SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 19: Lyapunov's Direct Method II

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Outline

- Recap last lecture
- Positive definite functions
- Computing $\dot{V}(x)$
- Examples

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Recap: State Space Model

- A nonlinear dynamic system can be represented by a set of nonlinear differential equations in the form

$$\dot{x} = f(x) + g(x)\,u$$
$$y = h(x)$$

which is called the state space model of the dynamic system.

- We denote by the
  - State space $\mathcal{X} \ni x$
  - Control space $\mathcal{U} \ni u$
  - Output space $\mathcal{Y} \ni y$

Special case: Linear time-invariant systems

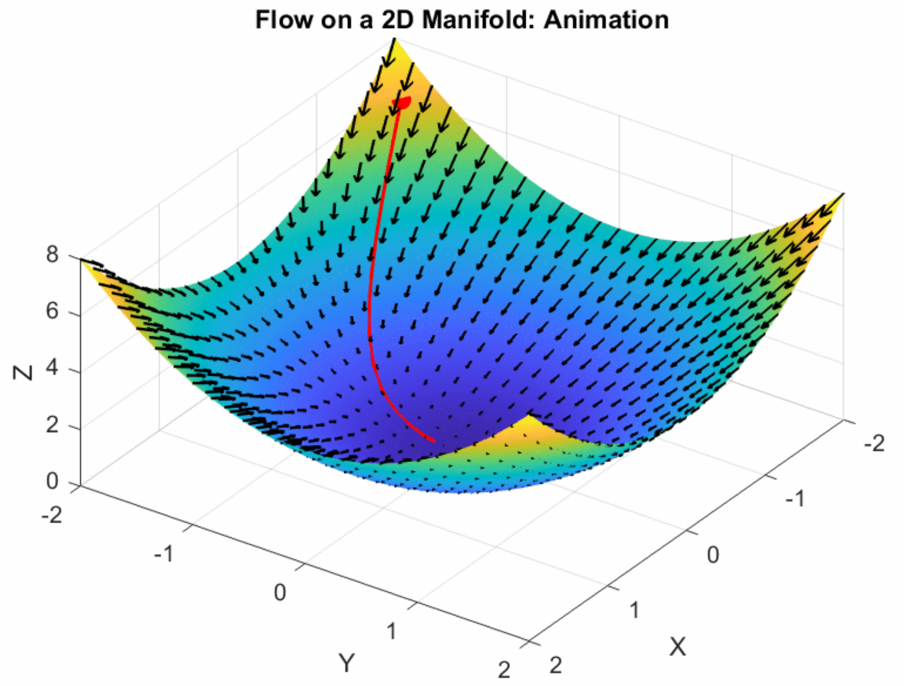
$$\dot{x} = Ax + Bu$$
$$y = C\,x$$

Recap: State Space Models – Mechanical Systems

System	State x	State Space $\mathcal{X}$
Mass-Spring-Damper	$(\xi, \dot{\xi})$	$\mathbb{R} \times \mathbb{R} \cong T\mathbb{R}$
Simple pendulum	$(\theta, \dot{\theta})$	$(-\pi, \pi] \times \mathbb{R} \cong T\mathbb{S}$
n-link Manipulator	$(q, \dot{q})$	TQ
Satellite	(R, ω)	$SO(3) \times \mathbb{R}^3 \cong TSO(3)$
Multirotor Aerial Vehicle	$(H, \mathcal{V})$	$SE(3) \times \mathbb{R}^6 \cong TSE(3)$

Recap: Geometric Nature of $\dot{x}(t) = f(x(t))$

- Euclidean case $\mathcal{X} = \mathbb{R}^{\boldsymbol{n}}$:
  - $x_t \in \mathbb{R}^{\boldsymbol{n}}$
  - $\dot{x}_t \in \mathbb{R}^{\boldsymbol{n}}$
  - $f: \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}^{\boldsymbol{n}}$
- Non-Euclidean case:
  - $x_t \in \mathcal{X}$
  - $\dot{x}_t \in T_{\boldsymbol{x}}\mathcal{X}$
  - $f: x_t \in \mathcal{X} \mapsto \dot{x}_t \in T_{\boldsymbol{x}}\mathcal{X}$



The solution of the dynamical system $x(t)$ is given by the integral curves of σ_f.

