n} \in so(3)\ \text{s.t.}\ R = \mathrm{e}^{\tilde{n}}$
  - $\exists \hat{n}_1\theta_1, \hat{n}_2\theta_2 \in \mathbb{R}^3 \text{ with } \hat{n}_1\theta_1 \neq \hat{n}_2\theta_2 \text{ s.t. } \mathrm{e}^{\tilde{n}_1\theta_1} = \mathrm{e}^{\tilde{n}_2\theta_2}$

Exponential map of SO(3)

- The exponential map

$$\exp\colon so(3) \rightarrow SO(3)$$

$$\tilde{n}\theta \mapsto e^{\tilde{n}\theta}$$

maps elements of the Lie algebra to elements of the Lie group.

Given any pair $(\hat{n}, \theta) \in \mathbb{S}^2 \times (-\pi, \pi]$, the matrix exponential $e^{\tilde{n}\theta} \in SO(3)$ of $\tilde{n}\theta \in so(3)$, with $\tilde{n} \coloneqq S(\hat{n})$, is given by:

$$e^{\tilde{n}\theta} = I_3 + \sin\theta\,\tilde{n} + (1 - \cos\theta)\tilde{n}^2$$

Rodrigues's formula

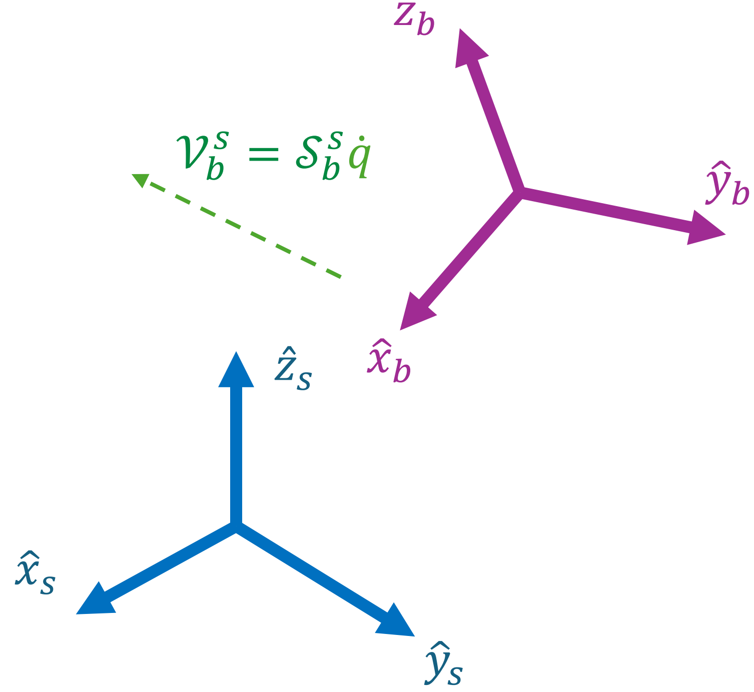
For the proof, see Pg. 83-84 of Lynch & Park

Outline

- Motivation and Terminology
- Displacement Subgroups
- Exponential map of a Lie group
  - SO(3)
  - SE(3)

Screw interpretation of Twists

- Just as an angular velocity ω_b^s can be viewed as $\hat{n}_b^s \dot{\theta} \in \mathbb{R}^3$ where $\hat{n}_b^s$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about it,
- A twist $\mathcal{V}_b^s$ can interpreted in terms of a screw axis $\mathcal{S}_b^s \in \mathbb{R}^6$ and a velocity $\dot{q}$ about the screw axis.



