SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 15: Dynamics of Fixed-base Manipulators

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Outline

- Recap last lecture
- Links Geometric Jacobian
- Twist-Wrench Duality
- Parent-Child Dynamics
- Reduced manipulator dynamics

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Recap: Exponential Coordinates of R.B. Motion

- Twist - $\mathbb{R}^6$

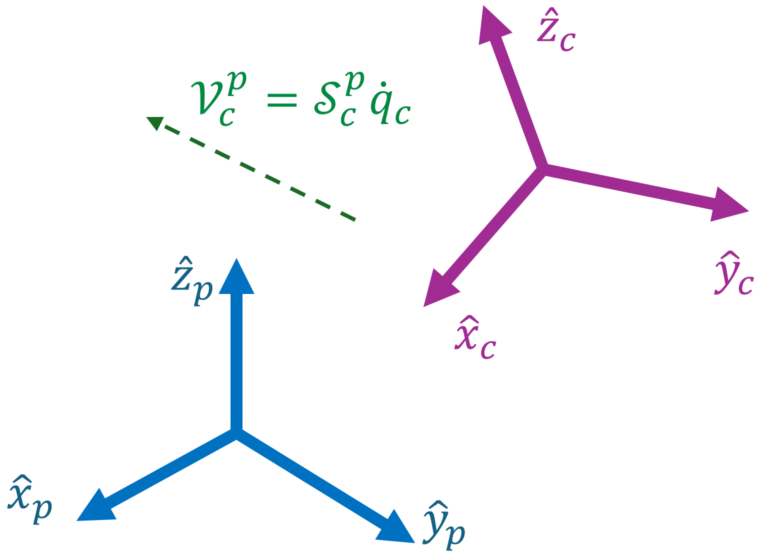
$$\mathcal{V}_c^{*,p} = \mathcal{S}_c^{*,p}\dot{q}_c$$

- Homogeneous transformation - $SE(3)$

$$H_c^p(q_c) = e^{\tilde{\mathcal{S}}_c^{p,p} q_c} H_c^p(0) \quad \text{or} \quad H_c^p(q_c) = H_c^p(0) e^{\tilde{\mathcal{S}}_c^{c,p} q_c}$$

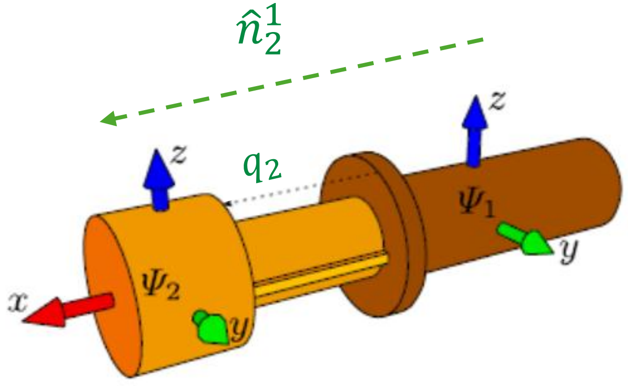
- Exponential map

$$\exp: se(3) \rightarrow SE(3)$$
$$\tilde{\mathcal{S}} q \mapsto e^{\tilde{\mathcal{S}} q}$$

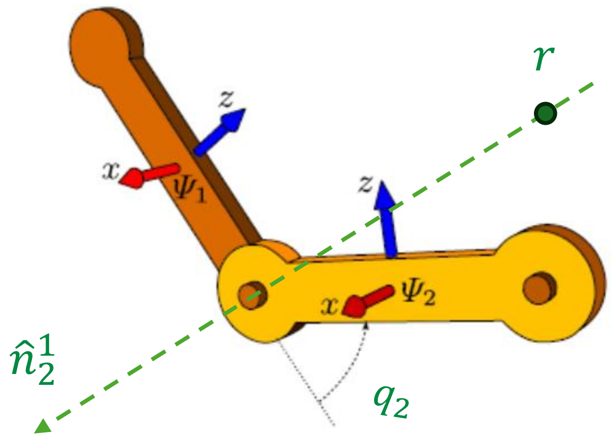


Recap: Modeling Ideal Joints

- Prismatic joints
  - $\mathcal{S}_c^{*,p} = \begin{pmatrix} 0 \\ \hat{n}_c^{*,p} \end{pmatrix} \in \mathbb{R}^6,$
  - $\hat{n}_c^{*,p}$ is the translation axis
  - $\dot{q}_c$ is the linear velocity along the screw axis



- Revolute joints
  - $\mathcal{S}_c^{*,p} = \begin{pmatrix} \hat{n}_c^{*,p} \\ -\hat{n}_c^{*,p} \wedge r^* \end{pmatrix} \in \mathbb{R}^6,$
  - $\hat{n}_c^{p}$ is the rotation axis
  - $\dot{q}_c$ is the angular velocity around the screw axis
  - r is any point on the screw axis



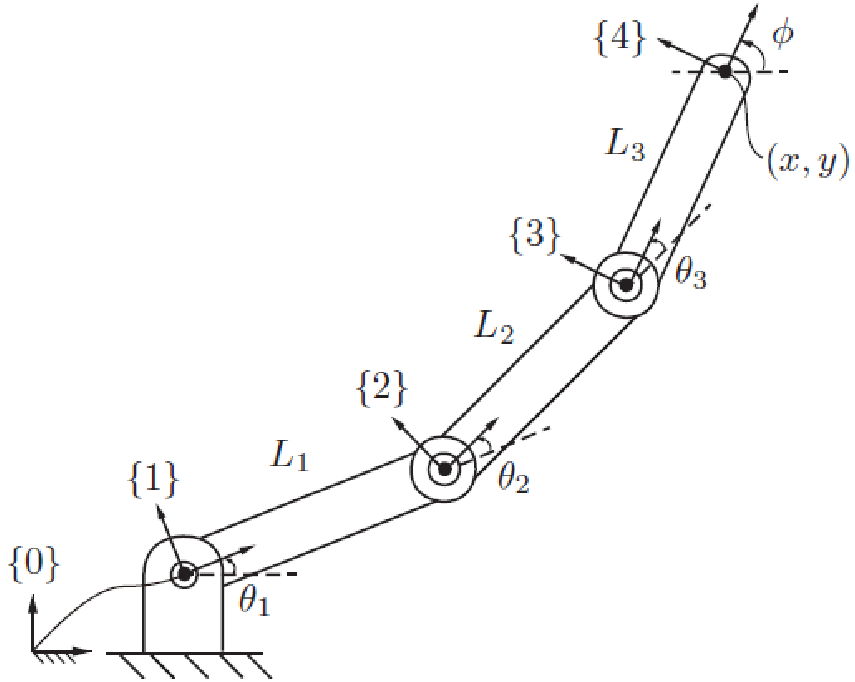
Recap: Forward Kinematics

- The forward kinematics of a robot refers to the calculation of the position and orientation of its end-effector frame given:
  - The robot's geometry (link lengths, joint types, etc.)
  - The joint configurations $q \coloneqq (q_1, \dots, q_n) \in Q$

$$FK: Q \to \mathbb{W} \subseteq SE(3)$$
$$q \mapsto H_e^0$$

Joint/Configuration Space ($\dim Q = n$)

Workspace

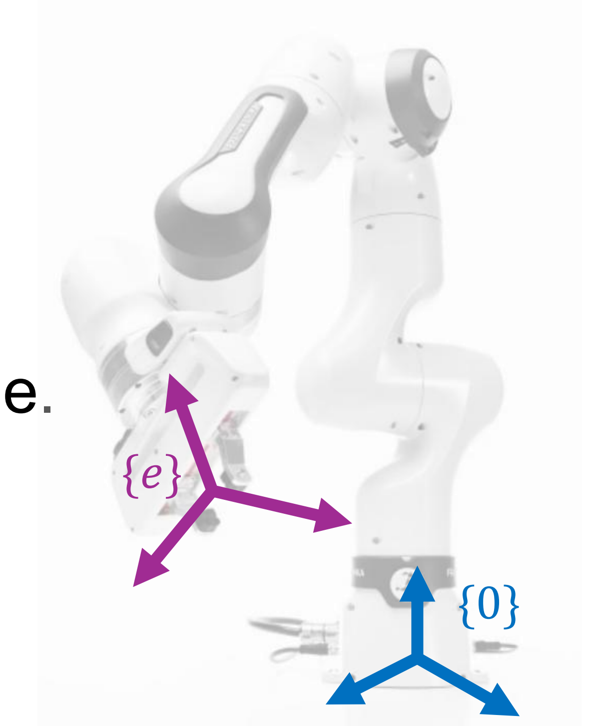


Recap: Product of Exponentials (PoE) Formula

- Thus, the forward kinematics usually has the spatial form

$$H_e^0(q) = e^{\tilde{\mathcal{S}}_1^{0,0} q_1} e^{\tilde{\mathcal{S}}_2^{0,1} q_2} \dots e^{\tilde{\mathcal{S}}_n^{0,n-1} q_n} H_e^0(0)$$

- This is called the product of exponentials formula.
- You need to provide:
  - The end-effector pose $H_e^0(0)$ when the robot is at its *home* configuration.
  - The screw axes $\mathcal{S}_1^0, \mathcal{S}_2^1, \cdots, \mathcal{S}_n^{n-1}$, corresponding to the joint motions at *home* configuration, expressed in base $\{0\}$ frame.
  - The joint configurations q.



