

# SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 16: Dynamics of Fixed-base Manipulators II



# Outline

- Recap last lecture
- Modeling External Wrenches
- Software Implementation



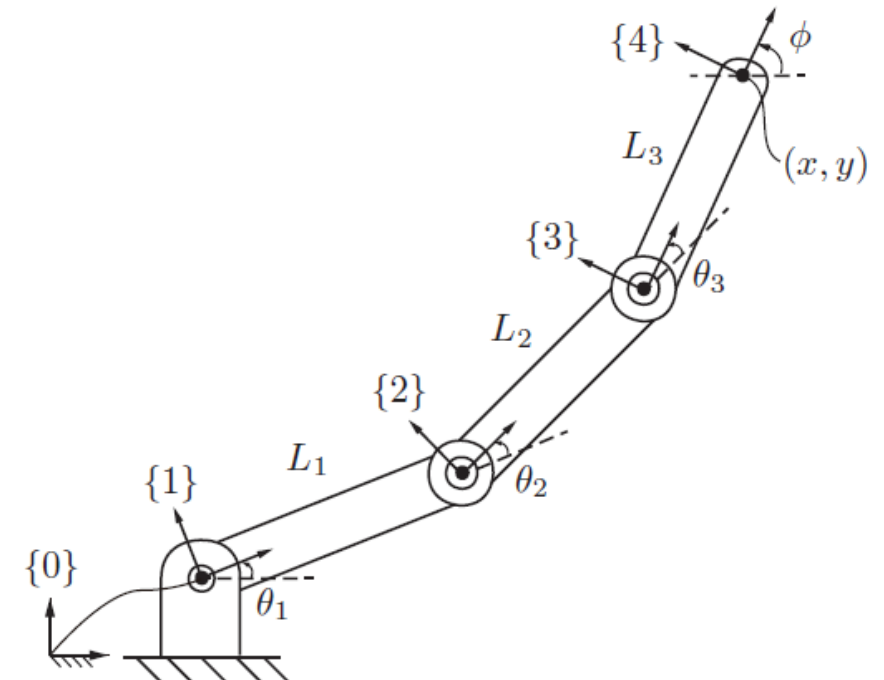
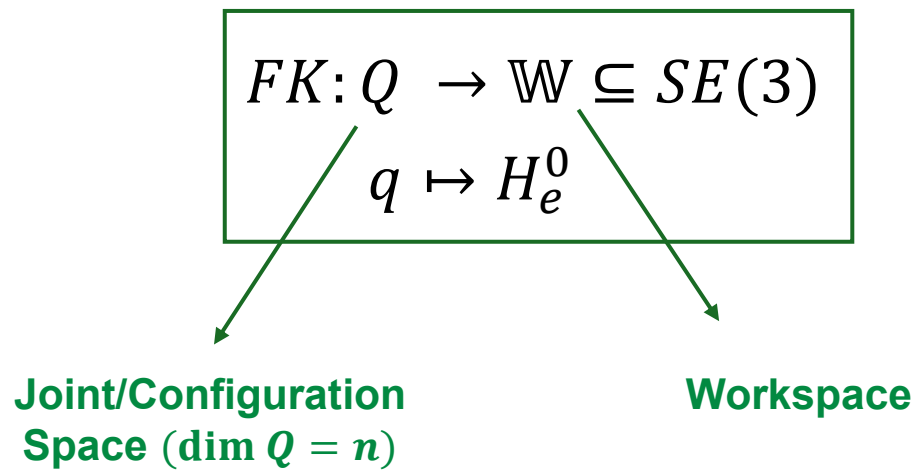
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# Recap: Forward Kinematics

- The **forward kinematics** of a robot refers to the calculation of the position and orientation of its end-effector frame given:
  - The robot's geometry (link lengths, joint types, etc.)
  - The joint configurations  $q := (q_1, \dots, q_n) \in Q$

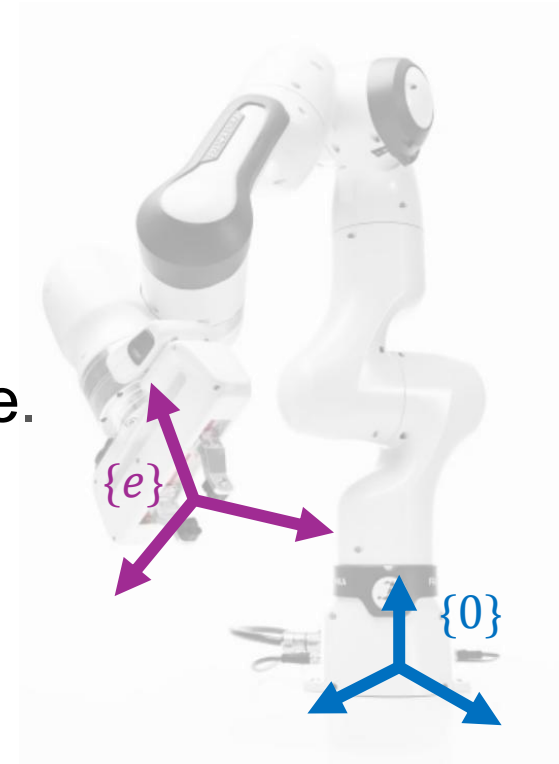


# Recap: Product of Exponentials (PoE) Formula

- Thus, the forward kinematics usually has the **spatial** form

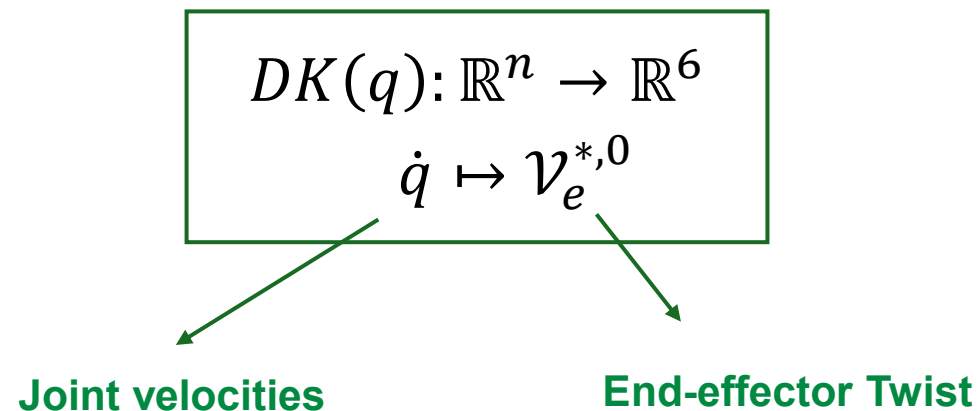
$$H_e^0(q) = e^{\tilde{\mathcal{S}}_1^{0,0} q_1} e^{\tilde{\mathcal{S}}_2^{0,1} q_2} \dots e^{\tilde{\mathcal{S}}_n^{0,n-1} q_n} H_e^0(0)$$