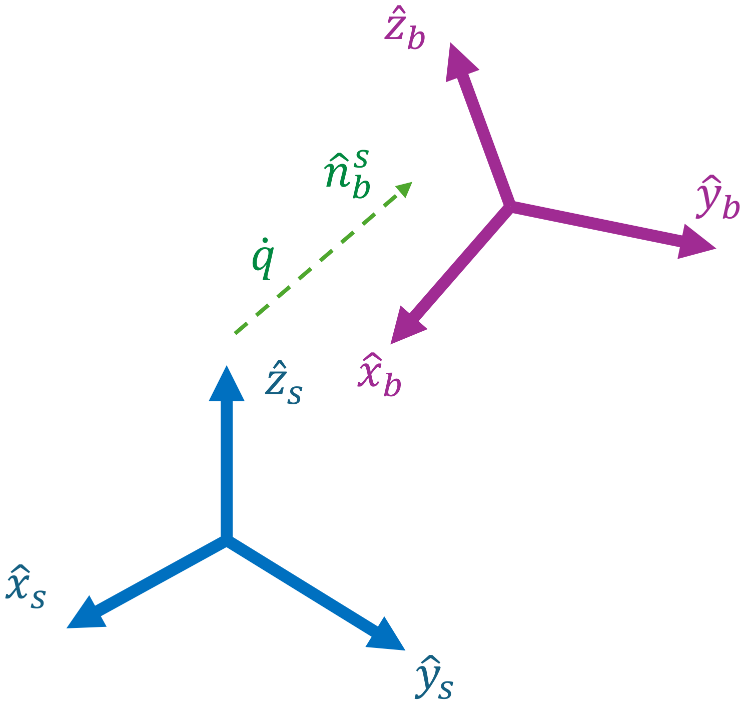
Screw interpretation of Twists

- We define the screw axis $\mathcal{S}_b^s$ and $\dot{q}$ in the following manner:
  a) Pure translation

$$\mathcal{S}_b^s = \begin{pmatrix} 0 \\ \hat{n}_b^s \end{pmatrix} \in \mathbb{R}^6,$$

$\hat{n}_b^s$ is the translation axis

$\dot{q}$ is the linear velocity along the screw axis



Screw interpretation of Twists

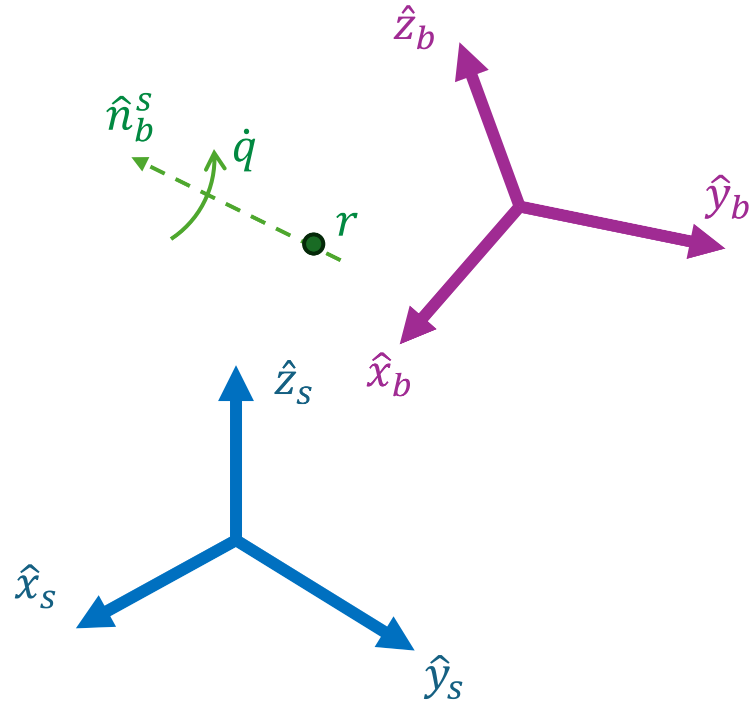
- We define the screw axis $\mathcal{S}_b^s$ and $\dot{q}$ in the following manner:
  b) Pure rotation

$$\mathcal{S}_b^s = \begin{pmatrix} \hat{n}_b^s \\ -\hat{n}_b^s \wedge r \end{pmatrix} \in \mathbb{R}^6,$$