Recap: Differential Kinematics

- The differential/velocity kinematics is the differential of this map which is a mapping from $\dot{q} \in T_q Q$ to $\dot{H}_e^0 \in T_{H_e^0} SE(3)$.
- Geometrically, we can equivalently represent it as a map from $\dot{q} \in T_q Q \cong \mathbb{R}^n$ to the end effector's twist $\mathcal{V}_e^{*,0} \in \mathbb{R}^6$

$$\begin{aligned} DK(q): \mathbb{R}^n &\to \mathbb{R}^6 \\ \dot{q} &\mapsto \mathcal{V}_e^{*,0} \end{aligned}$$

Joint velocities ($\dot{q}$)

End-effector Twist ($\mathcal{V}_e^{*,0}$)

Recap: Geometric Jacobian

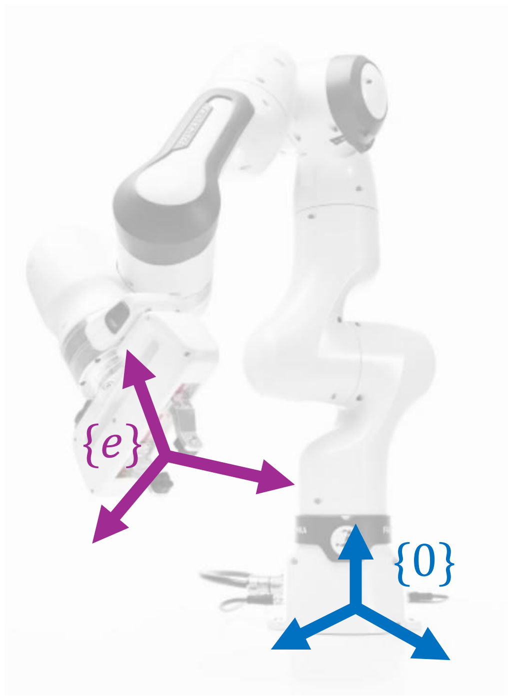
- The spatial Jacobian $J_e^{0,0}(q) \in \mathbb{R}^{6\times n}$ relates the joint rates $\dot{\theta} \in \mathbb{R}^n$ to the spatial end effector's twist $\mathcal{V}_n^{0,0} \in \mathbb{R}^6$ by

$$\mathcal{V}_e^{0,0} = J_e^{0,0}(q)\dot{q}$$

- The i-th column of $J_e^{0,0}(q)$ is given for $i \in \{2, \cdots, n\}$ by

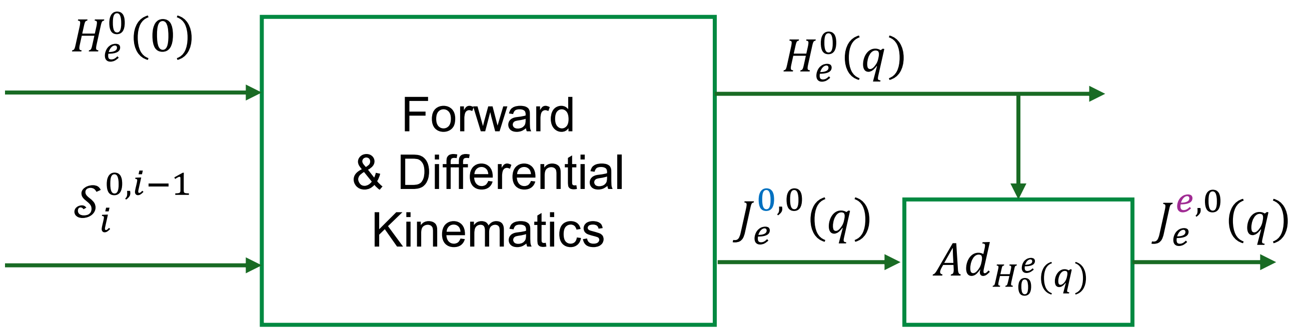
$$J_i^0(q_{1:i-1}) = \mathrm{Ad}_{e^{\tilde{\mathcal{S}}_1^{0,0} q_1} e^{\tilde{\mathcal{S}}_2^{0,1} q_2} \ldots e^{\tilde{\mathcal{S}}_{i-1}^{0,i-2} q_{n-1}}} \mathcal{S}_i^{0,i-1}$$

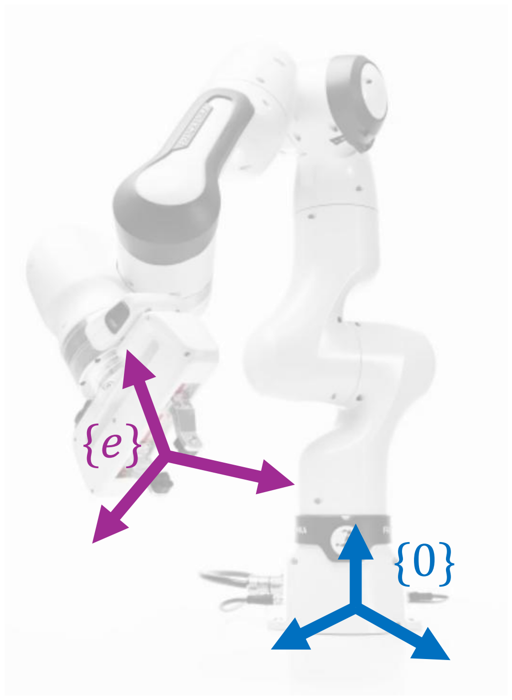
with the 1st column $J_1^0 = \mathcal{S}_1^{0,0}$.



Recap: Summary

- The above approach is highly systematic and can be easily programmable.
- The only inputs needed is the end-effector initial poses $H_e^0(0)$ and the constant screw axes $\mathcal{S}_i^{0,i-1}$ for joints in the home configuration.





- Practice problems in Homework 4 (Included in Midterm Exam)
- MATLAB Implementation next Lecture (Used for Project Deliverables)

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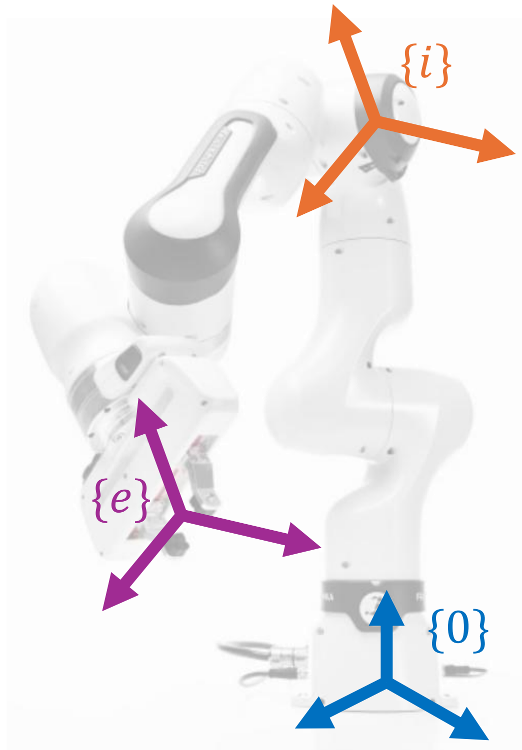
Recall: Pose of Link i

- To compute forward kinematics using PoE formula, you need stationary frame $\{0\}$ and a frame at the end-effector $\{e\}$ only.
- However, it is common to define a frame at every link.
- Typically, either at:
  - Parent Joint
  - Link’s CoM
- The pose of link $\{i\}$ can be computed by

$$H_i^0(q_{1:i}) = e^{\tilde{S}_1^{0,0} q_1} e^{\tilde{S}_2^{0,1} q_2} \dots e^{\tilde{S}_i^{0,i-1} q_i} H_i^0(0)$$

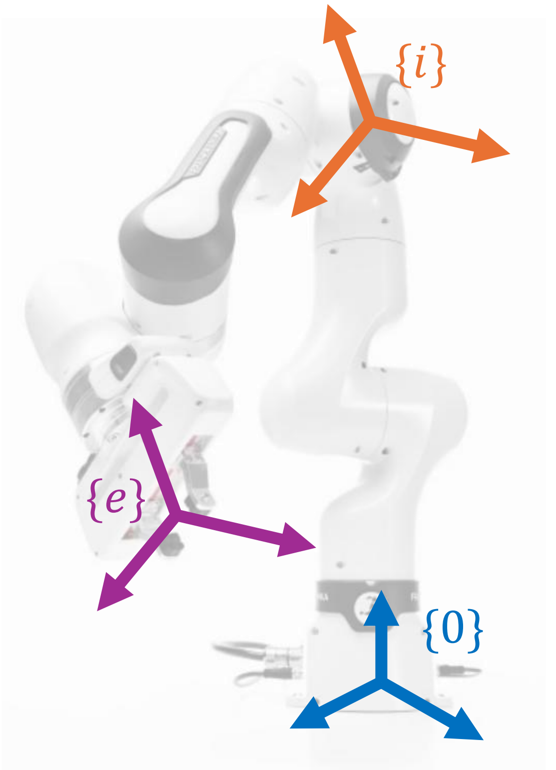
$$q_{1:i} \coloneqq (q_1, \cdots, q_i)$$



Link body twist

- If we repeat same steps we took for deriving end-effector Jacobian but for the i-th link, we reach that

$$\mathcal{V}_i^{0,0} = \mathcal{V}_1^{0,0} + \mathcal{V}_2^{0,1} + \cdots + \mathcal{V}_i^{0,i-1} + 0$$



Recall: $\mathcal{V}_e^{0,0} = \mathcal{V}_1^{0,0} + \mathcal{V}_2^{0,1} + \cdots + \mathcal{V}_n^{0,n-1}$

Link body twist

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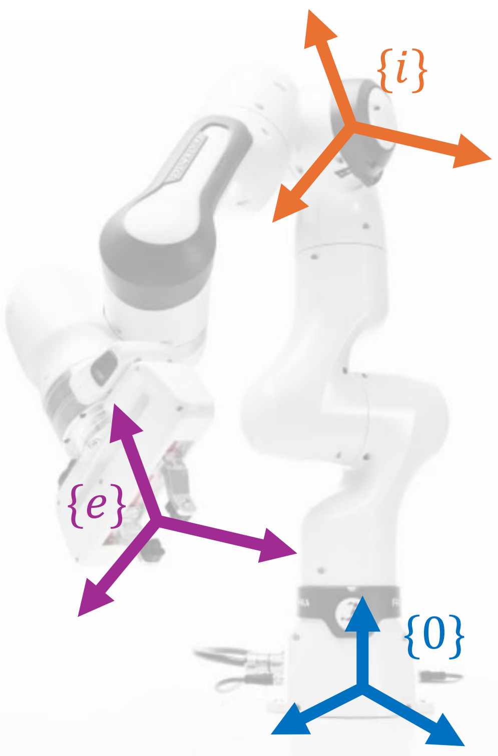
$$\mathcal{V}_i^{0,0} = \mathcal{V}_1^{0,0} + \mathcal{V}_2^{0,1} + \cdots + \mathcal{V}_i^{0,i-1} + 0$$

$$= \left(J_1^0,\ J_2^0(q_1),\ \cdots,\ J_i^0(q_{1:i-1}),\ 0_{6\times(n-i)}\right)\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_i \\ \dot{q}_{i+1:n} \end{pmatrix}$$

$$= J_i^{0,0}(q)\dot{q}$$



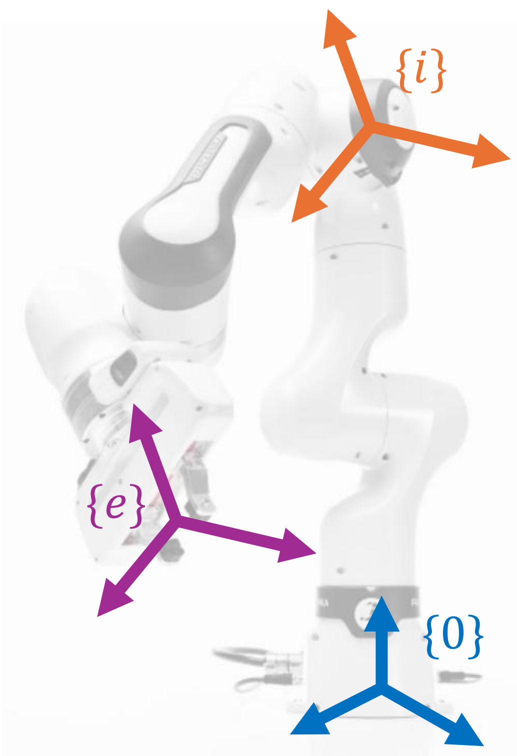
Recall: $\mathcal{V}_e^{0,0} = \mathcal{V}_1^{0,0} + \mathcal{V}_2^{0,1} + \cdots + \mathcal{V}_n^{0,n-1}$