- This is called the product of exponentials formula.
- You need to provide:
  - The end-effector pose  $H_e^0(0)$  when the robot is at its *home* configuration.
  - The screw axes  $\mathcal{S}_1^0, \mathcal{S}_2^1, \dots, \mathcal{S}_n^{n-1}$ , corresponding to the joint motions at *home* configuration, expressed in base  $\{0\}$  frame.
  - The joint configurations  $q$ .



# Recap: Differential Kinematics

- The differential/velocity kinematics is the differential of this map which is a mapping from  $\dot{q} \in T_q Q$  to  $\dot{H}_e^0 \in T_{H_e^0} SE(3)$ .
- **Geometrically**, we can equivalently represent it as a map from  $\dot{q} \in T_q Q \cong \mathbb{R}^n$  to the end effector's twist  $\mathcal{V}_e^{*,0} \in \mathbb{R}^6$



# Recap: Geometric Jacobian

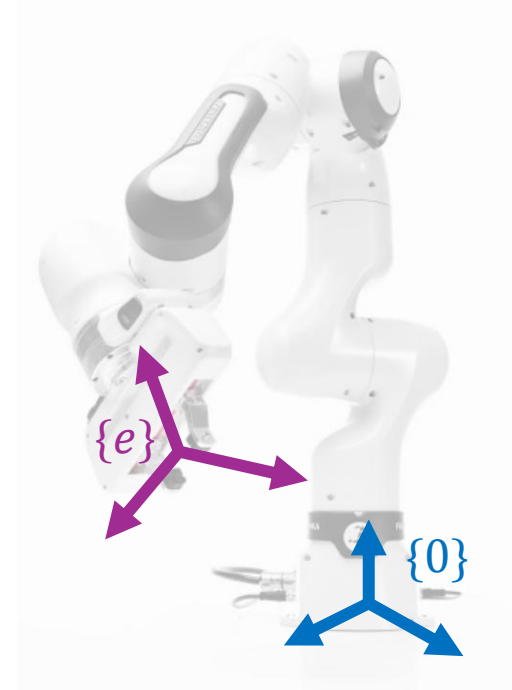
- The spatial Jacobian  $J_e^{0,0}(q) \in \mathbb{R}^{6 \times n}$  relates the joint rates  $\dot{\theta} \in \mathbb{R}^n$  to the spatial end effector's twist  $\mathcal{V}_n^{0,0} \in \mathbb{R}^6$  by

$$\mathcal{V}_e^{0,0} = J_e^{0,0}(q)\dot{q}$$

- The  $i$ -th column of  $J_e^{0,0}(q)$  is given for  $i \in \{2, \dots, n\}$  by

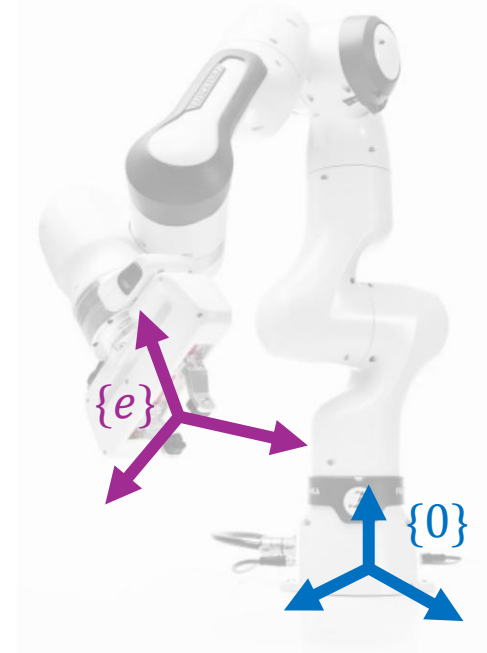
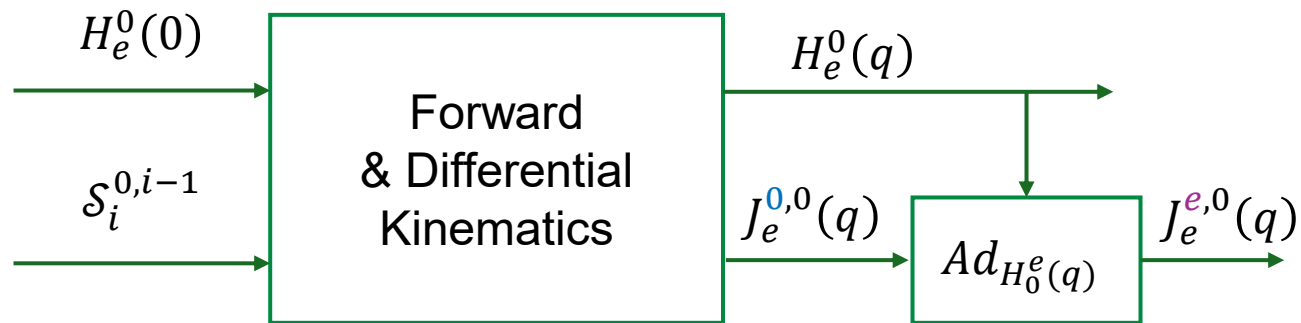
$$J_i^0(q_{1:i-1}) = \text{Ad}_{e^{\tilde{s}_1^{0,0} q_1} e^{\tilde{s}_2^{0,1} q_2} \dots e^{\tilde{s}_{i-1}^{0,i-2} q_{n-1}}} \mathcal{S}_i^{0,i-1}$$

with the 1<sup>st</sup> column  $J_1^0 = \mathcal{S}_1^{0,0}$ .



# Recap: Summary

- The above approach is highly systematic and can be easily programmable.
- The only inputs needed is the end-effector initial poses  $H_e^0(0)$  and the constant screw axes  $\mathcal{S}_i^{0,i-1}$  for joints in the home configuration.



