SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 18: Lyapunov's Direct Method I

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Outline

- Recap last lecture
- Case study: Pendulum
- Lyapunov stability analysis
- Positive definite functions

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Recap: State Space Model

- A nonlinear dynamic system can be represented by a set of nonlinear differential equations in the form

$$\dot{x} = f(x) + g(x)\,u$$
$$y = h(x)$$

which is called the state space model of the dynamic system.

- We denote by the
  - State space $\mathcal{X} \ni x$
  - Control space $\mathcal{U} \ni u$
  - Output space $\mathcal{Y} \ni y$

Special case: Linear time-invariant systems

$$\dot{x} = Ax + Bu$$
$$y = C\,x$$

Recap: Control Objectives

- The control input is designed in general as a function of the output $u = \beta(y)$ to achieve:
  - Regulation/Stabilization $x(t) \to x_d$ as $t \to \infty$
  - Tracking $x(t) \to x_d(t), \dot{x}(t) \to \dot{x}_d(t)$ as $t \to \infty$
  - Interaction

- The design of the control system is based on analyzing the stability of the closed-loop system:

$$\dot{x} = f_{\mathrm{cl}}(x)$$

where $f_{\mathrm{cl}}(x) \coloneqq f(x) + g(x) \cdot \beta\big(h(x)\big)$

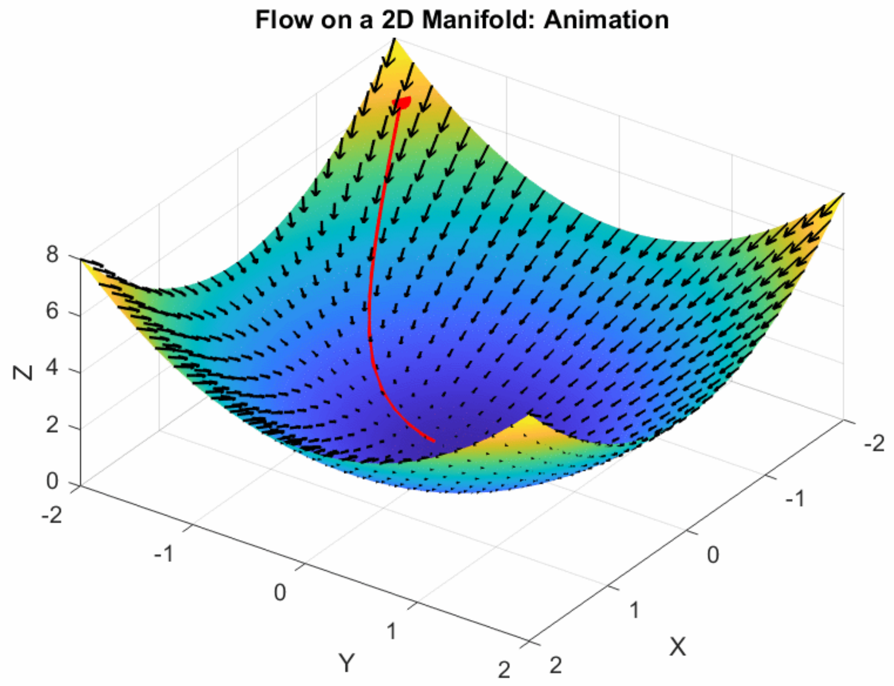
Recap: State Space Models – Mechanical Systems

System	State x	State Space $\mathcal{X}$
Mass-Spring-Damper	$(\xi, \dot{\xi})$	$\mathbb{R} \times \mathbb{R} \cong T\mathbb{R}$
Simple pendulum	$(\theta, \dot{\theta})$	$(-\pi, \pi] \times \mathbb{R} \cong T\mathbb{S}$
n-link Manipulator	$(q, \dot{q})$	TQ
Satellite	(R, ω)	$SO(3) \times \mathbb{R}^3 \cong TSO(3)$
Multirotor Aerial Vehicle	$(H, \mathcal{V})$	$SE(3) \times \mathbb{R}^6 \cong TSE(3)$

For mechanical systems in general, $x = (q, \dot{q}) \in TQ$, where $q \in Q$ represents a configuration variable of the mechanical system and $\dot{q} \in T_q Q$ denotes a velocity-like variable.