Recap: Equilibrium Points

- Given $\sigma_f \in \Gamma(T\mathcal{X})$, a point $x_* \in \mathcal{X}$ in the state space is called an *equilibrium point for σ_f* if and only if

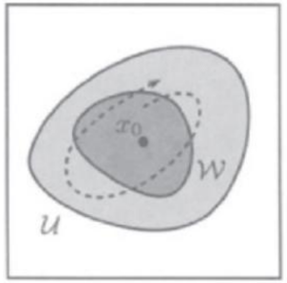
$$f(x_*) = 0$$

- Intuitively, an equilibrium point is a state x_* at which the system state remains for all time, once it reaches it.

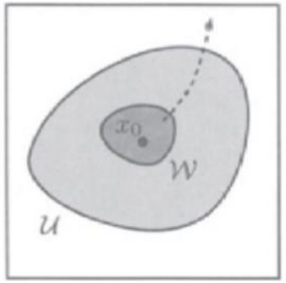
Recap: Stability of Equilibrium Points

- An equilibrium point x_* of σ_f is said to be:

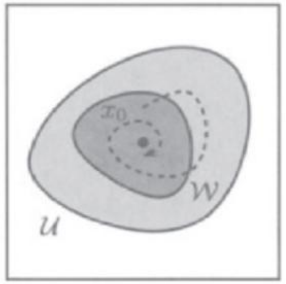
• Stable	If integral curves of σ_f stay "close" to x_*
• Unstable	If it is not stable
• Locally asymptotically stable	If it is stable and integral curves of σ_f converge to x_* only within a region $\mathrm{U} \subset \mathcal{X}$
• Globally asymptotically stable	If it is stable and integral curves of σ_f converge to x_* for all $x \in \mathcal{X}$



Stable x_0



Unstable x_0



Locally asym. stable x_0

Recap: Lyapunov stability

- The most useful and general approach for studying the stability of nonlinear dynamical systems is the theory introduced in the late 19th century by the Russian mathematician Lyapunov.

- His work introduced two methods:
  - **Indirect Method:** studies nonlinear local stability around an equilibrium point x_* from stability properties of its linear approximation.
  - **Direct Method:** not restricted to local motion. Stability of nonlinear system is studied by proposing a scalar "energy-like" function $V: \mathcal{X} \rightarrow \mathbb{R}$ for the system and examining its time variation

Aleksandr Mikhailovich Lyapunov (1857-1918) was a Russian mathematician, mechanic and physicist.
https://en.wikipedia.org/wiki/Aleksandr_Lyapunov

Recap: Lyapunov's direct method

$\dot{x} = f(x)$

x^* is an Eq. point of the system

Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) \leq 0$

$V(x)$ is positive definite

$\dot{V}(x)$ is negative semi-definite

Asymptotically Stable

Find a Lyapunov function $V(x)$ such that:

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$

$\dot{V}(x)$ is negative definite

Globally Asymptotically Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$
- $V(x) \to \infty$ as $\|x - x^*\| \to \infty$

$V(x)$ is radially unbounded

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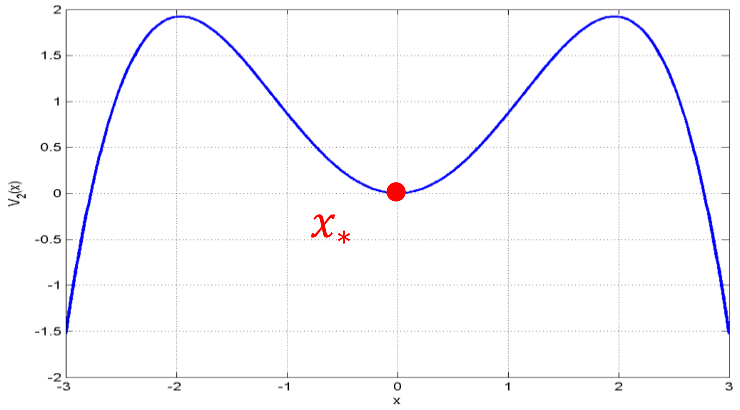
Positive Definite Functions (Formal Definition)