- Practice problems in Homework 4 (Included in Midterm Exam)
- MATLAB Implementation next Lecture (Used for Project Deliverables)

# Recap: Pose & Body Twist of Link $i$

- The pose of link  $\{i\}$  can be computed by

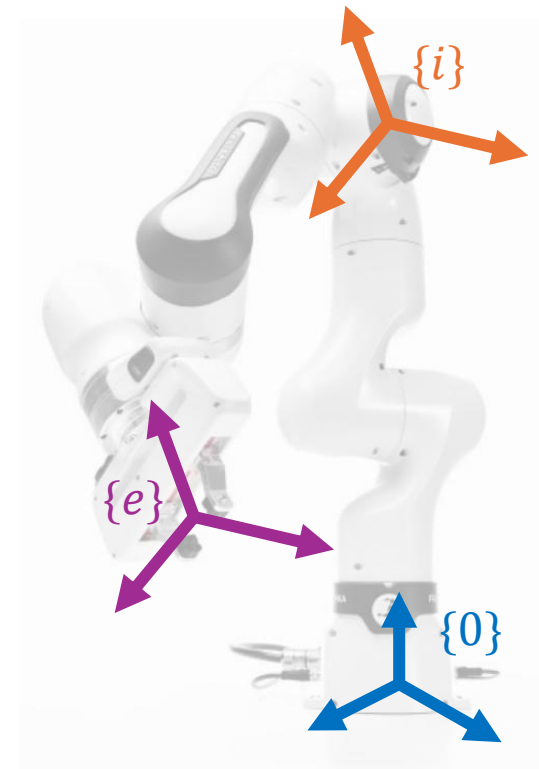
$$H_i^0(q_{1:i}) = e^{\tilde{\mathcal{S}}_1^{0,0} q_1} e^{\tilde{\mathcal{S}}_2^{0,1} q_2} \dots e^{\tilde{\mathcal{S}}_i^{0,i-1} q_i} H_i^0(0)$$

$$q_{1:i} := (q_1, \dots, q_i)$$

- The body twist of link  $\{i\}$  is computed by

$$\mathcal{V}_i^{i,0} = J_i^{i,0}(q) \dot{q}$$

with  $J_i^{i,0}(q) \in \mathbb{R}^{6 \times n}$  the geometric body Jacobian of link  $i$ .



# Recap: Wrench-Twist Duality

- **Relative Twist at Joint**

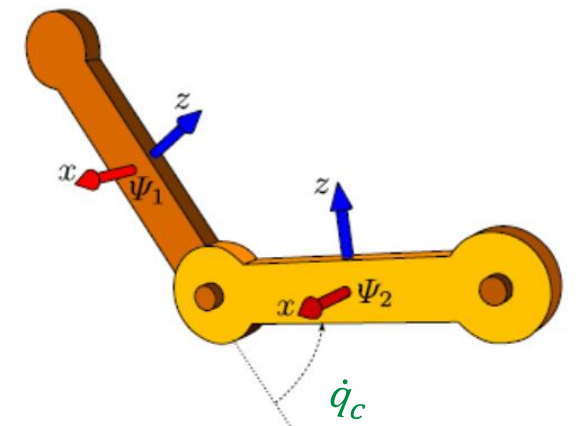
$$\mathcal{V}_c^{c,p} = \mathcal{S}_c^{c,p} \dot{q}_c$$

- **Constraint Wrench at Joint**

$$\tau_c = (\mathcal{S}_c^{c,p})^\top \mathcal{W}_p^{c,c}$$

- **Power**

$$(\mathcal{W}_p^{c,c})^\top \mathcal{V}_c^{c,p} = \tau_c \dot{q}_c$$



# Recap: Wrench-Twist Duality

- **Link Body twist**

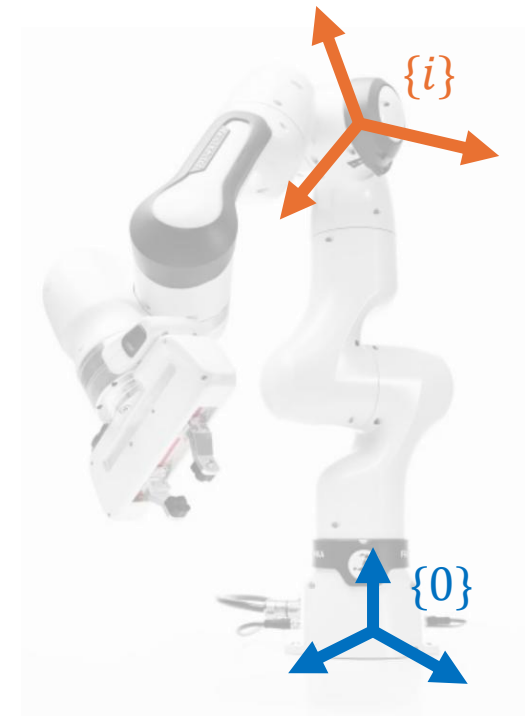
$$\mathcal{V}_i^{i,0} = J_i^{i,0}(q)\dot{q}$$

- **External Wrench at Link**

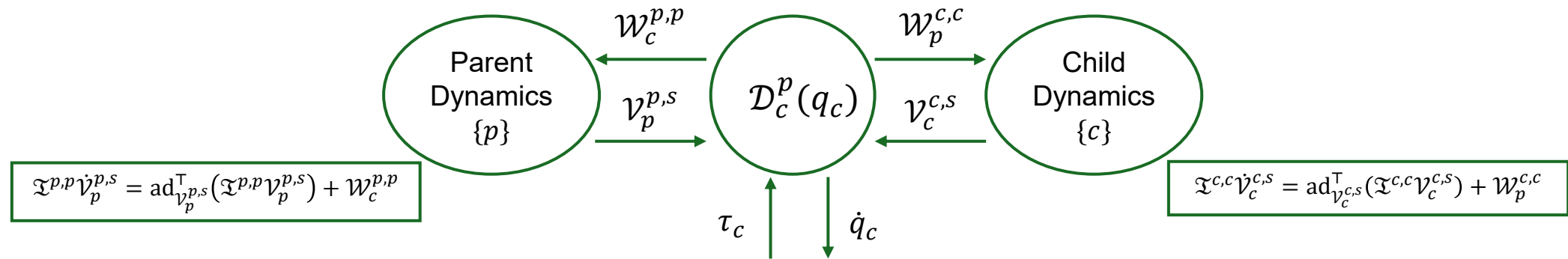
$$\tau_{\text{ext}} = \left( J_i^{i,0}(q) \right)^\top \mathcal{W}_{\text{ext}}^{i,i}$$