Recap: Geometric Nature of $\dot{x}(t) = f(x(t))$

- Euclidean case $\mathcal{X} = \mathbb{R}^{\boldsymbol{n}}$:
  - $x_t \in \mathbb{R}^{\boldsymbol{n}}$
  - $\dot{x}_t \in \mathbb{R}^{\boldsymbol{n}}$
  - $f: \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}^{\boldsymbol{n}}$
- Non-Euclidean case:
  - $x_t \in \mathcal{X}$
  - $\dot{x}_t \in T_{\boldsymbol{x}}\mathcal{X}$
  - $f: x_t \in \mathcal{X} \mapsto \dot{x}_t \in T_{\boldsymbol{x}}\mathcal{X}$



The solution of the dynamical system $x(t)$ is given by the integral curves of σ_f.

Integral Curves
While $f(x)$ represents the velocity of a particle at every point, the integral curve represents the trajectory of a particle moving along this velocity field.

Recap: Equilibrium Points

- Given $\sigma_f \in \Gamma(T\mathcal{X})$, a point $x_* \in \mathcal{X}$ in the state space is called an *equilibrium point for σ_f* if and only if

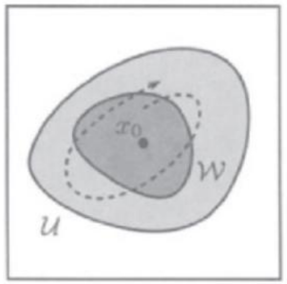
$$f(x_*) = 0$$

- Intuitively, an equilibrium point is a state x_* at which the system state remains for all time, once it reaches it.

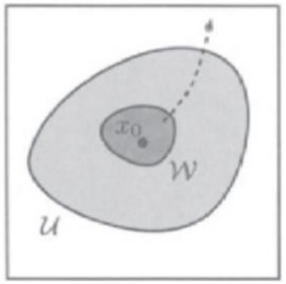
Recap: Stability of Equilibrium Points

- An equilibrium point x_* of σ_f is said to be:

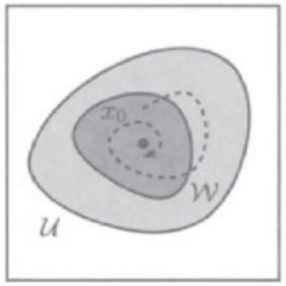
• Stable	If integral curves of σ_f stay "close" to x_*
• Unstable	If it is not stable
• Locally asymptotically stable	If it is stable and integral curves of σ_f converge to x_* only within a region $\mathrm{U} \subset \mathcal{X}$
• Globally asymptotically stable	If it is stable and integral curves of σ_f converge to x_* for all $x \in \mathcal{X}$



Stable x_0



Unstable x_0



Locally asym. stable x_0

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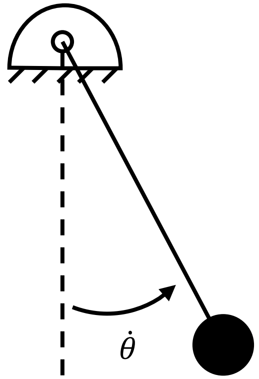
Equilibrium points of pendulum

- Consider the state-space model of the pendulum given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 \end{pmatrix} =: f(x)$$

where $x_1 = \theta, x_2 = \dot{\theta}$.



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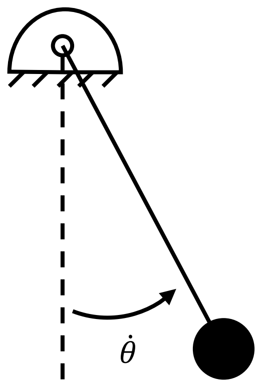
where $x_1 = \theta, x_2 = \dot{\theta}$.

- To find the equilibrium points x_*, we set $f(x) = 0$:

$$x_2 = 0 \quad , \quad -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 = 0,$$



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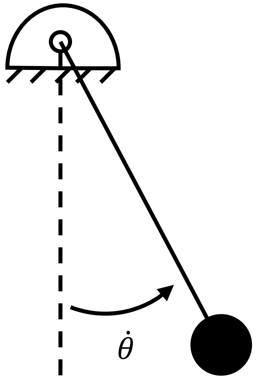
where $x_1 = \theta, x_2 = \dot{\theta}$.