Link body twist

- We then can relate the body twist of the i-th link as

$$\mathcal{V}_i^{i,0} = \underbrace{\mathrm{Ad}_{H_0^i(q)} J_i^{0,0}(q)}_{=:\, J_i^{i,0}(q)} \dot{q}$$

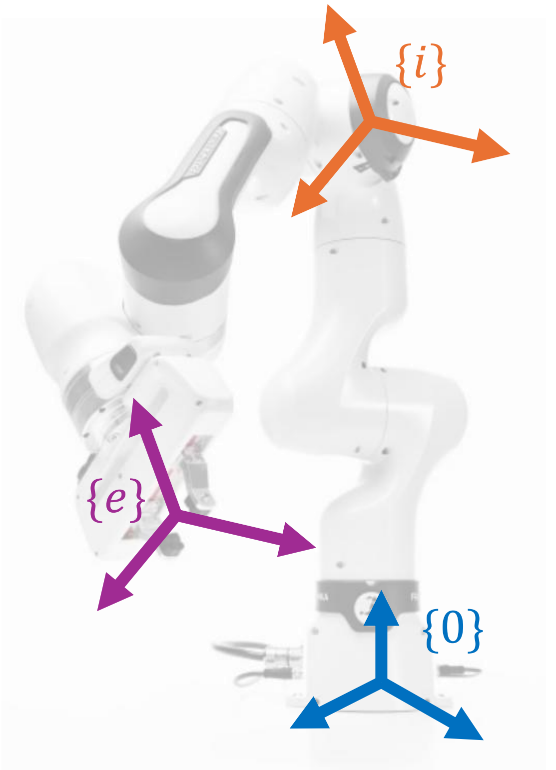
- We refer to $J_i^{i,0}(q) \in \mathbb{R}^{6\times n}$ as the geometric body Jacobian of link i.



Kinetic Energy of Fixed-base Manipulators

- As an application of the above constructions, let us investigate how the kinetic energy of an n-link fixed-base manipulator looks like.
- Recall that

$$E_{\text{kin}} = \frac{1}{2} \sum_{i=1}^{n} \left(\mathcal{V}_i^{i,0}\right)^{\top} \mathfrak{I}^{i,i} \mathcal{V}_i^{i,0}$$



Recall that $\mathfrak{I}^{*,i} \in \mathbb{R}^{6\times 6}$ is the generalized Inertia of the body attached to $\{i\}$ expressed in $\{*\}$

Kinetic Energy of Fixed-base Manipulators

- As an application of the above constructions, let us investigate how the kinetic energy of an n-link fixed-base manipulator looks like.
- Recall that

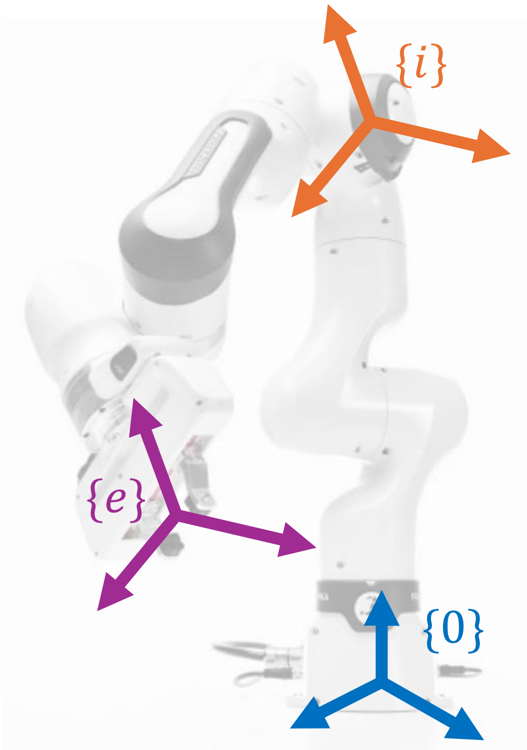
$$E_{\mathrm{kin}} = \frac{1}{2}\sum_{i=1}^{n}\left(\mathcal{V}_i^{i,0}\right)^{\top}\mathfrak{I}^{i,i}\mathcal{V}_i^{i,0}$$

- With the definition of $J_i^{i,0}(q)$, we have that

$$E_{\mathrm{kin}} = \frac{1}{2}\sum_{i=1}^{n}\dot{q}^{\top}\left(J_i^{i,0}(q)\right)^{\top}\mathfrak{I}^{i,i}J_i^{i,0}(q)\dot{q} = \frac{1}{2}\dot{q}^{\top}M(q)\dot{q}$$

where $M(q) \in \mathbb{R}^{n\times n}$ is called the mass matrix:

$$M(q) := \sum_{i=1}^{n}\left(J_i^{i,0}(q)\right)^{\top}\mathfrak{I}^{i,i}J_i^{i,0}(q)$$



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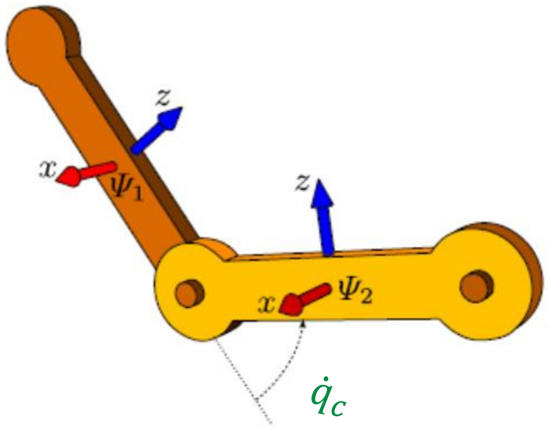
Wrench-Twist Duality

- **Relative Twist at Joint**

$$\mathcal{V}_c^{c,p} = \mathcal{S}_c^{c,p}\dot{q}_c$$

- **Link Body twist**

$$\mathcal{V}_i^{i,0} = J_i^{i,0}(q)\dot{q}$$



Wrench-Twist Duality

- **Relative Twist at Joint**

$$\mathcal{V}_c^{c,p} = \mathcal{S}_c^{c,p} \dot{q}_c$$

- **Constraint Wrench at Joint**

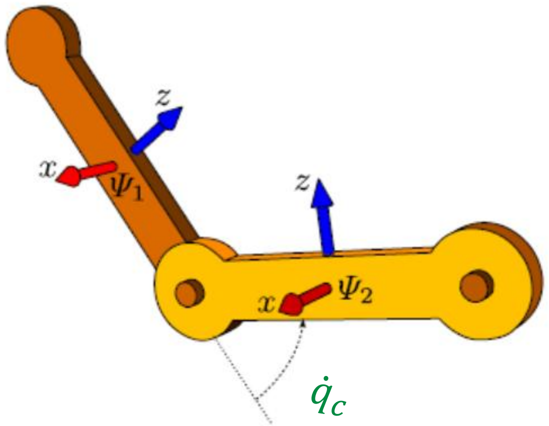
$$\tau_c = \left(\mathcal{S}_c^{c,p}\right)^\top \mathcal{W}_{\mathrm{p}}^{c,c}$$

- **Power**

$$\left(\mathcal{W}_{\mathrm{p}}^{c,c}\right)^\top \mathcal{V}_c^{c,p} = \tau_c \dot{q}_c$$

- **Link Body twist**

$$\mathcal{V}_i^{i,0} = J_i^{i,0}(q)\dot{q}$$



Wrench-Twist Duality

- **Relative Twist at Joint**

$$\mathcal{V}_c^{c,p} = \mathcal{S}_c^{c,p}\dot{q}_c$$

- **Constraint Wrench at Joint**

$$\tau_c = \left(\mathcal{S}_c^{c,p}\right)^{\top}\mathcal{W}_{\mathrm{p}}^{c,c}$$

- **Power**

$$\left(\mathcal{W}_{\mathrm{p}}^{c,c}\right)^{\top}\mathcal{V}_c^{c,p} = \tau_c\dot{q}_c$$

- **Link Body twist**

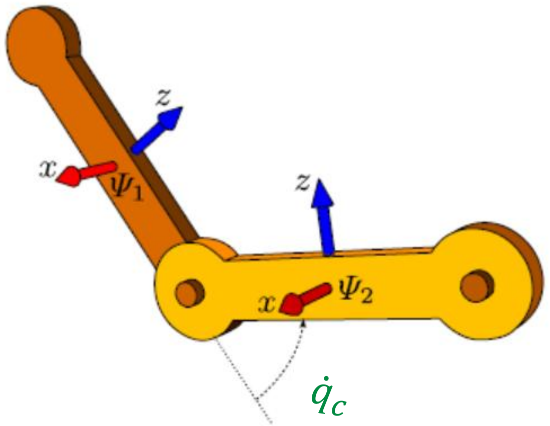
$$\mathcal{V}_i^{i,0} = J_i^{i,0}(q)\dot{q}$$