- **Power**

$$\left( \mathcal{W}_{\text{ext}}^{i,i} \right)^\top \mathcal{V}_i^{i,0} = \tau_{\text{ext}}^\top \dot{q}$$

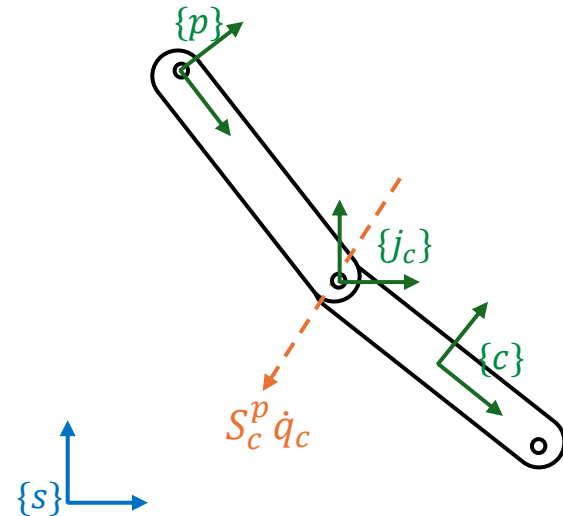


# Recap: Parent-Child Dynamic Model



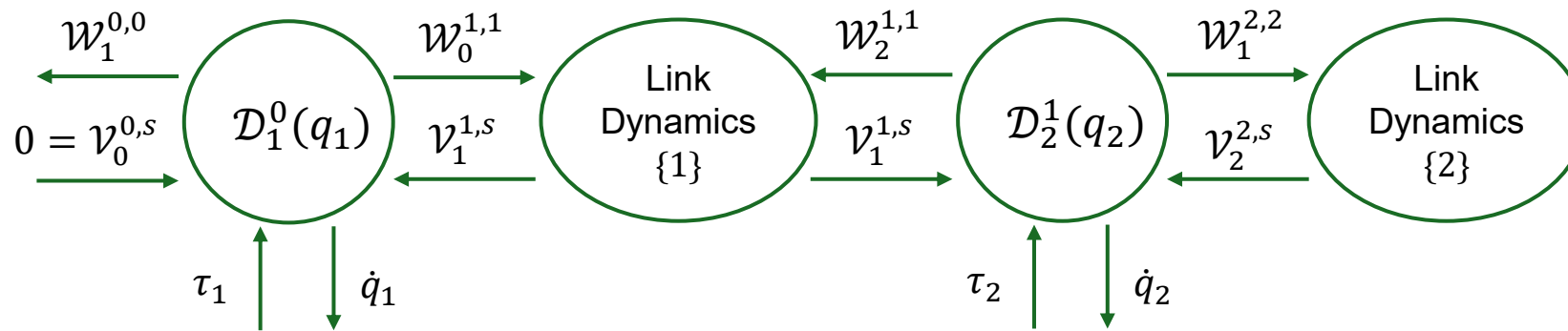
- Joint Constraints:  $\mathcal{D}_c^p(q_c) := \left\{ \begin{array}{l} \mathcal{V}_c^{c,s} = \text{Ad}_{H_p^c(q_c)} \mathcal{V}_p^{p,s} + S_c^{c,p} \dot{q}_c \\ \mathcal{W}_c^{p,p} = -\text{Ad}_{H_p^c(q_c)}^\top \mathcal{W}_p^{c,c} \\ \tau_c = (S_c^{c,p})^\top \mathcal{W}_p^{c,c} \end{array} \right\}$

- Relative pose:  $H_c^p(q_c) = H_c^p(0) e^{\tilde{S}_c^{c,p} q_c}$

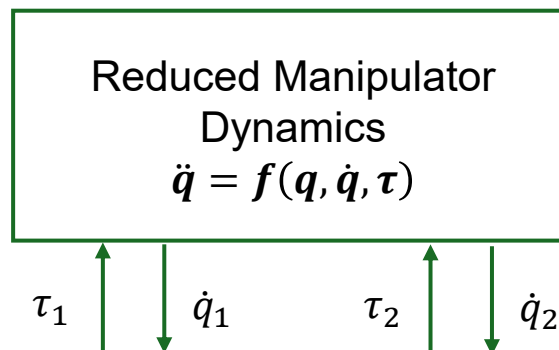


# Recap: From DAEs to ODEs

- In the case of a 2-link manipulator, the DAEs will take the form:



- For manipulators, we can eliminate the constraints wrenches to represent the dynamics as a set of Ordinary Differential Equations (ODEs) of the form.



# Recap: Dynamic Equations in Closed Form

- In general, the recursive Newton Euler approach can be assembled into the following set of matrix equations:
  - $\mathcal{V} = \mathcal{L}(q)\mathcal{S} \dot{q}$
  - $\dot{\mathcal{V}} = \mathcal{L}(q)\mathcal{A}(q)\mathcal{V} + \mathcal{L}(q)\mathcal{S} \ddot{q}$
  - $\mathcal{W} = \mathcal{L}^\top(q)\mathcal{I}\dot{\mathcal{V}} - \mathcal{L}^\top(q)\text{ad}_{\mathcal{V}}^\top \mathcal{I} \mathcal{V}$
  - $\tau = \mathcal{S}^\top \mathcal{W}$

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_1^{1,s} \\ \vdots \\ \mathcal{V}_n^{n,s} \end{pmatrix} \in \mathbb{R}^{6n}$$

$$\mathcal{W} = \begin{pmatrix} \mathcal{W}_0^{1,1} \\ \vdots \\ \mathcal{W}_{n-1}^{n,n} \end{pmatrix} \in (\mathbb{R}^{6n})^*$$

$$\dot{q} = \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \in \mathbb{R}^n$$

$$\tau = \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_n \end{pmatrix} \in (\mathbb{R}^n)^*$$