- To find the equilibrium points x_*, we set $f(x) = 0$, which simplifies to

$$x_2 = 0 \ , \ \ \sin x_1 = 0$$

  - Solving for $x_1 \in \mathbb{R}$, we get:

$$\sin x_1 = 0 \quad \Rightarrow \quad x_1 = k\pi, \qquad k \in \mathbb{Z}$$



Equilibrium points of pendulum

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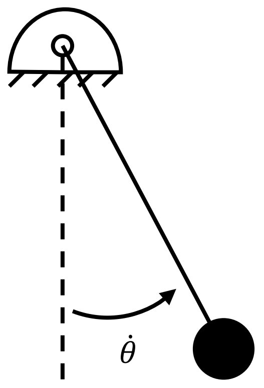
$$\sin x_1 \ = 0 \quad \Rightarrow \quad x_1 = k\pi, \qquad k \in \mathbb{Z}$$

  - However, recall that $x_1 = \theta \in (-\pi, \pi] \cong \mathbb{S}^1$, thus the system has only two equilibrium points



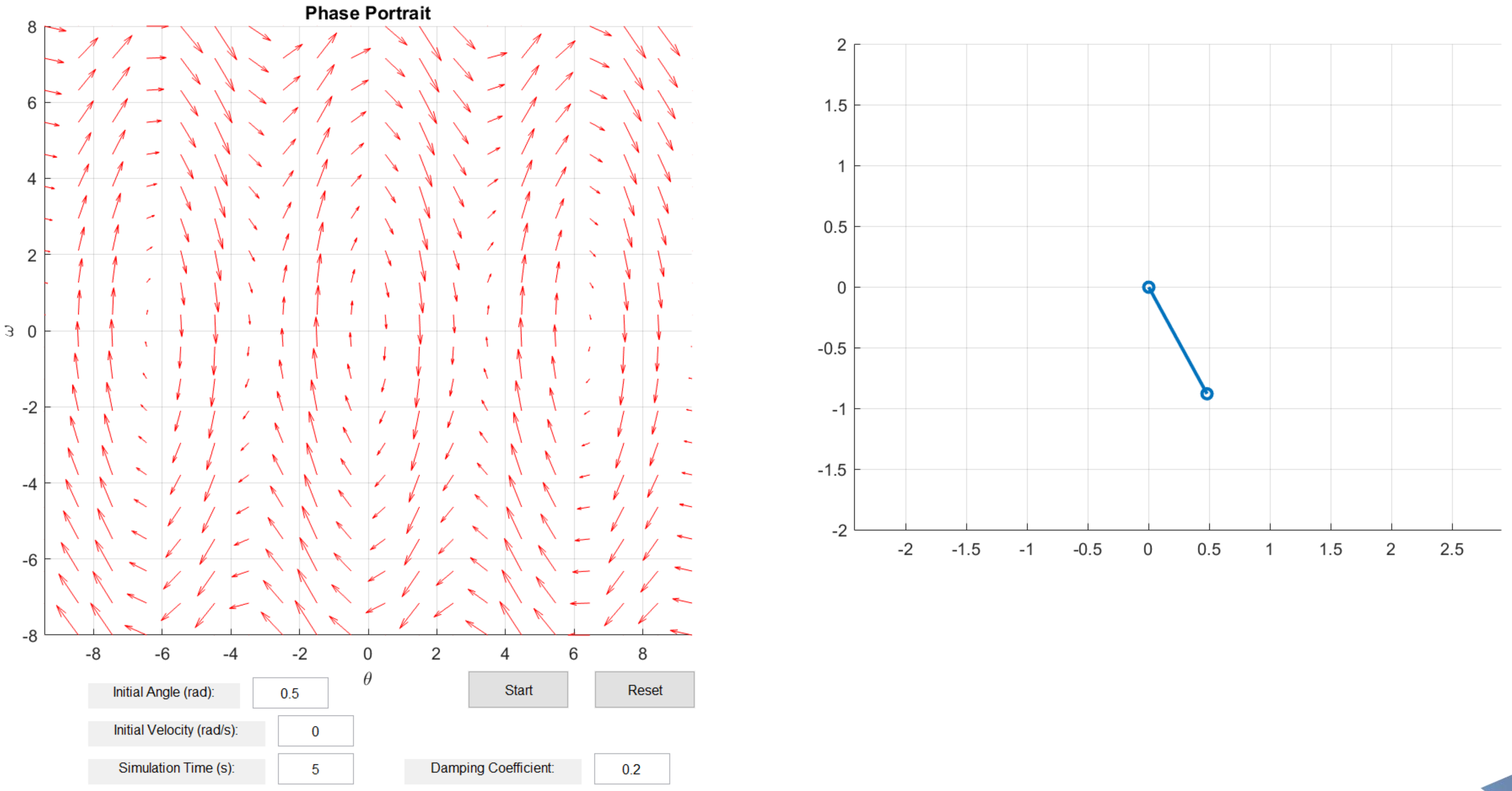
$$\boxed{x_* = (0,0)}$$

Downward position

$$\boxed{x_* = (\pi, 0)}$$

Upward position

MATLAB Code



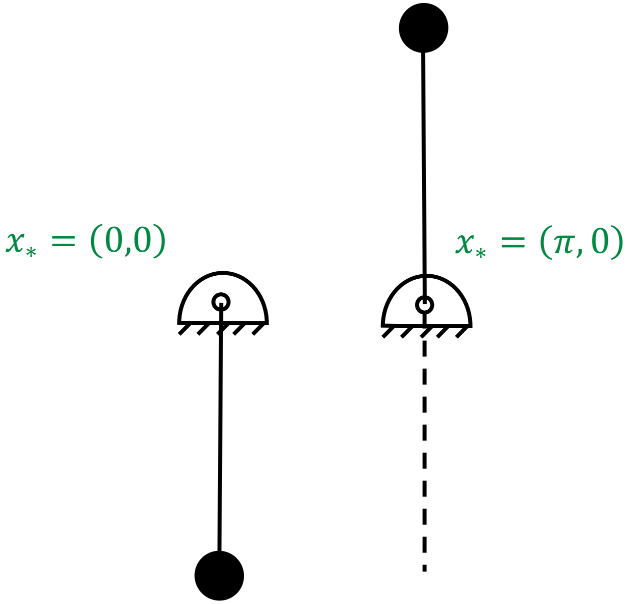
Phase Portrait
ω
θ
Initial Angle (rad):
0.5
Initial Velocity (rad/s):
0
Simulation Time (s):
5
Start
Reset
Damping Coefficient:
0.2

Equilibrium points of pendulum

- In summary, the state space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{b}{mL^2} x_2 - \frac{g}{L} \sin x_1 \end{pmatrix}$$

has an asymptotically stable equilibrium at $x_* = (0,0)$ and an unstable equilibrium at $x_* = (\pi, 0)$.



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Lyapunov stability

- The most useful and general approach for studying the stability of nonlinear dynamical systems is the theory introduced in the late 19th century by the Russian mathematician Lyapunov.

- His work introduced two methods:
  - **Indirect Method:** studies nonlinear local stability around an equilibrium point x_* from stability properties of its linear approximation.
  - **Direct Method:** not restricted to local motion. Stability of nonlinear system is studied by proposing a scalar "energy-like" function $V: \mathcal{X} \to \mathbb{R}$ for the system and examining its time variation

Aleksandr Mikhailovich Lyapunov (1857-1918) was a Russian mathematician, mechanic and physicist.
https://en.wikipedia.org/wiki/Aleksandr_Lyapunov

Lyapunov's indirect method

Read Only
Supplementary HW

- Consider the nonlinear system $\dot{x} = f(x)$ with equilibrium point x_*. The linearization of this system around the equilibrium point x_* is given by:

$$\dot{z} = A\,z$$

where $z := x - x_* \in \mathbb{R}^n$ and $A := J_f(x_*)$ is the Jacobian of $f(x)$ evaluated at the equilibrium points.

$$J_f(x) = \begin{pmatrix} df_1(x) \\ \vdots \\ df_n(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{pmatrix}$$