- **External Wrench at Link**

$$\tau_{\mathrm{ext}} = \left(J_i^{i,0}(q)\right)^{\top}\mathcal{W}_{\mathrm{ext}}^{i,i}$$

- **Power**

$$\left(\mathcal{W}_{\mathrm{ext}}^{i,i}\right)^{\top}\mathcal{V}_i^{i,0} = \tau_{\mathrm{ext}}^{\top}\dot{q}$$



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Parent-Child Dynamics

- Kinematics:
  - Relative Pose:

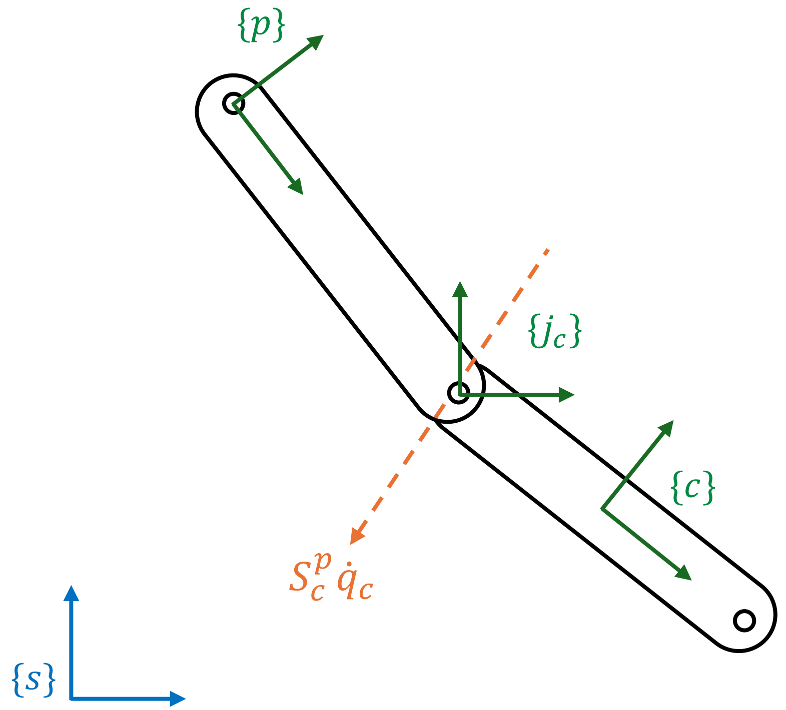
$$H_c^p(q_c) = e^{\tilde{\mathcal{S}}_c^{p,p} q_c} H_c^p(0)$$

  - Relative Twist:

$$\mathcal{V}_c^{*,p} = \mathcal{S}_c^{*,p} \dot{q}_c$$

  - Parent-Child Twist Relation:

$$\mathcal{V}_c^{*,s} = \mathcal{V}_p^{*,s} + \mathcal{V}_c^{*,p}$$



Parent-Child Dynamics

- Kinematics:
  - Relative Pose:

$$H_c^p(q_c) = e^{\tilde{\mathcal{S}}_c^{p,p} q_c} H_c^p(0)$$

  - Relative Twist:

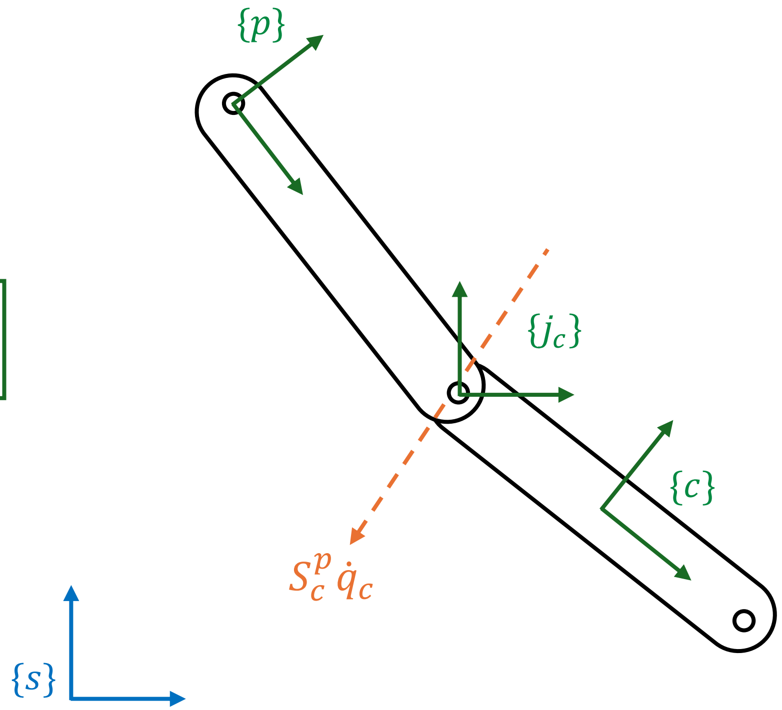
$$\mathcal{V}_c^{*,p} = \mathcal{S}_c^{*,p} \dot{q}_c$$

  - Parent-Child Twist Relation:

$$\mathcal{V}_c^{*,s} = \mathcal{V}_p^{*,s} + \mathcal{V}_c^{*,p}$$

  which can be rewritten as

$$\mathcal{V}_c^{c,s} = Ad_{H_p^c(q_c)} \mathcal{V}_p^{p,s} + \mathcal{S}_c^{c,p} \dot{q}_c$$



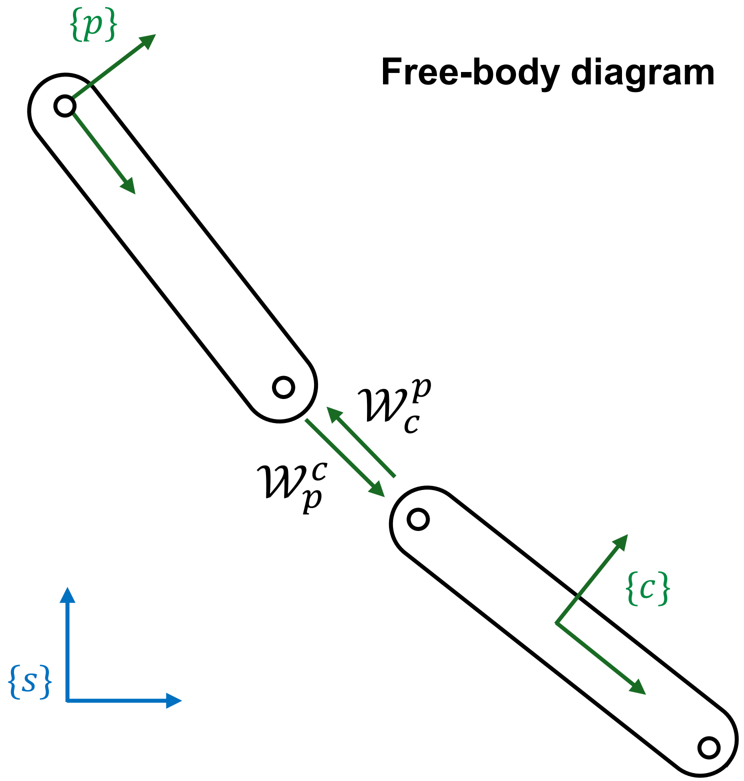
Parent-Child Dynamics

- Rigid Body Dynamics:
  - Parent

$$\mathfrak{I}^{p,p}\dot{\mathcal{V}}_p^{p,s} = \mathrm{ad}^{\mathsf{T}}_{\mathcal{V}_p^{p,s}}\left(\mathfrak{I}^{p,p}\mathcal{V}_p^{p,s}\right) + \mathcal{W}_c^{p,p}$$

  - Child

$$\mathfrak{I}^{c,c}\dot{\mathcal{V}}_c^{c,s} = \mathrm{ad}^{\mathsf{T}}_{\mathcal{V}_c^{c,s}}\left(\mathfrak{I}^{c,c}\mathcal{V}_c^{c,s}\right) + \mathcal{W}_p^{c,c}$$



$\mathcal{W}_c^{*,p}$: Reaction wrench from the child on the parent, expressed in $\{*\}$

$\mathcal{W}_p^{*,c}$: Reaction wrench from the parent on the child, expressed in $\{*\}$

Parent-Child Dynamics

- Rigid Body Dynamics:
  - Parent

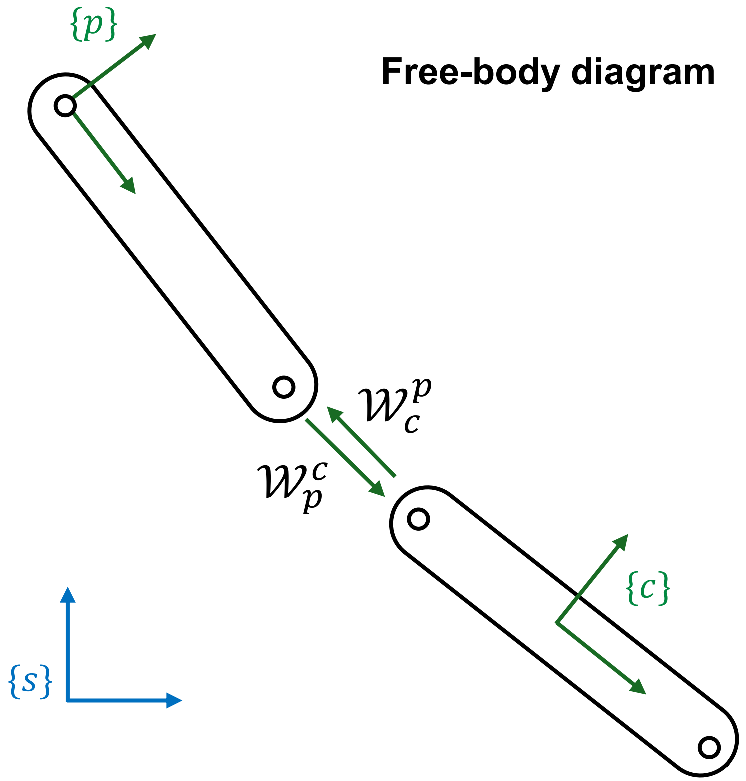
$$\mathfrak{I}^{p,p}\dot{\mathcal{V}}_p^{p,s} = \mathrm{ad}^{\top}_{\mathcal{V}_p^{p,s}}\left(\mathfrak{I}^{p,p}\mathcal{V}_p^{p,s}\right) + \mathcal{W}_c^{p,p}$$