- A function $\psi: \mathcal{X} \rightarrow \mathbb{R}$ is locally positive definite about $x_* \in \mathcal{X}$ if:
  - $\psi\,(x_*) = 0$
  - There exists a neighborhood $\mathcal{U} \subset \mathcal{X}$ that includes x_* (i.e., $x_* \in \mathcal{U}$) such that $\psi(x) > 0 \;\forall\; x \in \mathcal{U} \backslash \{x_*\}$.

- $\psi : \mathcal{X} \rightarrow \mathbb{R}$ is locally positive semi-definite if $\psi(x) \geq 0 \;\forall\; x \in \mathcal{U} \backslash \{x_*\}$.
- $\psi : \mathcal{X} \rightarrow \mathbb{R}$ is globally positive definite if $\psi(x) > 0 \;\forall\; x \in \mathcal{X} \backslash \{x_*\}$.
- $\psi : \mathcal{X} \rightarrow \mathbb{R}$ is locally negative (semi-)definite if $-\psi(x)$ is locally positive (semi-)definite.

Local Minima and Critical Zeros

- In other words, x_* is a strict local minimum of $\psi(x)$, and the function increases in every direction away from x_* within $\mathcal{U}$.
- Recall the classic result of calculus:
  - If ψ is differentiable, and x_* is a local minimum, then the gradient must vanish there i.e., $J_\psi(x_*) = 0$.
- We call points $x_* \in \mathcal{X}$ a critical zero for the smooth function $\psi: \mathcal{X} \to \mathbb{R}$ if $\psi\,(x_*) = 0$ and $J_\psi(x_*) = 0$.



Positive Definite Matrices

- A real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called a positive definite matrix if we have that

$$x^\top A\, x > 0\,, \quad \forall x \in \mathbb{R}^n \backslash \{0\}.$$

- $A \in \mathbb{R}^{n\times n}$ is called a positive semi-definite matrix if $x^\top A\, x \geq 0$.
- $A \in \mathbb{R}^{n\times n}$ is called a negative definite matrix if $x^\top A\, x < 0$.
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Positive Definite Matrices

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- Key property:
  - $A \succ 0$ is positive definite if and only if all its eigenvalues are positive.
  - $A \succcurlyeq 0$ is positive semi-definite if and only if all its eigenvalues are non-negative.
  - $A \prec 0$ is negative definite if and only if all its eigenvalues are negative.
  - $A \preccurlyeq 0$ is negative definite if and only if all its eigenvalues are non-positive.

Using Taylor Expansion for Positive Definiteness

- Suppose you have a smooth scalar function $V\colon \mathbb{R}^n \to \mathbb{R}$, and you want to check whether it's positive definite around $x_* \in \mathbb{R}^n$.
- You can expand $\mathrm{V}(x)$ as a Taylor series of order 2:

$$V(x) = V(x_*) + J_V(x_*)\,(x - x_*) + \frac{1}{2}(x - x_*)^\top H_V(x_*)(x - x_*) + \ \dots$$

where $H_V(x_*) \in \mathbb{R}^{n\times n}$ is called the Hessian matrix with the entry of the ith row and the jth column is $(H_V)_{ij} \coloneqq \frac{\partial^2 V}{\partial x_i \partial x_j}\colon \mathbb{R}^n \to \mathbb{R}$.

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- However, since x_* is a critical zero of V, the function is locally approximated by

$$V(x) \approx \frac{1}{2}(x - x_*)^\top H_V(x_*)(x - x_*)$$

Using Taylor Expansion for Positive Definiteness

$$V(x) \approx \frac{1}{2}(x - x_*)^\top H_V(x_*)(x - x_*)$$

- Therefore, we have that $V(x)$ is a locally positive definite function if and only if the Hessian $H_V(x_*)$ is a positive definite matrix, which can be assessed from its eigenvalues.