Lyapunov's indirect method

Read Only
Supplementary HW

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$$\dot{z} = A\,z$$

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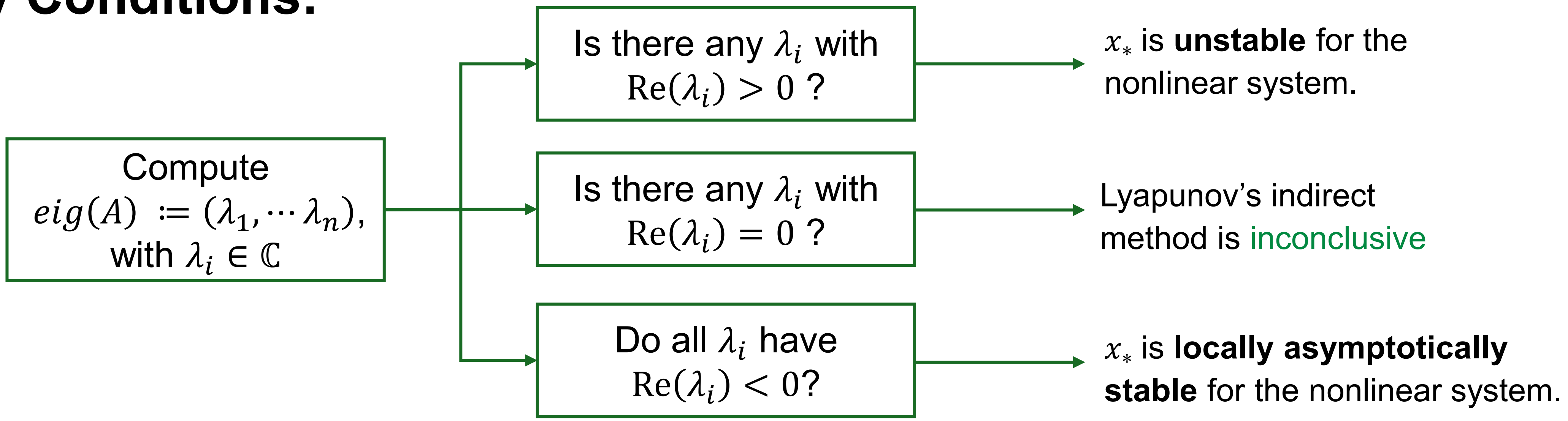
- **Stability Conditions:**



Lyapunov's direct method

$\dot{x} = f(x)$

x^* is an Eq. point of the system

Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) \leq 0$

($V(x^*) = 0$ and $V(x) > 0$: $V(x)$ is positive definite)

($\dot{V}(x) \leq 0$: $\dot{V}(x)$ is negative semi-definite)

Asymptotically Stable

Find a Lyapunov function $V(x)$ such that:

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$

($\dot{V}(x) < 0$: $\dot{V}(x)$ is negative definite)

Globally Asymptotically Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$
- $V(x) \to \infty$ as $\|x - x^*\| \to \infty$

($V(x) \to \infty$ as $\|x - x^*\| \to \infty$: $V(x)$ is radially unbounded)

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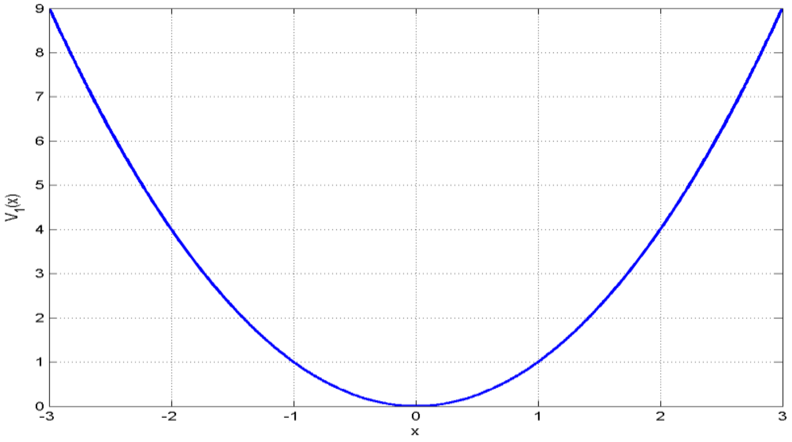
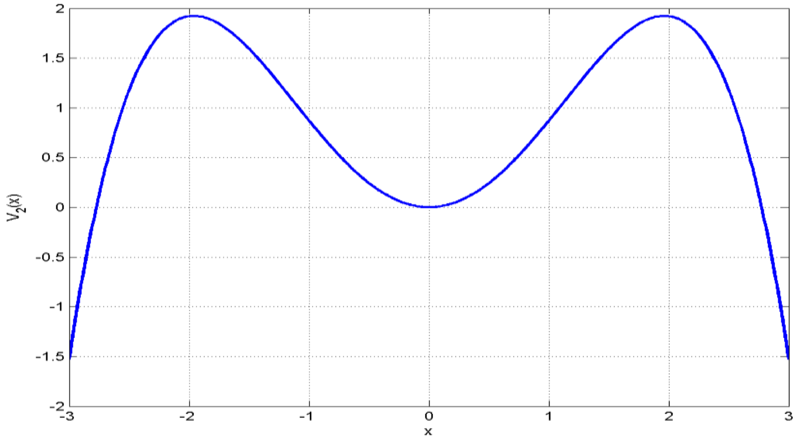
Positive Definite Functions

