SCE 594: Special Topics in Intelligent Automation & Robotics

Topic 4: Stability and control of mechanical systems

Lecture 17: Equilibrium points and Stability Notions

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Outline

- State Space Models
- State Space Models of Mechanical Systems
- Equilibrium points and Stability notions
- Case study: Pendulum

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State Space Model

- A nonlinear dynamic system can be represented by a set of nonlinear ordinary differential equations (ODEs) in the form

$$\dot{x} = f(x) + g(x)\,u$$

$$y = h(x)$$

which is called the state space model of the dynamic system.

- We denote by the
  - State space $\mathcal{X} \ni x$
  - Control space $\mathcal{U} \ni u$
  - Output space $\mathcal{Y} \ni y$

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- The function $f(x)$ is called the drift term, which describes how the system evolves when no control is applied (i.e., natural behavior of the system).

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- The function $g(x)$ is called the input mapping, which describes how the control affects the state evolution.

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- The function $h(x)$ is called the output mapping, which describes which part of the internal state is available as an output that you measure or observe.

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Special case: Linear time-invariant systems

$$\dot{x} = Ax + Bu$$
$$y = C\,x$$

$\mathcal{X} = \mathbb{R}^m, \mathcal{U} = \mathbb{R}^p, \mathcal{Y} = \mathbb{R}^q$

Control Objectives

- The control input is designed in general as a function of the output $u = \beta(y)$ to achieve:
  - Regulation/Stabilization $x(t) \rightarrow x_d$ as $t \rightarrow \infty$
  - Tracking $x(t) \rightarrow x_d(t)$, $\dot{x}(t) \rightarrow \dot{x}_d(t)$ as $t \rightarrow \infty$
  - Interaction

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- The design of the control system is based on analyzing the stability of the closed-loop system:

$$\dot{x} = f_{\text{cl}}(x)$$

where $f_{\text{cl}}(x) \coloneqq f(x) + g(x) \cdot \beta\big(h(x)\big)$

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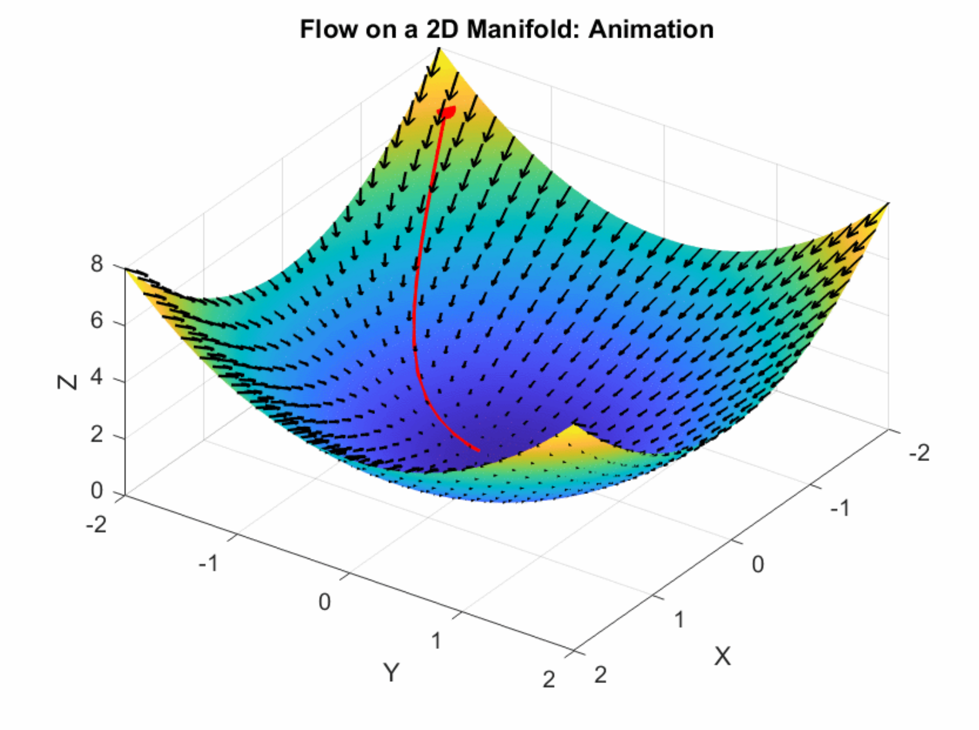
Geometric Nature of $\dot{x}(t) = f(x(t))$