  - Child

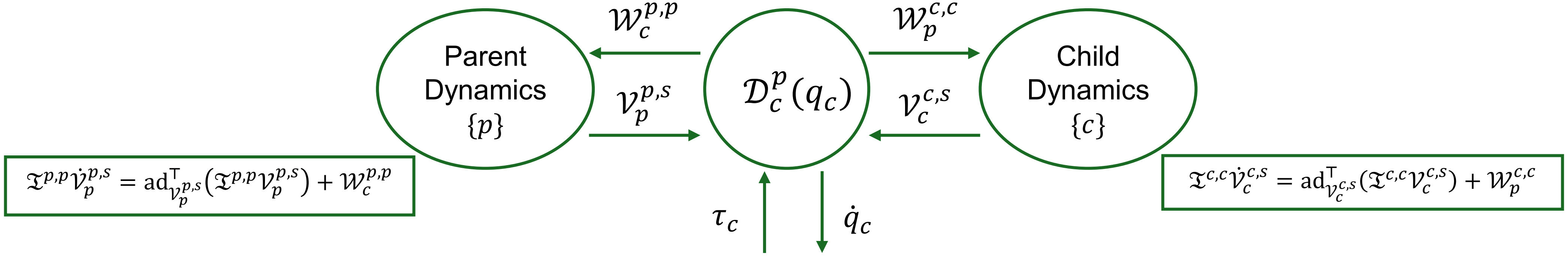
$$\mathfrak{I}^{c,c}\dot{\mathcal{V}}_c^{c,s} = \mathrm{ad}^{\top}_{\mathcal{V}_c^{c,s}}\left(\mathfrak{I}^{c,c}\mathcal{V}_c^{c,s}\right) + \mathcal{W}_p^{c,c}$$

- We have that
  - $\mathcal{W}_c^{p,p} = -\mathcal{W}_p^{p,c} = -\mathrm{Ad}^{\top}_{H_p^c(q_c)}\mathcal{W}_p^{c,c}$
  - $\left(S_c^{c,p}\right)^{\top}\mathcal{W}_p^{c,c} = \tau_c$

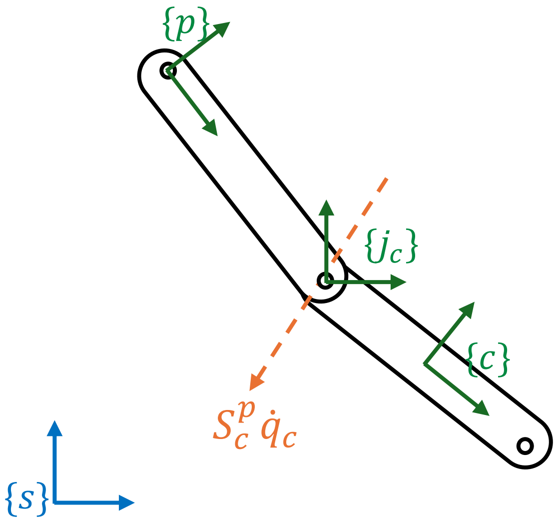
Actuator Torque



Parent-Child Dynamic Model

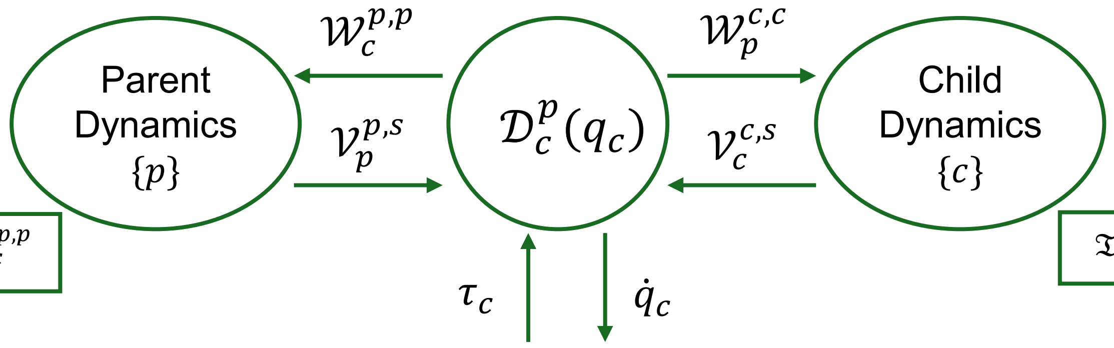


- Joint Constraints: $$\mathcal{D}_c^p(q_c) \coloneqq \begin{Bmatrix} \mathcal{V}_c^{c,s} = Ad_{H_p^c(q_c)}\mathcal{V}_p^{p,s} + \mathcal{S}_c^{c,p}\dot{q}_c \\ \mathcal{W}_c^{p,p} = -\mathrm{Ad}^{\top}_{H_p^c(q_c)}\mathcal{W}_p^{c,c} \\ \tau_c = \left(S_c^{c,p}\right)^{\top}\mathcal{W}_p^{c,c} \end{Bmatrix}$$
- Relative pose: $$H_c^p(q_c) = H_c^p(0)e^{\tilde{S}_c^{c,p}q_c}$$



Parent-Child Dynamic Model

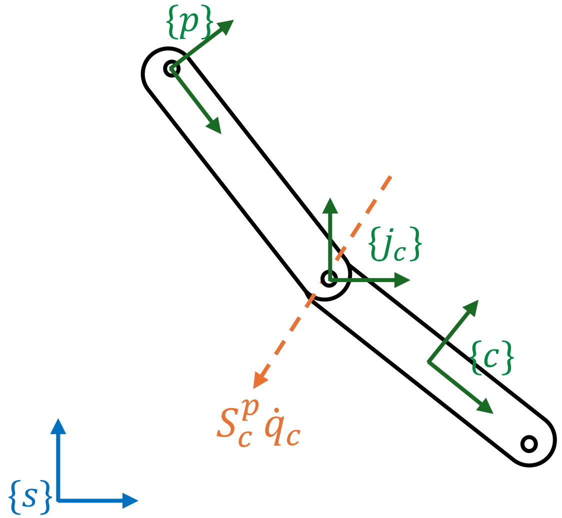
This is called a set of Differential-Algebraic Equations (DAEs). Principle of Multi-body software like Simulink (Simscape), Gazebo (Bullet, DART)



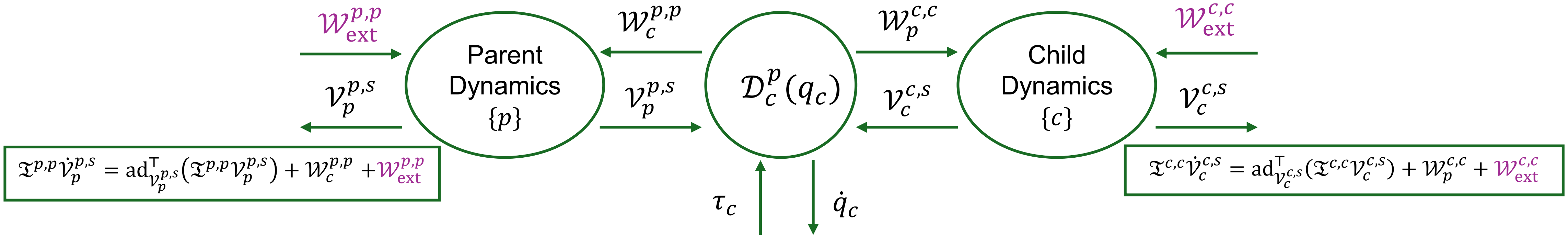
$$\mathfrak{I}^{p,p}\dot{\mathcal{V}}_p^{p,s} = \mathrm{ad}^{\top}_{\mathcal{V}_p^{p,s}}\left(\mathfrak{I}^{p,p}\mathcal{V}_p^{p,s}\right) + \mathcal{W}_c^{p,p}$$

$$\mathfrak{I}^{c,c}\dot{\mathcal{V}}_c^{c,s} = \mathrm{ad}^{\top}_{\mathcal{V}_c^{c,s}}\left(\mathfrak{I}^{c,c}\mathcal{V}_c^{c,s}\right) + \mathcal{W}_p^{c,c}$$

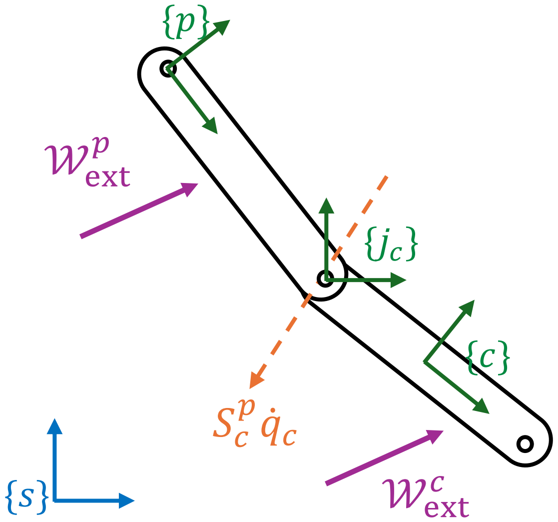
- Joint Constraints: $\mathcal{D}_c^p(q_c) := \begin{cases} \mathcal{V}_c^{c,s} = Ad_{H_p^c(q_c)}\mathcal{V}_p^{p,s} + \mathcal{S}_c^{c,p}\dot{q}_c \\ \mathcal{W}_c^{p,p} = -\mathrm{Ad}^{\top}_{H_p^c(q_c)}\mathcal{W}_p^{c,c} \\ \tau_c = \left(S_c^{c,p}\right)^{\top}\mathcal{W}_p^{c,c} \end{cases}$
- Relative pose: $H_c^p(q_c) = H_c^p(0)e^{\tilde{S}_c^{c,p}q_c}$