- A function $V(\boldsymbol{x})$ is said to be positive definite it
  - $V(\mathbf{0}) = 0$
  - $V(\boldsymbol{x}) > 0 \quad \forall \boldsymbol{x} \neq \mathbf{0}$
- Examples $x \in \mathbb{R}$:

$V_1(x) = x^2$

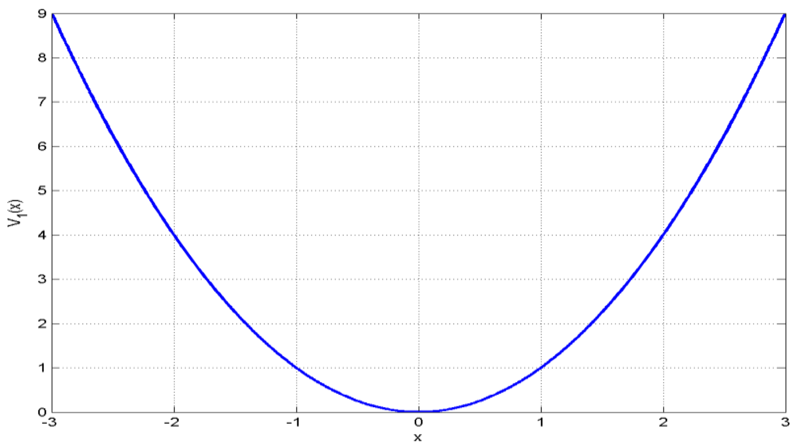


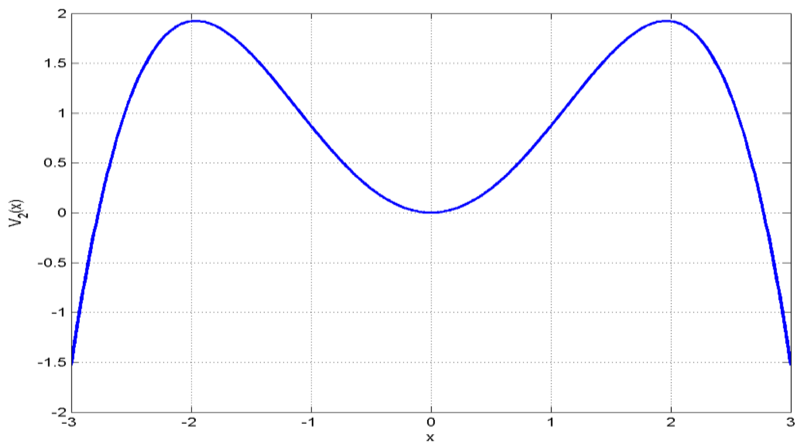
$V_2(x) = x^2$ - 0.13 x^4



Positive Definite Functions

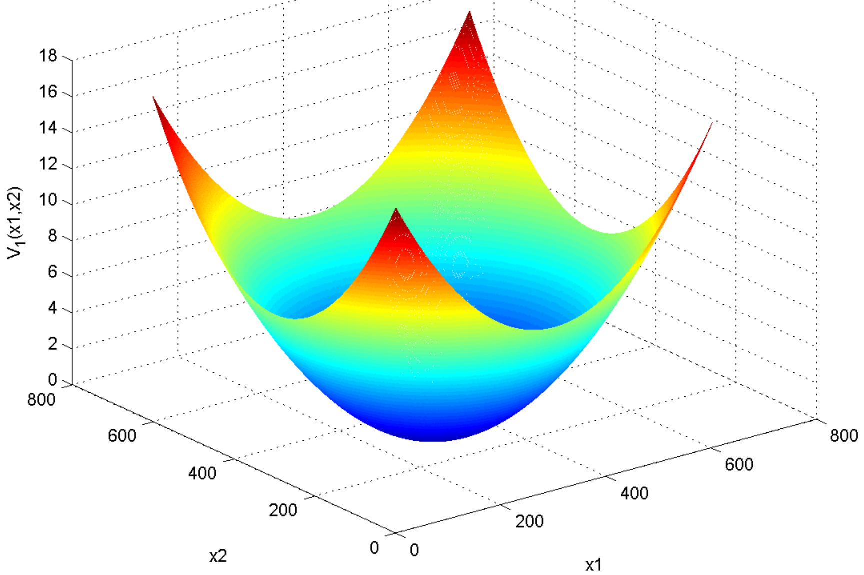
- Examples $x \in \mathbb{R}$:
  - $V_1(x) = x^2$
  - $V_1(0) = 0$
  - $V_1(x) > 0 \quad \forall x \neq 0$