- Euclidean case $\mathcal{X} = \mathbb{R}^{\boldsymbol{n}}$:
  - $x_t \in \mathbb{R}^{\boldsymbol{n}}$
  - $\dot{x}_t \in \mathbb{R}^{\boldsymbol{n}}$
  - $f: \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}^{\boldsymbol{n}}$

Geometric Nature of $\dot{x}(t) = f(x(t))$

- Non-Euclidean case:
  - $x_t \in \mathcal{X}$
  - $\dot{x}_t \in T_{\boldsymbol{x}}\mathcal{X}$
  - $f: x_t \in \mathcal{X} \mapsto \dot{x}_t \in T_{\boldsymbol{x}}\mathcal{X}$



The map f creates a vector field $\sigma_f \in \Gamma(T\mathcal{X})$ on the state space manifold $\sigma_f: \mathcal{X} \to T\mathcal{X}$ defined by:
$\sigma_f(x) \coloneqq \big(x, f(x)\big) \in T\mathcal{X}$, with $f(x) \in T_{\boldsymbol{x}}\mathcal{X}$

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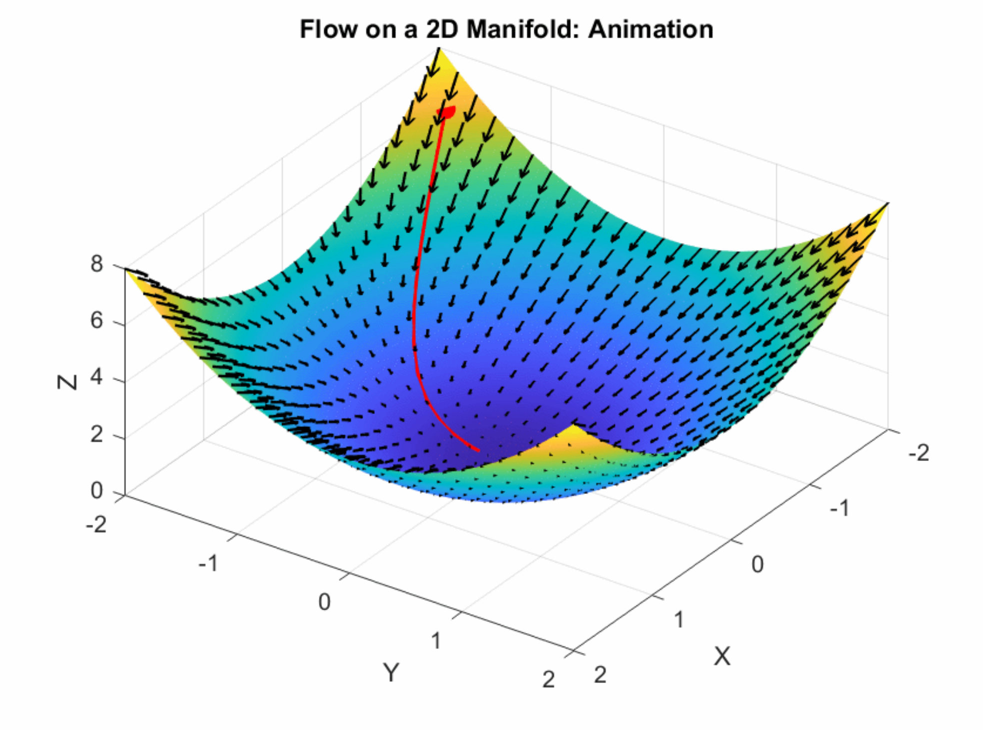

The solution of the dynamical system $x(t)$ is given by the integral curves of σ_f.