- This result is a basic result from what is known as Morse theory.

Note: Positive definite Hessian gives a sufficient, not necessary, condition !

Assignment 4

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Computing $\dot{V}(x)$

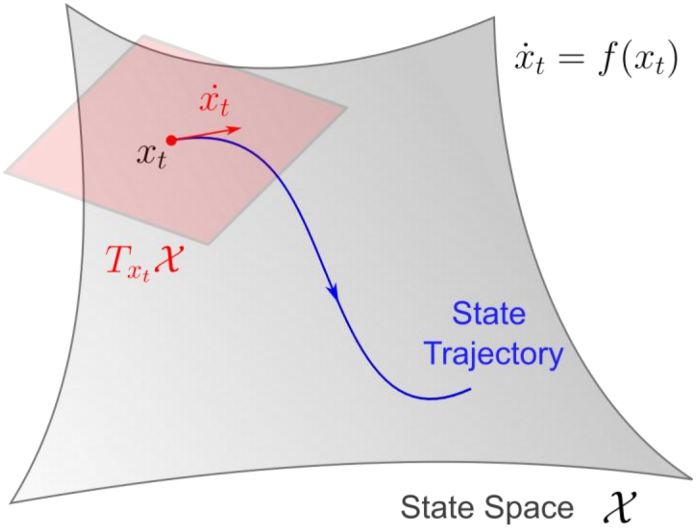
- We have in general for $x_t \in \mathcal{X}$ that $\dot{x}_t \in T_{x_t}\mathcal{X}$, with $\dot{x}_t = f(x_t)$
- For any smooth function $V: \mathcal{X} \to \mathbb{R}$, we have that

$$\dot{V}(x_t) = \langle dV(x_t) | \dot{x}_t \rangle = \langle dV(x_t) | f(x_t) \rangle$$

where $dV(x) \in T_x^*\mathcal{X}$ and

$$\langle \cdot \mid \cdot \rangle : T_x^*\mathcal{X} \times T_x\mathcal{X} \to \mathbb{R}$$

is the *duality product* on the tangent space of $\mathcal{X}$.



dV is called the differential of the scalar function V

Computing $\dot{V}(x)$: Case $\mathbb{R}^n$

- For the case $\mathcal{X} = \mathbb{R}^n$ we have that

$$\dot{V}(x) = \langle dV(x) | \dot{x} \rangle = dV(x)\dot{x}$$

where $dV(x) \in (\mathbb{R}^n)^*$ is given by

$$dV(x) = \left(\frac{\partial V}{\partial x^1}(x), \cdots, \frac{\partial V}{\partial x^n}(x) \right) \in \mathbb{R}^{1\times n}$$

while

$$\dot{x} = f(x) \rightarrow \begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^n \end{pmatrix} = \begin{pmatrix} f^1(x) \\ \vdots \\ f^n(x) \end{pmatrix} \in \mathbb{R}^n$$

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Therefore,

$$\dot{V}(x) = \frac{\partial V}{\partial x^1}(x) f^1(x) + \cdots + \frac{\partial V}{\partial x^n}(x) f^n(x) = \sum_{k=1}^{n} \frac{\partial V}{\partial x^k}(x) f^k(x)$$

Computing $\dot{V}(x)$: Case $\mathbb{R}^n$

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- It is more common to write the above using the gradient vector

$$\nabla V(x) = \begin{pmatrix} \frac{\partial V}{\partial x^1}(x) \\ \vdots \\ \frac{\partial V}{\partial x^n}(x) \end{pmatrix} \in \mathbb{R}^n$$

Computing $\dot{V}(x)$: Case $\mathbb{R}^n$

- For the case $\mathcal{X} = \mathbb{R}^n$ we have that

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The differential and gradient are dual to each other
$dV(x) = \nabla V^\top(x)$

Then, we have that

$$\dot{V}(x) = \nabla V(x)^\top f(x)$$

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