  - Therefore $V_1(x)$ is positive definite **globally**
  - $V_2(x) = x^2$ - 0.13 x^4
  - $V_2(0) = 0$
  - $V_2(x) > 0 \quad (\text{but not } \forall x \neq 0)$
  - Therefore $V_2(x)$ is positive definite only **locally** in the region $|x| < 2.77$.

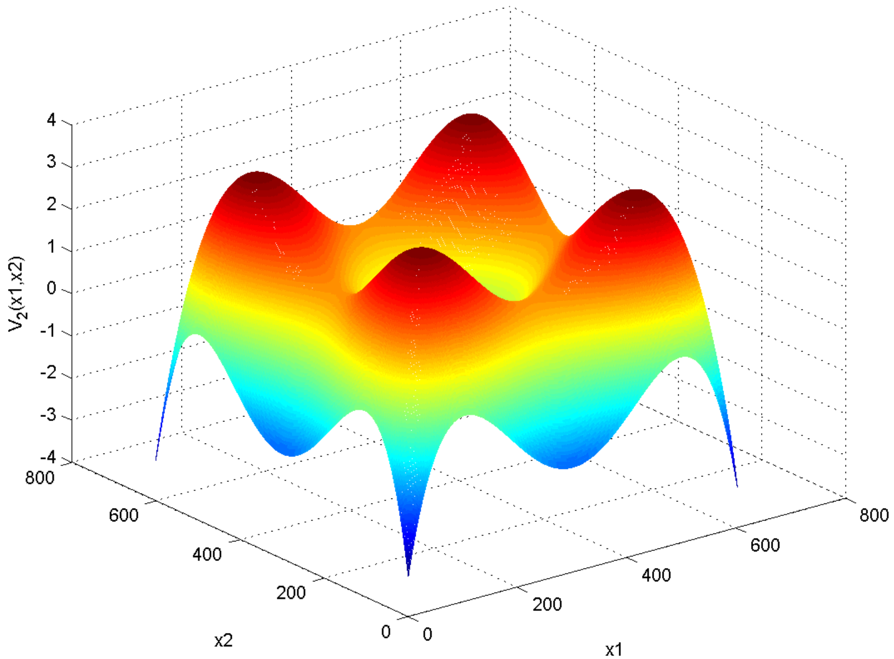




Positive Definite Functions

- Examples $x \in \mathbb{R}^2$:
  - $V_1(x) = x_1^2 + x_2^2$
  - $V_2(x) = x_1^2 + x_2^2 - 0.13(x_1^4 + x_2^4)$





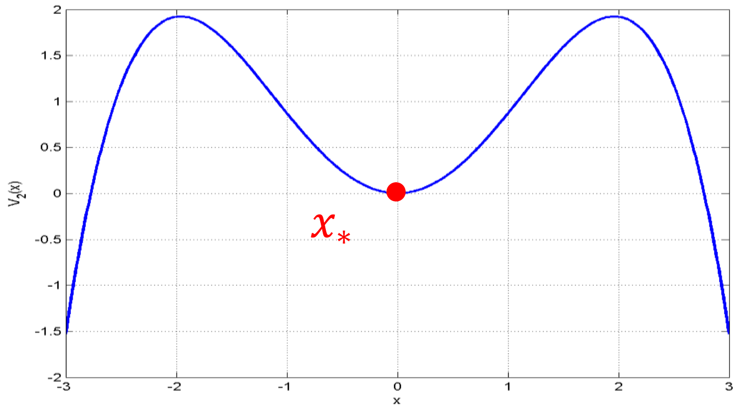
Positive Definite Functions (Formal Definition)