# Recap: Dynamic Equations in Closed Form

- In general, the recursive Newton Euler approach can be assembled into the following set of matrix equations:
  - $\mathcal{V} = \mathcal{L}(q)\mathcal{S} \dot{q}$
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  - $\mathcal{W} = \mathcal{L}^\top(q)\mathfrak{I}\dot{\mathcal{V}} - \mathcal{L}^\top(q)\text{ad}_{\mathcal{V}}^\top \mathfrak{I} \mathcal{V}$
  - $\tau = \mathcal{S}^\top \mathcal{W}$

$$\mathcal{L}(q) = \begin{pmatrix} I & 0 & \dots & \dots & 0 \\ \text{Ad}_{H_1^2}(q) & I & \dots & \dots & 0 \\ \text{Ad}_{H_1^3}(q) & \text{Ad}_{H_2^3}(q) & I & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \text{Ad}_{H_1^n}(q) & \text{Ad}_{H_2^n}(q) & \text{Ad}_{H_3^n}(q) & \dots & I \end{pmatrix} \in \mathbb{R}^{6n \times 6n}$$

$$\mathcal{S} = \begin{pmatrix} S_1^{1,0} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & S_n^{n,n-1} \end{pmatrix} \in \mathbb{R}^{6n \times n}$$

$$\text{ad}_{\mathcal{V}} = \begin{pmatrix} \text{ad}_{\mathcal{V}_1^{1,s}} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \text{ad}_{\mathcal{V}_n^{n,s}} \end{pmatrix} \in \mathbb{R}^{6n \times 6n}$$

$$\mathfrak{I} = \begin{pmatrix} \mathfrak{I}^{1,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathfrak{I}^{n,n} \end{pmatrix} \in \mathbb{R}^{6n \times 6n}$$



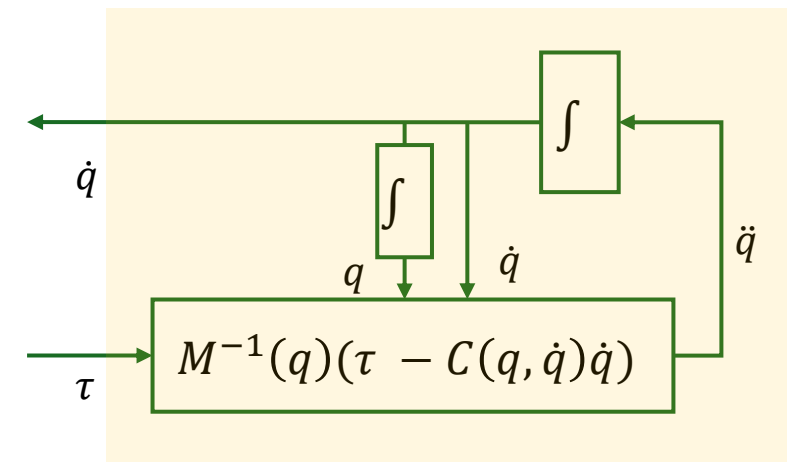
# Recap: Reduced $n$ -link manipulator dynamics

- Finally, we reach the form

$$\tau = M(q)\ddot{q} + C(q, \dot{q})\dot{q}$$

$$M(q) := J^T(q)\mathfrak{L}J(q) \in \mathbb{R}^{n \times n}$$

$$C(q, \dot{q}) := J^T(q)\mathfrak{L}L(q)\dot{A}(q)J(q) - J^T(q)\text{ad}_{J(q)\dot{q}}^T \mathfrak{L}J(q) \in \mathbb{R}^{n \times n}$$



$M(q)$  : Mass (Inertia) matrix

$C(q, \dot{q})$ : Coriolis and Centripetal Matrix



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- **Modeling External Wrenches**
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# Dynamics $n$ -link manipulator

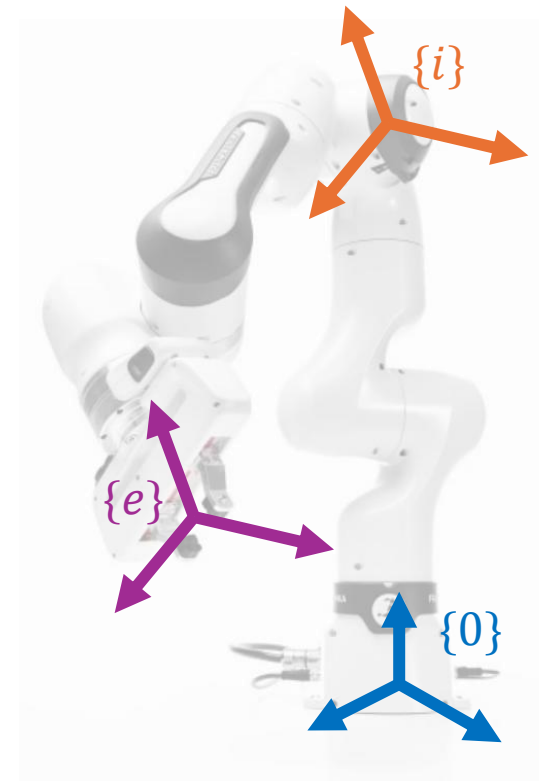
- The dynamic equation

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau$$

$M(q)$  : Mass (Inertia) matrix  
 $C(q, \dot{q})$ : Coriolis and Centripetal Matrix

describes only the inertial effects of the manipulator.

- Missing External Wrenches:
  - Torques due to interaction at end-effector
  - Torques due to gravity



# Adding External Wrenches

- If we now consider the case that there is an external wrench  $\mathcal{W}_{\text{ext}}^{i,i}$  applied to each link, then the dynamic equations would be:

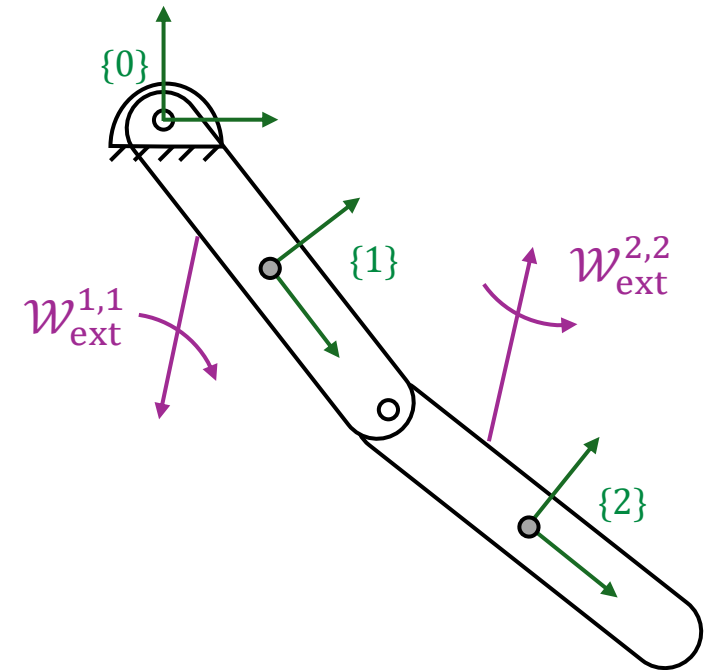
- $\mathcal{V} = \mathcal{L}(q)\mathcal{S} \dot{q}$

- $\dot{\mathcal{V}} = \mathcal{L}(q)\dot{\mathcal{A}}(q)\mathcal{V} + \mathcal{L}(q)\mathcal{S} \ddot{q}$

- $\mathcal{W} = \mathcal{L}^T(q)\mathfrak{I}\dot{\mathcal{V}} - \mathcal{L}^T(q)\text{ad}_{\mathcal{V}}^T \mathfrak{I}\mathcal{V} - \mathcal{L}^T(q)\mathcal{W}_{\text{ext}}$

- $\tau = \mathcal{S}^T \mathcal{W}$

where  $\mathcal{W}_{\text{ext}} := \begin{pmatrix} \mathcal{W}_{\text{ext}}^{1,1} \\ \vdots \\ \mathcal{W}_{\text{ext}}^{n,n} \end{pmatrix} \in (\mathbb{R}^{6n})^*$ .



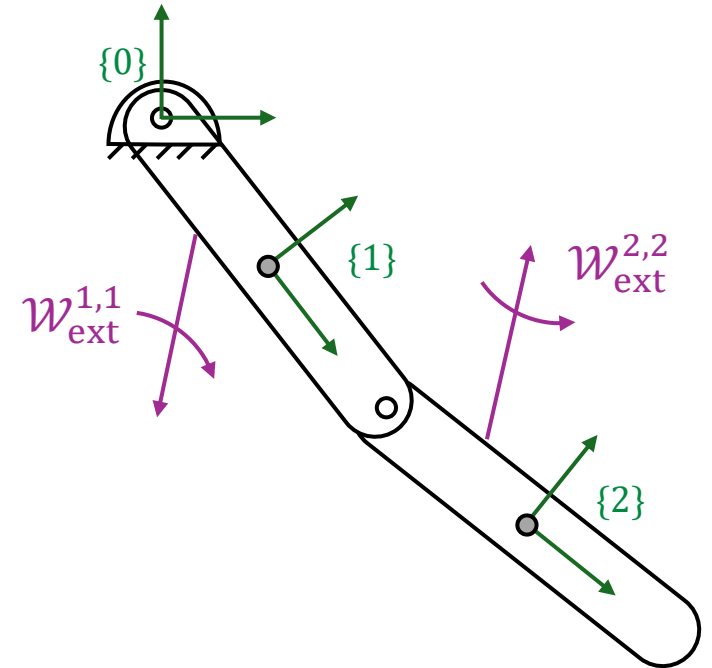
# Adding External Wrenches

- The reduced dynamics then take the form

- $M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau + \mathcal{J}^T(q)\mathcal{W}_{\text{ext}}$

- $\mathcal{V} = \mathcal{J}(q)\dot{q}$

where  $\mathcal{J}(q) := \mathcal{L}(q)\mathcal{S} \in \mathbb{R}^{6n \times n}$  and the second equation is an (extra) output equation.

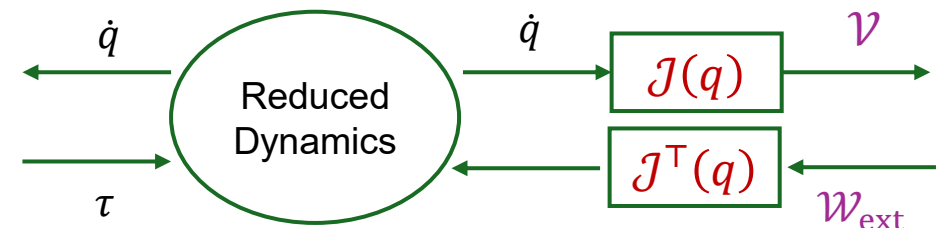


# Geometric Jacobian

- If we examine carefully the mapping  $\mathcal{J}(q)$  one can see:

$$\mathcal{J}(q) = \begin{pmatrix} S_1^{1,0} & 0 & \dots & \dots & 0 \\ \text{Ad}_{H_1^2(q)} S_1^{1,0} & S_2^{2,1} & \dots & \dots & 0 \\ \text{Ad}_{H_1^3(q)} S_1^{1,0} & \text{Ad}_{H_2^3(q)} S_2^{2,1} & S_3^{3,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \text{Ad}_{H_1^n(q)} S_1^{1,0} & \text{Ad}_{H_2^n(q)} S_2^{2,1} & \text{Ad}_{H_3^n(q)} S_3^{3,2} & \dots & S_n^{n,n-1} \end{pmatrix} = \begin{pmatrix} \mathcal{J}_1^{1,0} \\ \mathcal{J}_2^{2,0}(q) \\ \mathcal{J}_3^{3,0}(q) \\ \vdots \\ \mathcal{J}_n^{n,0}(q) \end{pmatrix}$$

- We see that  $\mathcal{J}(q)$  is a collection of  $n$  geometric Jacobians, with its  $i$ -th block equivalent to the geometric body\* Jacobian of the  $i$ -th link.