Parent-Child Dynamic Model with external wrenches

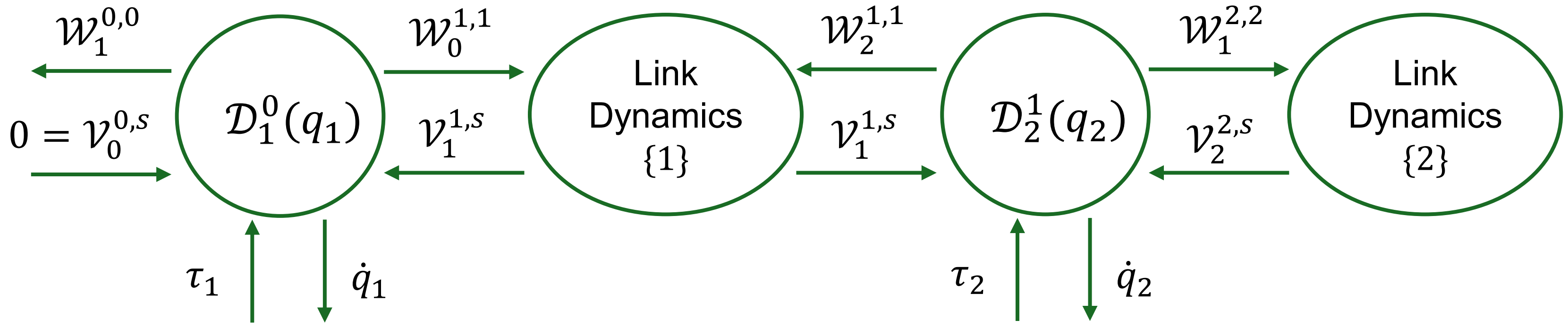


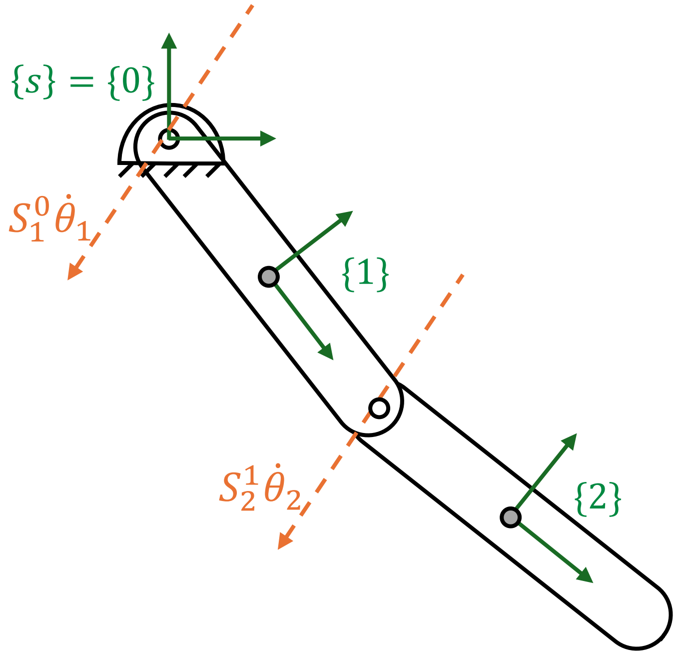
- Joint Constraints: $$\mathcal{D}_c^p(q_c) := \left\{ \begin{matrix} \mathcal{V}_c^{c,s} = Ad_{H_p^c(q_c)}\mathcal{V}_p^{p,s} + \mathcal{S}_c^{c,p}\dot{q}_c \\ \mathcal{W}_c^{p,p} = -\mathrm{Ad}^{\top}_{H_p^c(q_c)}\mathcal{W}_p^{c,c} \\ \tau_c = \left(S_c^{c,p}\right)^{\top}\mathcal{W}_p^{c,c} \end{matrix} \right\}$$
- Relative pose: $$H_c^p(q_c) = H_c^p(0)e^{\tilde{S}_c^{c,p}q_c}$$



Example: Two-Link Manipulator DAE

- In the case of a 2-link manipulator, the DAEs will take the form:

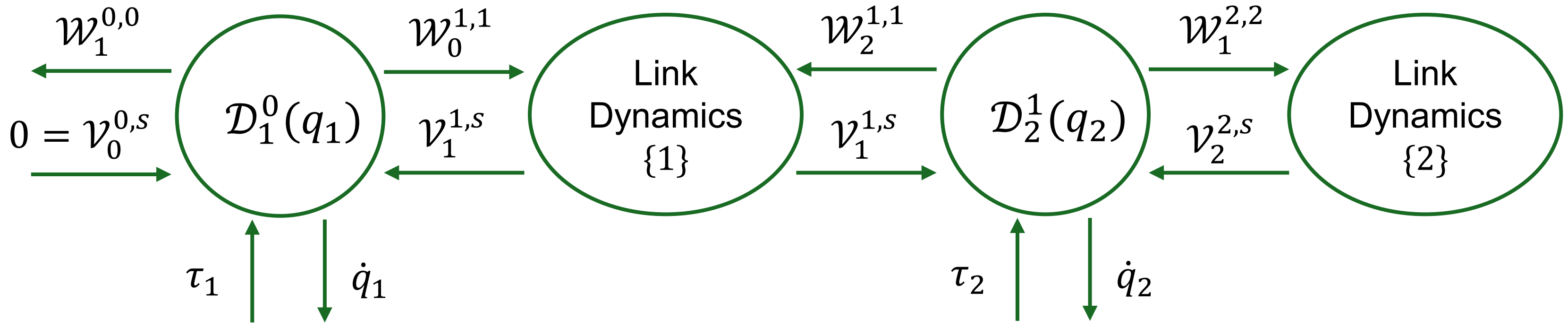




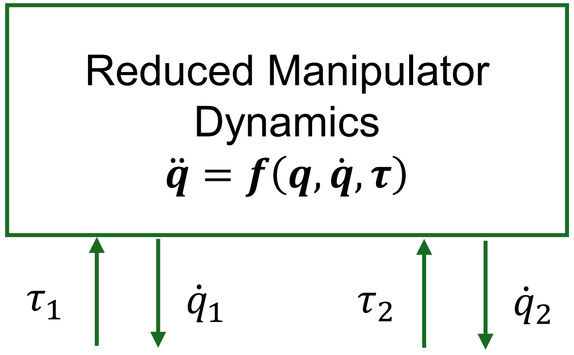
This is generalizable to an n-link multi-body system.

From DAEs to ODEs

- In the case of a 2-link manipulator, the DAEs will take the form:



- For manipulators, we can eliminate the constraints wrenches to represent the dynamics as a set of Ordinary Differential Equations (ODEs) of the form.



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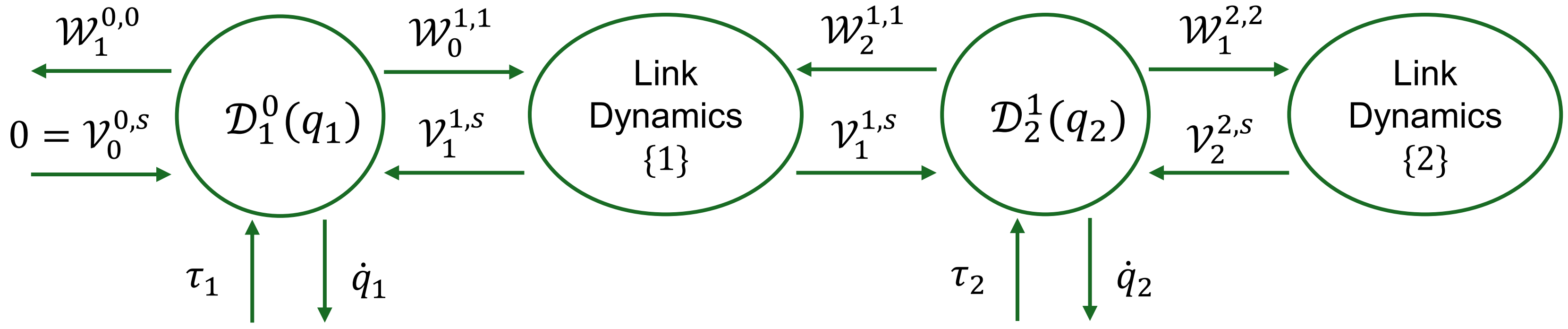

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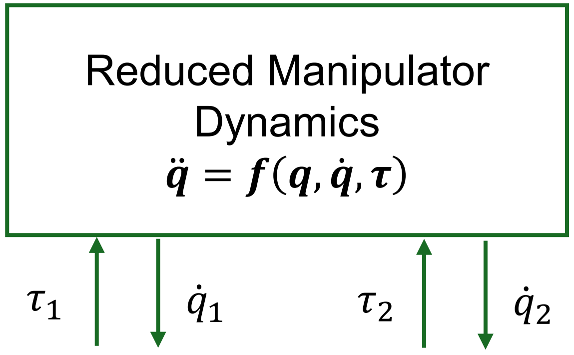


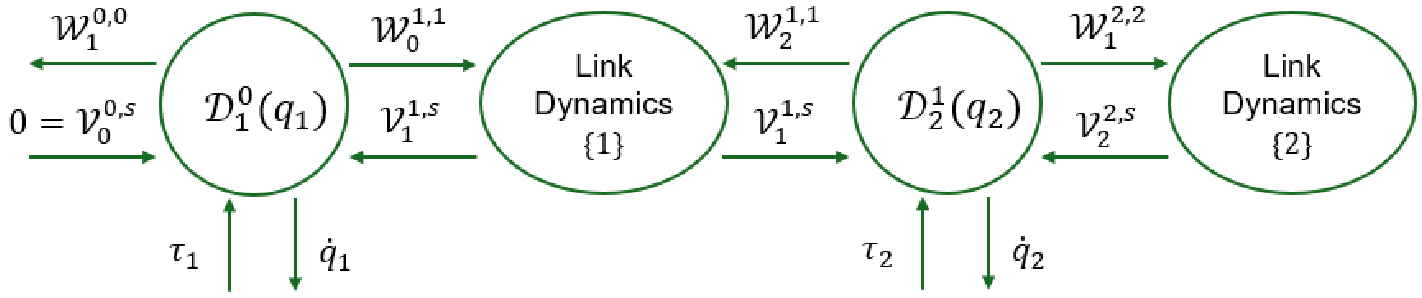
We will use the Recursive-Newton-Euler Algorithm (RNEA)

Outline

- Recap last lecture
- Links Geometric Jacobian
- Twist-Wrench Duality
- Parent-Child Dynamics
- Reduced manipulator dynamics