$\hat{n}_b^s$ is the rotation axis

$\dot{q}$ is the angular velocity around the screw axis

r is any point on the screw axis



For a third type (screw motion), see Pg. 102-04 of Lynch & Park

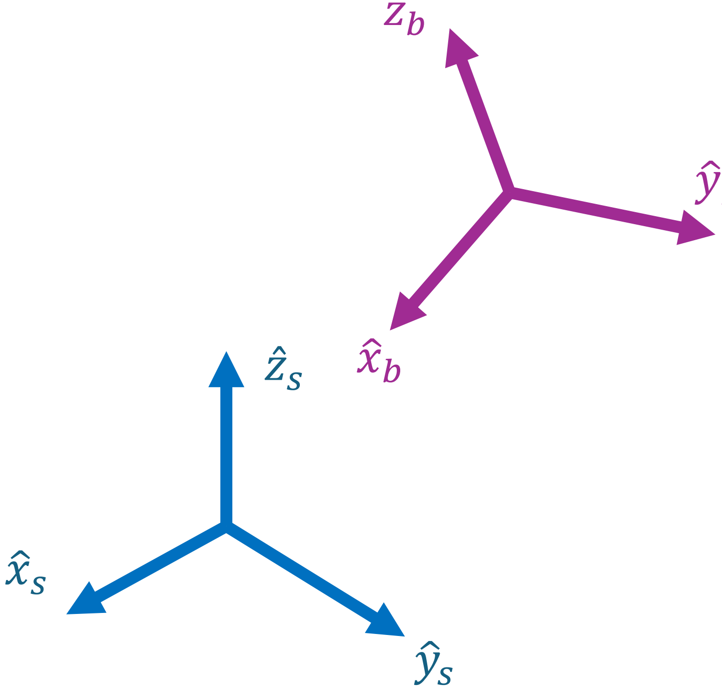
Exp. coordinates of homogeneous transformations

- By analogy to the exponential coordinates $(\hat{n}_b^s, \theta)$ for rotations, we can define the six-dimensional exponential coordinates of a homogeneous transformation H_b^s by $(\mathcal{S}_b^s, q)$ such that

$$\boxed{H_b^s(q) = e^{\tilde{\mathcal{S}}_b^{s,s} q} H_b^s(0)}$$

or

$$\boxed{H_b^s(q) = H_b^s(0) e^{\tilde{\mathcal{S}}_b^{b,s} q}}$$



$\tilde{\mathcal{S}}_b^{*,s} \in se(3)$ is $\mathcal{S}_b^s$ expressed in $\{*\}$

Exponential map of SE(3)

- The exponential map

$$\begin{aligned}\exp\colon se(3) &\to SE(3)\\ \tilde{S}q &\mapsto e^{\tilde{S}q}\end{aligned}$$

maps elements of the Lie algebra to elements of the Lie group.

- It is surjective, but not injective

Exponential map of SE(3)

- The exponential map

$$\begin{aligned} \exp\colon se(3) &\to SE(3) \\ \tilde{\mathcal{S}}q &\mapsto e^{\tilde{\mathcal{S}}q} \end{aligned}$$

maps elements of the Lie algebra to elements of the Lie group.

<u>Case 1: (Pure translation)</u>

Let $\mathcal{S} := (0, v) \in \mathbb{R}^6$ be a screw axis with $\|v\| = 1$. Then for any $q \in \mathbb{R}$ along that screw axis, we have that

$$e^{\tilde{\mathcal{S}}q} = \begin{pmatrix} I & vq \\ 0 & 1 \end{pmatrix}$$

Exponential map of SE(3)

- The exponential map

$$\begin{aligned} \exp : se(3) &\to SE(3) \\ \tilde{\mathcal{S}} q &\mapsto e^{\tilde{\mathcal{S}} q} \end{aligned}$$

maps elements of the Lie algebra to elements of the Lie group.

Case 2: (Pure rotation)

Let $\mathcal{S} := (\omega, v) \in \mathbb{R}^6$ be a screw axis with $\|\omega\| = 1$. Then for any $q \in \mathbb{R}$ along that screw axis, we have that

$$e^{\tilde{\mathcal{S}} q} = \begin{pmatrix} e^{\tilde{\omega} q} & (Iq + (1 - \cos q)\tilde{\omega} + (q - \sin q)\tilde{\omega}^2)v \\ 0 & 1 \end{pmatrix}$$

Uses Rodrigues formula