# Mass Matrix Revisited

- Consequently, the mass matrix can be computed efficiently by

$$M(q) := J^T(q)\mathfrak{I}J(q) = \sum_{i=1}^n \left( J_i^{i,0}(q) \right)^T \mathfrak{I}^{i,i} J_i^{i,0}(q)$$

$$J(q) = \begin{pmatrix} J_1^{1,0} \\ J_2^{2,0}(q) \\ J_3^{3,0}(q) \\ \vdots \\ J_n^{n,0}(q) \end{pmatrix} \quad \mathfrak{I} = \begin{pmatrix} \mathfrak{I}^{1,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathfrak{I}^{n,n} \end{pmatrix}$$



# Adding External Wrenches

- The reduced dynamics can be equivalently then expressed as

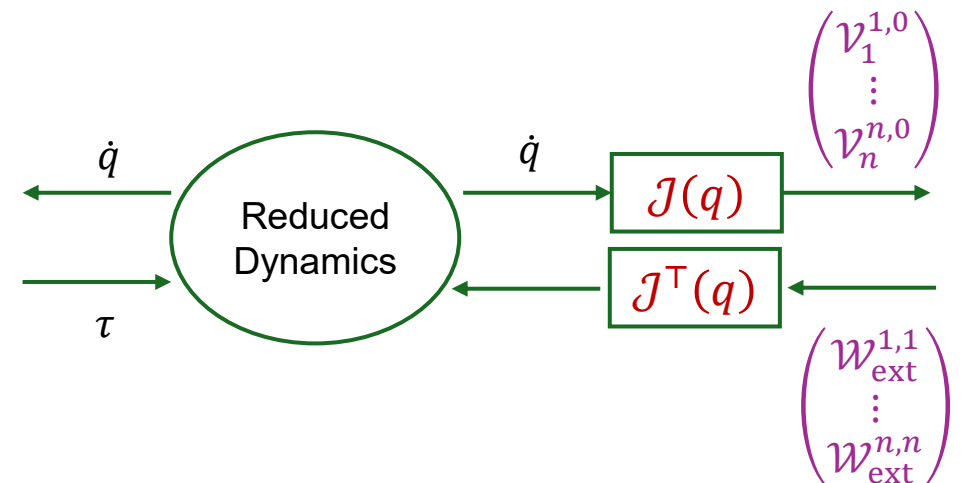
- $M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau + \left[ J_1^{1,0\top}(q) \mathcal{W}_{\text{ext}}^{1,1} + J_2^{2,0\top}(q) \mathcal{W}_{\text{ext}}^{2,2} + \dots + J_n^{n,0\top}(q) \mathcal{W}_{\text{ext}}^{n,n} \right]$

- $\mathcal{V}_1^{1,0} = J_1^{1,0}(q)\dot{q}$

- $\mathcal{V}_2^{2,0} = J_2^{2,0}(q)\dot{q}$

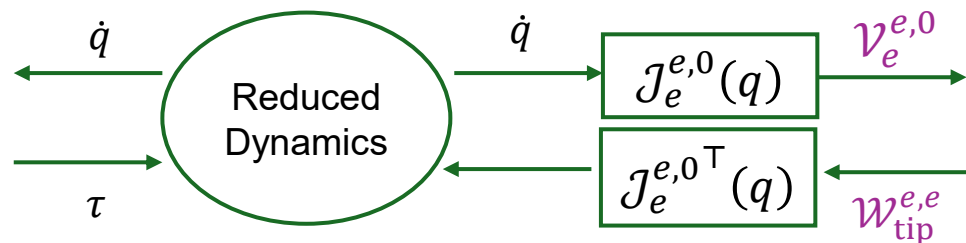
⋮

- $\mathcal{V}_n^{n,0} = J_n^{n,0}(q)\dot{q}$



# Adding End-effector wrench

- In the presence of an interaction wrench
  - $M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau + J_e^{e,0\top}(q) \mathcal{W}_{\text{tip}}^{e,e}$
  - $\mathcal{V}_e^{e,0} = J_e^{e,0}(q)\dot{q}$



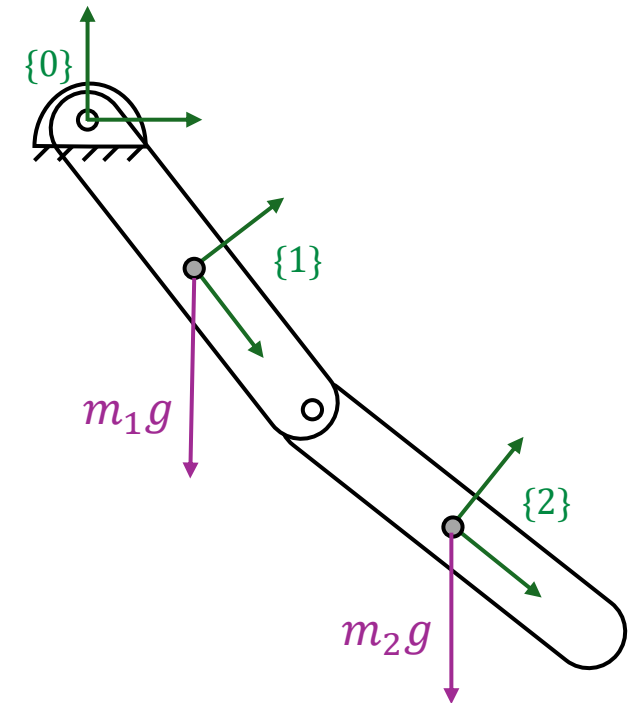
$\mathcal{W}_{\text{tip}}^{e,e}$  : wrench applied by the environment to the end-effector



# Adding Gravity to 2-Link Manipulator

- One can use the above formulation to add gravity:
  - $M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau + \left[ J_1^{1,0\top}(q) \mathcal{W}_{\text{grv}}^{1,1} + J_2^{2,0\top}(q) \mathcal{W}_{\text{grv}}^{2,2} \right]$
  - $\mathcal{V}_1^{1,0} = J_1^{1,0}(q)\dot{q}$
  - $\mathcal{V}_2^{2,0} = J_2^{2,0}(q)\dot{q}$

where  $\mathcal{W}_{\text{grv}}^{i,i} = \begin{pmatrix} 0 \\ m_i R_0^i g^0 \end{pmatrix} \in (\mathbb{R}^6)^*$ .