- A function $\psi: \mathcal{X} \rightarrow \mathbb{R}$ is locally positive definite about $x_* \in \mathcal{X}$ if:
  - $\psi\,(x_*) = 0$
  - There exists a neighborhood $\mathcal{U} \subset \mathcal{X}$ that includes x_* (i.e., $x_* \in \mathcal{U}$) such that $\psi(x) > 0\ \forall\ x \in \mathcal{U} \backslash \{x_*\}$.

- $\psi : \mathcal{X} \rightarrow \mathbb{R}$ is locally positive semi-definite if $\psi(x) \geq 0\ \forall\ x \in \mathcal{U} \backslash \{x_*\}$.
- $\psi : \mathcal{X} \rightarrow \mathbb{R}$ is globally positive definite if $\psi(x) > 0\ \forall\ x \in \mathcal{X} \backslash \{x_*\}$.
- $\psi : \mathcal{X} \rightarrow \mathbb{R}$ is locally negative (semi-)definite if $-\psi(x)$ is locally positive (semi-)definite.

Local Minima and Critical Zeros

- In other words, x_* is a strict local minimum of $\psi(x)$, and the function increases in every direction away from x_* within $\mathcal{U}$.
- Recall the classic result of calculus:
  - If ψ is differentiable, and x_* is a local minimum, then the gradient must vanish there i.e., $J_\psi(x_*) = 0$.
- We call points $x_* \in \mathcal{X}$ a critical zero for the smooth function $\psi: \mathcal{X} \to \mathbb{R}$ if $\psi(x_*) = 0$ and $J_\psi(x_*) = 0$.



Positive Definite Matrices

- A real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called a positive definite matrix if we have that

$$x^\top A\, x > 0\,, \quad \forall x \in \mathbb{R}^n \backslash \{0\}.$$

- $A \in \mathbb{R}^{n\times n}$ is called a positive semi-definite matrix if $x^\top A\, x \geq 0$.
- $A \in \mathbb{R}^{n\times n}$ is called a negative definite matrix if $x^\top A\, x < 0$.
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Positive Definite Matrices

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- Key property:
  - $A \succ 0$ is positive definite if and only if all its eigenvalues are positive.
  - $A \succcurlyeq 0$ is positive semi-definite if and only if all its eigenvalues are non-negative.
  - $A \prec 0$ is negative definite if and only if all its eigenvalues are negative.
  - $A \preccurlyeq 0$ is negative definite if and only if all its eigenvalues are non-positive.

Using Taylor Expansion for Positive Definiteness

- Suppose you have a smooth scalar function $V: \mathbb{R}^n \to \mathbb{R}$, and you want to check whether it's positive definite around $x_* \in \mathbb{R}^n$.
- You can expand $\mathrm{V}(x)$ as a Taylor series of order 2:

$$V(x) = V(x_*) + J_V(x_*)\,(x - x_*) + \frac{1}{2}(x - x_*)^\top H_V(x_*)(x - x_*) + \ ...$$

where $H_V(x_*) \in \mathbb{R}^{n \times n}$ is called the Hessian matrix with the entry of the ith row and the jth column is $(H_V)_{ij} \coloneqq \frac{\partial^2 V}{\partial x_i \partial x_j}: \mathbb{R}^n \to \mathbb{R}$.

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- However, since x_* is a critical zero of V, the function is locally approximated by

$$V(x) \approx \frac{1}{2}(x - x_*)^\top H_V(x_*)(x - x_*)$$

Using Taylor Expansion for Positive Definiteness

$$V(x) \approx \frac{1}{2}(x - x_*)^\top H_V(x_*)(x - x_*)$$

- Therefore, we have that $V(x)$ is a locally positive definite function if and only if the Hessian $H_V(x_*)$ is a positive definite matrix, which can be assessed from its eigenvalues.

- This result is a basic result from what is known as Morse theory.

Note: Positive definite Hessian gives a sufficient, not necessary, condition !

Assignment 4