Integral Curves
While $f(x)$ represents the velocity of a particle at every point, the integral curve represents the trajectory of a particle moving along this velocity field.

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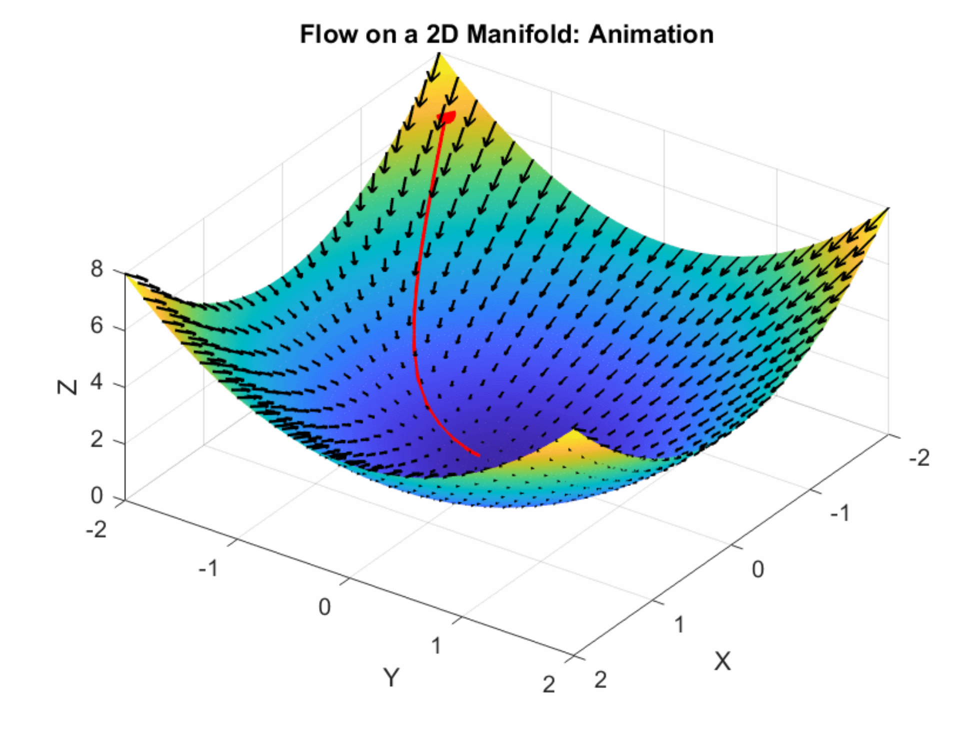


Stability analysis

By analyzing properties of f or σ_f, we can infer how the system's state $x(t)$ will evolve with time, without explicitly computing the solution as a function of time.

Equilibrium Points

- Given $\sigma_f \in \Gamma(T\mathcal{X})$, a point $x_* \in \mathcal{X}$ in the state space is called an *equilibrium point for σ_f* if and only if

$$f(x_*) = 0$$

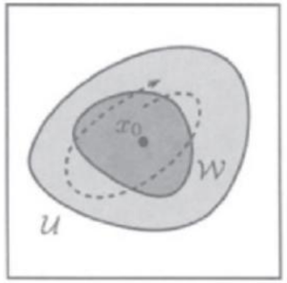
- Intuitively, an equilibrium point is a state x_* at which the system state remains for all time, once it reaches it.

- Equilibrium points are also referred to as "fixed points" or "critical points" of the system $\dot{x}(t) = f(x(t))$.