{i}: must be placed at CoM of i-th link to use above formula

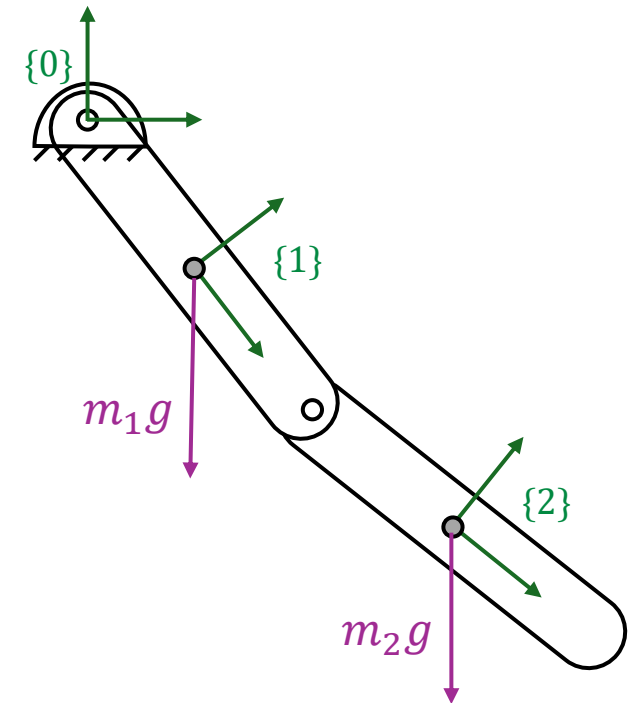


# Dynamic Equation of $n$ -link manipulator

- A popular form of the reduced dynamics is

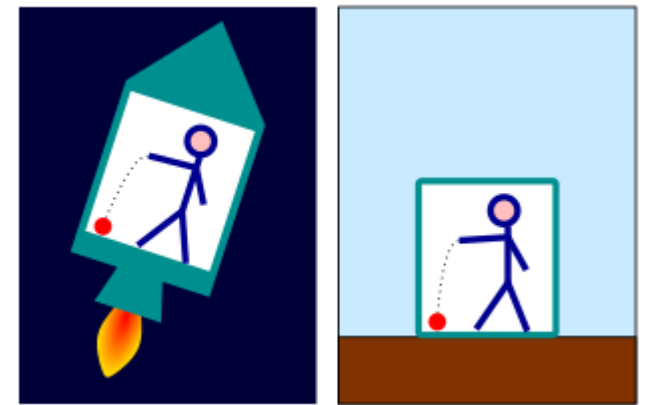
$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

where  $g(q) := -J^T(q)\mathcal{W}_{\text{grv}} \in (\mathbb{R}^n)^*$  are the torques due to gravity.



# Einstein's equivalence principle

- An alternative way of modeling gravity is to consider the stationary frame “free-falling”.
- Einstein's equivalence principle can be summarized as follows:
  - *“No local experiment can distinguish a uniform gravitational field from an accelerating reference frame.”*
- *Consequently*, All physical laws in a freely falling laboratory take the same form as in an inertial frame with no gravity.



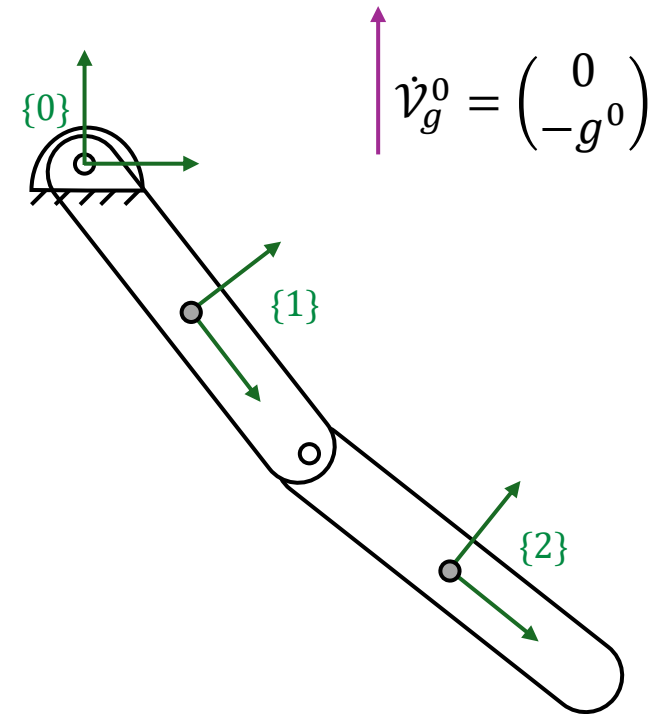
A falling object behaves exactly the same on Earth or in an equivalent accelerating frame of reference.



# Adding Gravity

- By considering the stationary frame free-falling, let  $\dot{\mathcal{V}}_g$  denote the acceleration of the “world” relative to the frame  $\{0\}$ .
- One has elegantly that the gravity torques are computed by

$$g(q) := J^T(q) \mathfrak{I} \mathcal{L}(q) \dot{\mathcal{V}}_{\text{base}}$$



$$\text{where } \dot{\mathcal{V}}_{\text{base}} := \begin{pmatrix} \text{Ad}_{H_0^1(q_1)} \dot{\mathcal{V}}_g^0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{6n}$$



# Summary: Manipulator Dynamics

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau + J_e^{e,0\top}(q) \mathcal{W}_{tip}^{e,e}$$

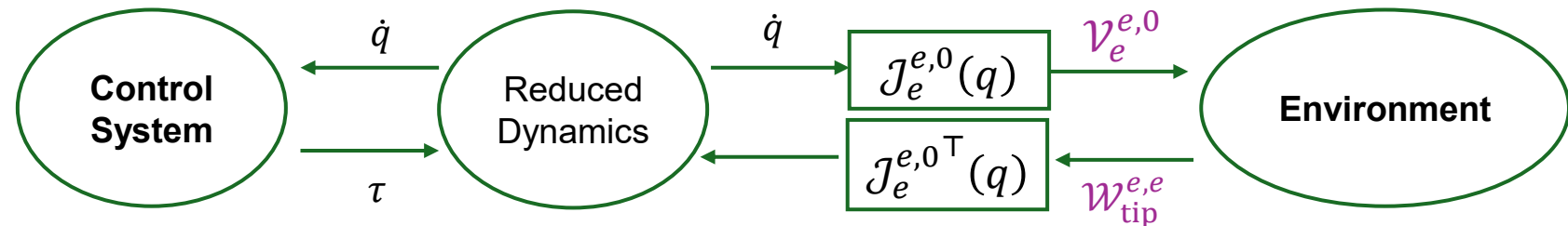
Inertia effects

Coriolis & Centripetal effects

Gravity torque

Control torque

End-effector Wrench



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