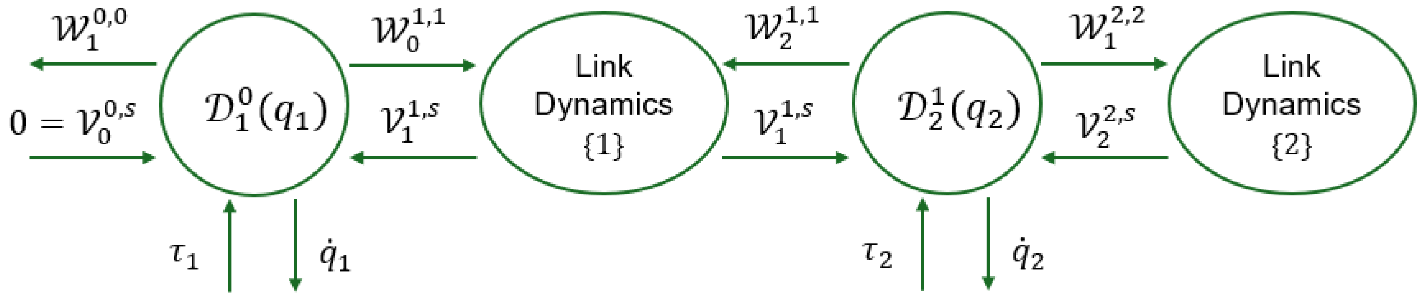
Recursive Newton-Euler Algorithm

- RNEA applies Newton–Euler equations recursively along the kinematic tree in an approach involving a forward and backward pass.
- Forward pass:
  - Compute link twists $\mathcal{V}_i^{i,s}$ and link accelerations $\dot{\mathcal{V}}_i^{i,s}$
    - $\mathcal{V}_i^{i,s} = \mathrm{Ad}_{H_{i-1}^{i}} \mathcal{V}_{i-1}^{i-1,s} + S_i^{i,i-1} \dot{q}_i$
    - $\dot{\mathcal{V}}_i^{i,s} = \mathrm{Ad}_{H_{i-1}^{i}} \dot{\mathcal{V}}_{i-1}^{i-1,s} + \frac{\mathrm{d}}{\mathrm{dt}} \mathrm{Ad}_{H_{i-1}^{i}} \mathcal{V}_{i-1}^{i-1,s} + S_i^{i,i-1} \ddot{q}_i$



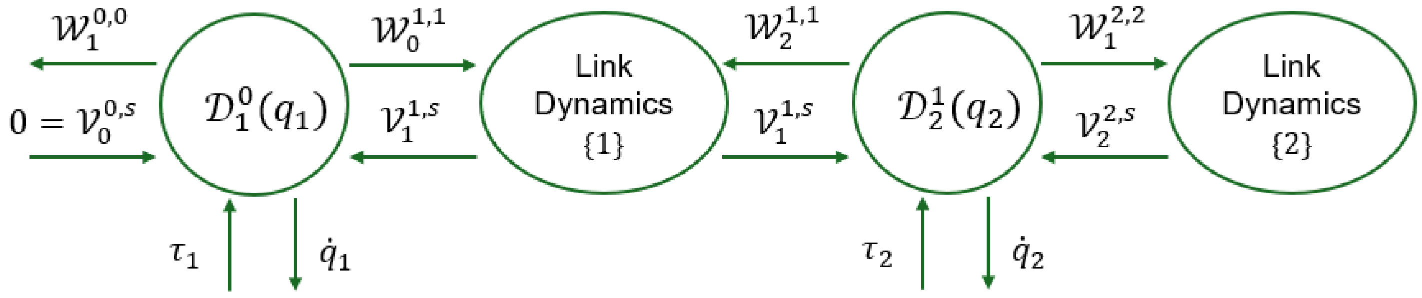
Recursive Newton-Euler Algorithm

- RNEA applies Newton–Euler equations recursively along the kinematic tree in an approach involving a forward and backward pass.
- Backward pass:
  - Compute constraint wrenches
    - $\mathcal{W}_{i-1}^{i,i} = \mathfrak{I}^{i,i}\dot{\mathcal{V}}_i^{i,s} - \mathrm{ad}_{\mathcal{V}_i^{i,0}}^{\top}\left(\mathfrak{I}^{i,i}\mathcal{V}_i^{i,s}\right) + \mathrm{Ad}_{H_i^{i+1}}^{\top}\mathcal{W}_i^{i+1,i+1}$
  - Project them onto joint subspaces to get torques
    - $\tau_i = \left(S_i^{i,i-1}\right)^{\top}\mathcal{W}_{i-1}^{i,i}$



Recursive Newton-Euler Algorithm

- RNEA applies Newton–Euler equations recursively along the kinematic tree in an approach involving a forward and backward pass.
- We will now show how this recursive algorithm can be cast into a linear set of equations that have a compact closed form expression.



Dynamic Equations in Closed Form

- First, let us rewrite the recursive equation
  - $\mathcal{V}_i^{i,s} = \mathrm{Ad}_{H_{i-1}^{i}} \mathcal{V}_{i-1}^{i-1,s} + S_i^{i,i-1} \dot{q}_i$

compactly as

- $\mathcal{V} = \mathcal{A}(q)\mathcal{V} + \mathcal{S}\, \dot{q}$

with

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_1^{1,s} \\ \vdots \\ \mathcal{V}_n^{n,s} \end{pmatrix} \in \mathbb{R}^{6n}$$

$$\dot{q} = \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \in \mathbb{R}^{n}$$

$$\mathcal{A}(q) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathrm{Ad}_{H_1^2(q_2)} & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{Ad}_{H_2^3(q_3)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{Ad}_{H_{n-1}^n(q_n)} & 0 \end{pmatrix} \in \mathbb{R}^{6n \times 6n}$$

$$\mathcal{S} = \begin{pmatrix} S_1^{1,0} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & S_n^{n,n-1} \end{pmatrix} \in \mathbb{R}^{6n \times n}$$

Dynamic Equations in Closed Form

- Similarly, we can rewrite
  - $\dot{\mathcal{V}}_i^{i,s} = \mathrm{Ad}_{H_{i-1}^i}\dot{\mathcal{V}}_{i-1}^{i-1,s} + \frac{\mathrm{d}}{\mathrm{dt}}\mathrm{Ad}_{H_{i-1}^i}\mathcal{V}_{i-1}^{i-1,s} + S_i^{i,i-1}\ddot{q}_i$

compactly as

  - $\dot{\mathcal{V}} = \mathcal{A}(q)\dot{\mathcal{V}} + \dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{S}\,\ddot{q}$

with

$$\dot{\mathcal{V}} = \begin{pmatrix} \dot{\mathcal{V}}_1^{1,s} \\ \vdots \\ \dot{\mathcal{V}}_n^{n,s} \end{pmatrix} \in \mathbb{R}^{6n}$$

$$\ddot{q} = \begin{pmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{pmatrix} \in \mathbb{R}^{n}$$

Dynamic Equations in Closed Form

- Similarly, we can rewrite
  - $\mathcal{W}_{i-1}^{i,i} = \mathfrak{T}^{i,i}\dot{\mathcal{V}}_i^{i,s} - ad^{\top}_{\mathcal{V}_i^{i,0}}\left(\mathfrak{T}^{i,i}\mathcal{V}_i^{i,s}\right) + \mathrm{Ad}^{\top}_{H_i^{i+1}}\mathcal{W}_i^{i+1,i+1}$
  - $\tau_i = \left(S_i^{i,i-1}\right)^{\top}\mathcal{W}_{i-1}^{i,i}$

compactly as

- $\mathcal{W} = \mathfrak{T}\dot{\mathcal{V}} - \mathrm{ad}^{\top}_{\mathcal{V}}\,\mathfrak{T}\,\mathcal{V} + \mathcal{A}^{\top}(q)\mathcal{W}$
- $\tau = \mathcal{S}^{\top}\mathcal{W}$

with

$$\mathcal{W} = \begin{pmatrix} \mathcal{W}_0^{1,1} \\ \vdots \\ \mathcal{W}_{n-1}^{n,n} \end{pmatrix} \in (\mathbb{R}^{6n})^*$$

$$\mathfrak{T} = \begin{pmatrix} \mathfrak{T}^{1,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathfrak{T}^{n,n} \end{pmatrix} \in \mathbb{R}^{6n\times 6n}$$

$$\tau = \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_n \end{pmatrix} \in (\mathbb{R}^{n})^*$$

$$\mathrm{ad}_{\mathcal{V}} = \begin{pmatrix} \mathrm{ad}_{\mathcal{V}_1^{1,s}} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathrm{ad}_{\mathcal{V}_n^{n,s}} \end{pmatrix} \in \mathbb{R}^{6n\times 6n}$$

Dynamic Equations in Closed Form

- In general, the recursive dynamics can be assembled into the following set of matrix equations:
  - $\mathcal{V} = \mathcal{A}(q)\mathcal{V} + \mathcal{S}\,\dot{q}$
  - $\dot{\mathcal{V}} = \mathcal{A}(q)\dot{\mathcal{V}} + \dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{S}\,\ddot{q}$
  - $\mathcal{W} = \mathfrak{T}\dot{\mathcal{V}} - \mathrm{ad}_{\mathcal{V}}^{\top}\,\mathfrak{T}\,\mathcal{V} + \mathcal{A}^{\top}(q)\mathcal{W}$
  - $\tau = \mathcal{S}^{\top}\mathcal{W}$

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  - $\tau = \mathcal{S}^{\top}\mathcal{W}$
- Equivalent to
  - $[I - \mathcal{A}(q)]\mathcal{V} = \mathcal{S}\,\dot{q}$
  - $[I - \mathcal{A}(q)]\dot{\mathcal{V}} = \dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{S}\,\ddot{q}$
  - $[I - \mathcal{A}(q)]^{\top}\mathcal{W} = \mathfrak{I}\dot{\mathcal{V}} - \mathrm{ad}_{\mathcal{V}}^{\top}\,\mathfrak{I}\,\mathcal{V}$
  - $\tau = \mathcal{S}^{\top}\mathcal{W}$

Dynamic Equations in Closed Form

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- Equivalent to
  - $[I - \mathcal{A}(q)]\mathcal{V} = \mathcal{S}\,\dot{q}$
  - $[I - \mathcal{A}(q)]\dot{\mathcal{V}} = \dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{S}\,\ddot{q}$
  - $[I - \mathcal{A}(q)]^{\top}\mathcal{W} = \mathfrak{I}\dot{\mathcal{V}} - \mathrm{ad}_{\mathcal{V}}^{\top}\,\mathfrak{I}\,\mathcal{V}$
  - $\tau = \mathcal{S}^{\top}\mathcal{W}$