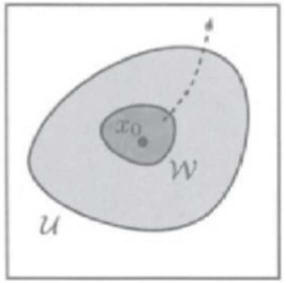
Stability of Equilibrium Points

- An equilibrium point x_* of σ_f is said to be:

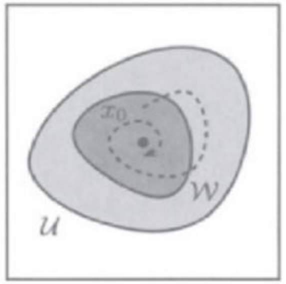
Stable	If integral curves of σ_f stay "close" to x_*
Unstable	If it is not stable
Locally asymptotically stable	If it is stable and integral curves of σ_f converge to x_* only within a region $\mathrm{U} \subset \mathcal{X}$
Globally asymptotically stable	If it is stable and integral curves of σ_f converge to x_* for all $x \in \mathcal{X}$



Stable x_0



Unstable x_0



Locally asym. stable x_0

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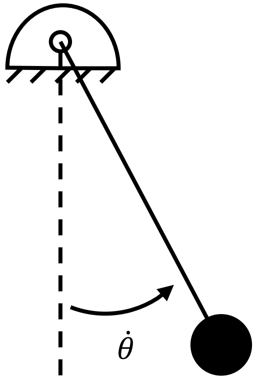
Equilibrium points of pendulum

- Consider the state-space model of the pendulum given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 \end{pmatrix} =: f(x)$$

where $x_1 = \theta, x_2 = \dot{\theta}$.



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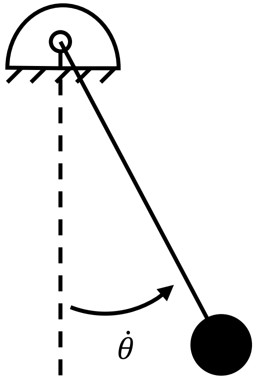
- To find the equilibrium points x_*, we set $f(x) = 0$:

$$x_2 = 0 \quad , \quad -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 = 0,$$



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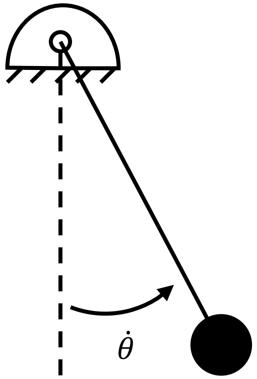
- To find the equilibrium points x_*, we set $f(x) = 0$, which simplifies to

$$x_2 = 0 \ , \ \ \sin x_1 = 0$$

  - Solving for $x_1 \in \mathbb{R}$, we get:

$$\sin x_1 = 0 \quad \Rightarrow \quad x_1 = k\pi, \qquad k \in \mathbb{Z}$$



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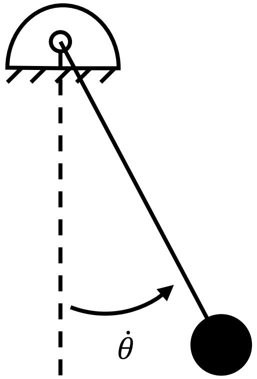
  - However, recall that $x_1 = \theta \in (-\pi, \pi] \cong \mathbb{S}^1$, thus the system has only two equilibrium points