Let $\mathcal{L}(q) \coloneqq [I - \mathcal{A}(q)]^{-1}$ which has the closed form expression:

$$\mathcal{L}(q) = \begin{pmatrix} I & 0 & \cdots & \cdots & 0 \\ \mathrm{Ad}_{H_1^2(q)} & I & \cdots & \cdots & 0 \\ \mathrm{Ad}_{H_1^3(q)} & \mathrm{Ad}_{H_2^3(q)} & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathrm{Ad}_{H_1^n(q)} & \mathrm{Ad}_{H_2^n(q)} & \mathrm{Ad}_{H_3^n(q)} & \cdots & I \end{pmatrix} \in \mathbb{R}^{6n\times 6n}$$

Dynamic Equations in Closed Form

- In general, the recursive dynamics can be assembled into the following set of matrix equations:
  - $\mathcal{V} = \mathcal{A}(q)\mathcal{V} + \mathcal{S}\,\dot{q}$
  - $\dot{\mathcal{V}} = \mathcal{A}(q)\dot{\mathcal{V}} + \dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{S}\,\ddot{q}$
  - $\mathcal{W} = \mathfrak{I}\dot{\mathcal{V}} - \mathrm{ad}_{\mathcal{V}}^{\top}\,\mathfrak{I}\,\mathcal{V} + \mathcal{A}^{\top}(q)\mathcal{W}$
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- Equivalent to
  - $\mathcal{V} = \mathcal{L}(q)\mathcal{S}\,\dot{q}$
  - $\dot{\mathcal{V}} = \mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{L}(q)\mathcal{S}\,\ddot{q}$
  - $\mathcal{W} = \mathcal{L}^{\top}(q)\mathfrak{I}\dot{\mathcal{V}} - \mathcal{L}^{\top}(q)\mathrm{ad}_{\mathcal{V}}^{\top}\,\mathfrak{I}\,\mathcal{V}$
  - $\tau = \mathcal{S}^{\top}\mathcal{W}$

Dynamic Equations in Closed Form

- The compact equations
  - $\mathcal{V} \ = \mathcal{L}(q)\mathcal{S}\,\dot{q}$
  - $\dot{\mathcal{V}} = \mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{L}(q)\mathcal{S}\,\ddot{q}$
  - $\mathcal{W} = \mathcal{L}^\top(q)\mathfrak{I}\dot{\mathcal{V}} - \mathcal{L}^\top(q)\mathrm{ad}_{\mathcal{V}}^\top\,\mathfrak{I}\,\mathcal{V}$
  - $\tau = \mathcal{S}^\top\mathcal{W}$

can be combined to get

- $\tau = \mathcal{S}^\top\mathcal{L}^\top(q)\mathfrak{I}\dot{\mathcal{V}} - \mathcal{S}^\top\mathcal{L}^\top(q)\mathrm{ad}_{\mathcal{V}}^\top\,\mathfrak{I}\,\mathcal{V}$

$$= \mathcal{S}^\top\mathcal{L}^\top(q)\mathfrak{I}\big(\mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{L}(q)\mathcal{S}\,\dot{q} + \mathcal{L}(q)\mathcal{S}\,\ddot{q}\big) - \mathcal{S}^\top\mathcal{L}^\top(q)\mathrm{ad}_{\mathcal{L}(q)\mathcal{S}\,\dot{q}}^\top\,\mathfrak{I}\mathcal{L}(q)\mathcal{S}\,\dot{q}$$

Dynamic Equations in Closed Form

- The compact equations
  - $\mathcal{V} = \mathcal{L}(q)\mathcal{S}\,\dot{q}$
  - $\dot{\mathcal{V}} = \mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{L}(q)\mathcal{S}\,\ddot{q}$
  - $\mathcal{W} = \mathcal{L}^\top(q)\mathfrak{T}\dot{\mathcal{V}} - \mathcal{L}^\top(q)\mathrm{ad}_{\mathcal{V}}^\top\,\mathfrak{T}\,\mathcal{V}$
  - $\tau = \mathcal{S}^\top\mathcal{W}$

can be combined to get

- $\tau = \mathcal{S}^\top\mathcal{L}^\top(q)\mathfrak{T}\dot{\mathcal{V}} - \mathcal{S}^\top\mathcal{L}^\top(q)\mathrm{ad}_{\mathcal{V}}^\top\,\mathfrak{T}\,\mathcal{V}$

$$= \mathcal{S}^\top\mathcal{L}^\top(q)\mathfrak{T}\big(\mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{L}(q)\mathcal{S}\,\dot{q} + \mathcal{L}(q)\mathcal{S}\,\ddot{q}\big) - \mathcal{S}^\top\mathcal{L}^\top(q)\mathrm{ad}_{\mathcal{L}(q)\mathcal{S}\,\dot{q}}^\top\,\mathfrak{T}\mathcal{L}(q)\mathcal{S}\,\dot{q}$$

Introduce $\mathcal{J}(q) \coloneqq \mathcal{L}(q)\mathcal{S}$

Dynamic Equations in Closed Form

- The compact equations
  - $\mathcal{V} \;= \mathcal{L}(q)\mathcal{S}\,\dot{q}$
  - $\dot{\mathcal{V}} = \mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{L}(q)\mathcal{S}\,\ddot{q}$
  - $\mathcal{W} = \mathcal{L}^{\top}(q)\mathfrak{I}\dot{\mathcal{V}} - \mathcal{L}^{\top}(q)\mathrm{ad}_{\mathcal{V}}^{\top}\,\mathfrak{I}\,\mathcal{V}$
  - $\tau = \mathcal{S}^{\top}\mathcal{W}$

can be combined to get

- $\tau = \mathcal{S}^{\top}\mathcal{L}^{\top}(q)\mathfrak{I}\dot{\mathcal{V}} - \mathcal{S}^{\top}\mathcal{L}^{\top}(q)\mathrm{ad}_{\mathcal{V}}^{\top}\,\mathfrak{I}\,\mathcal{V}$

$$= \mathcal{J}^{\top}(q)\mathfrak{I}\big(\mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{J}(q)\dot{q} + \mathcal{J}(q)\ddot{q}\big) - \mathcal{J}^{\top}(q)\mathrm{ad}_{\mathcal{J}(q)\dot{q}}^{\top}\,\mathfrak{I}\mathcal{J}(q)\dot{q}$$

Dynamic Equations in Closed Form

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  - $\dot{\mathcal{V}} = \mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{L}(q)\mathcal{S}\,\ddot{q}$
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- $\tau = \mathcal{S}^\top\mathcal{L}^\top(q)\mathfrak{I}\dot{\mathcal{V}} - \mathcal{S}^\top\mathcal{L}^\top(q)\mathrm{ad}_{\mathcal{V}}^\top\,\mathfrak{I}\,\mathcal{V}$

  $= \mathcal{J}^\top(q)\mathfrak{I}\big(\mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{J}(q)\dot{q} + \mathcal{J}(q)\ddot{q}\big) - \mathcal{J}^\top(q)\mathrm{ad}_{\mathcal{J}(q)\dot{q}}^\top\,\mathfrak{I}\mathcal{J}(q)\dot{q}$

  $= \mathcal{J}^\top(q)\mathfrak{I}\mathcal{J}(q)\ddot{q} + \Big(\mathcal{J}^\top(q)\mathfrak{I}\mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{J}(q) - \mathcal{J}^\top(q)\mathrm{ad}_{\mathcal{J}(q)\dot{q}}^\top\,\mathfrak{I}\mathcal{J}(q)\Big)\,\dot{q}$

Dynamic Equations in Closed Form

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$$= \mathcal{J}^\top(q)\mathfrak{I}\big(\mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{J}(q)\dot{q} + \mathcal{J}(q)\ddot{q}\big) - \mathcal{J}^\top(q)\mathrm{ad}_{\mathcal{J}(q)\dot{q}}^\top\,\mathfrak{I}\mathcal{J}(q)\dot{q}$$

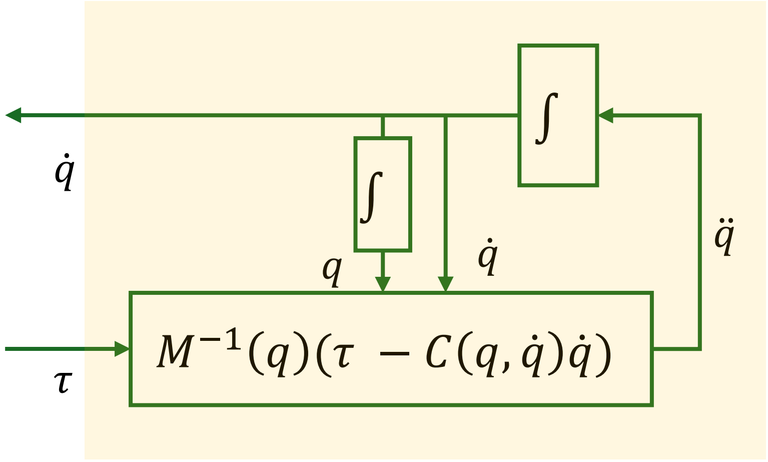
$$= \underbrace{\mathcal{J}^\top(q)\mathfrak{I}\mathcal{J}(q)}_{=:M(q)}\,\ddot{q} + \underbrace{\left(\mathcal{J}^\top(q)\mathfrak{I}\mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{J}(q) - \mathcal{J}^\top(q)\mathrm{ad}_{\mathcal{J}(q)\dot{q}}^\top\,\mathfrak{I}\mathcal{J}(q)\right)}_{=:C(q,\dot{q})}\dot{q}$$

Reduced n-link manipulator dynamics

- Finally, we reach the form

$$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q}$$

$$M(q) \coloneqq \mathcal{J}^{\top}(q)\mathfrak{I}\mathcal{J}(q) \in \mathbb{R}^{n\times n}$$
$$C(q,\dot{q}) \coloneqq \mathcal{J}^{\top}(q)\mathfrak{I}\mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{J}(q) - \mathcal{J}^{\top}(q)\mathrm{ad}^{\top}_{\mathcal{J}(q)\dot{q}}\mathfrak{I}\mathcal{J}(q) \in \mathbb{R}^{n\times n}$$



Reduced Dynamics