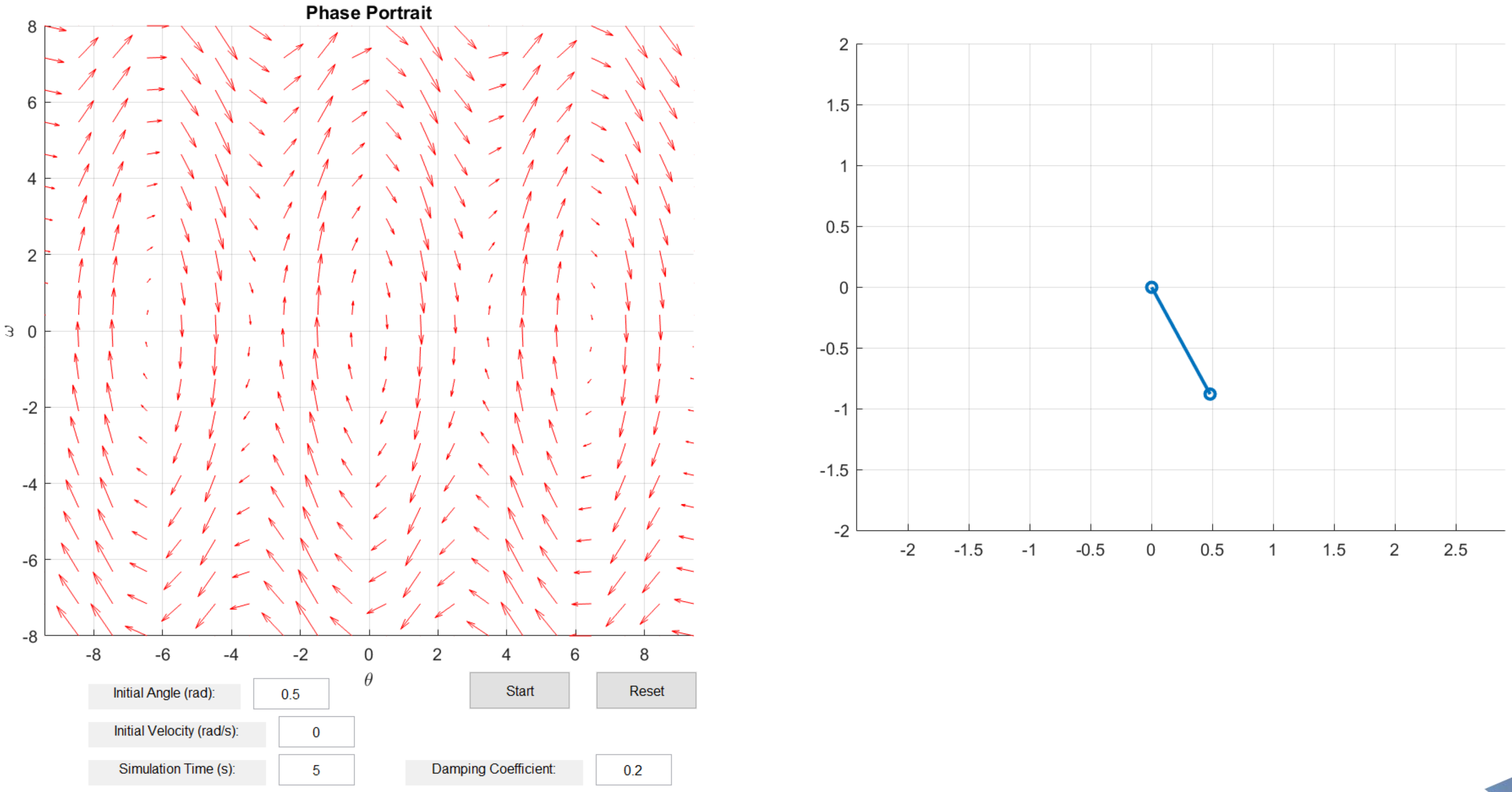
$$\boxed{x_* = (0,0)}$$

Downward position

$$\boxed{x_* = (\pi, 0)}$$

Upward position

MATLAB Code



Initial Angle (rad): 0.5

Initial Velocity (rad/s): 0

Simulation Time (s): 5

Damping Coefficient: 0.2

Equilibrium points of pendulum

- In summary, the state space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 \end{pmatrix}$$

has an asymptotically stable equilibrium at $x_* = (0,0)$ and an unstable equilibrium at $x_* = (\pi, 0)$.