

SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 20: La Salle's Invariance Principle & PD Control on \mathbb{R}^3



Outline

- Recap last lectures
- La Salle's Invariance Principle
- PD Control of a Point Mass on \mathbb{R}^3
- Energy balancing formulation



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Recap: State Space Model

- A nonlinear dynamic system can be represented by a set of nonlinear differential equations in the form

$$\begin{aligned}\dot{x} &= f(x) + g(x) u \\ y &= h(x)\end{aligned}$$

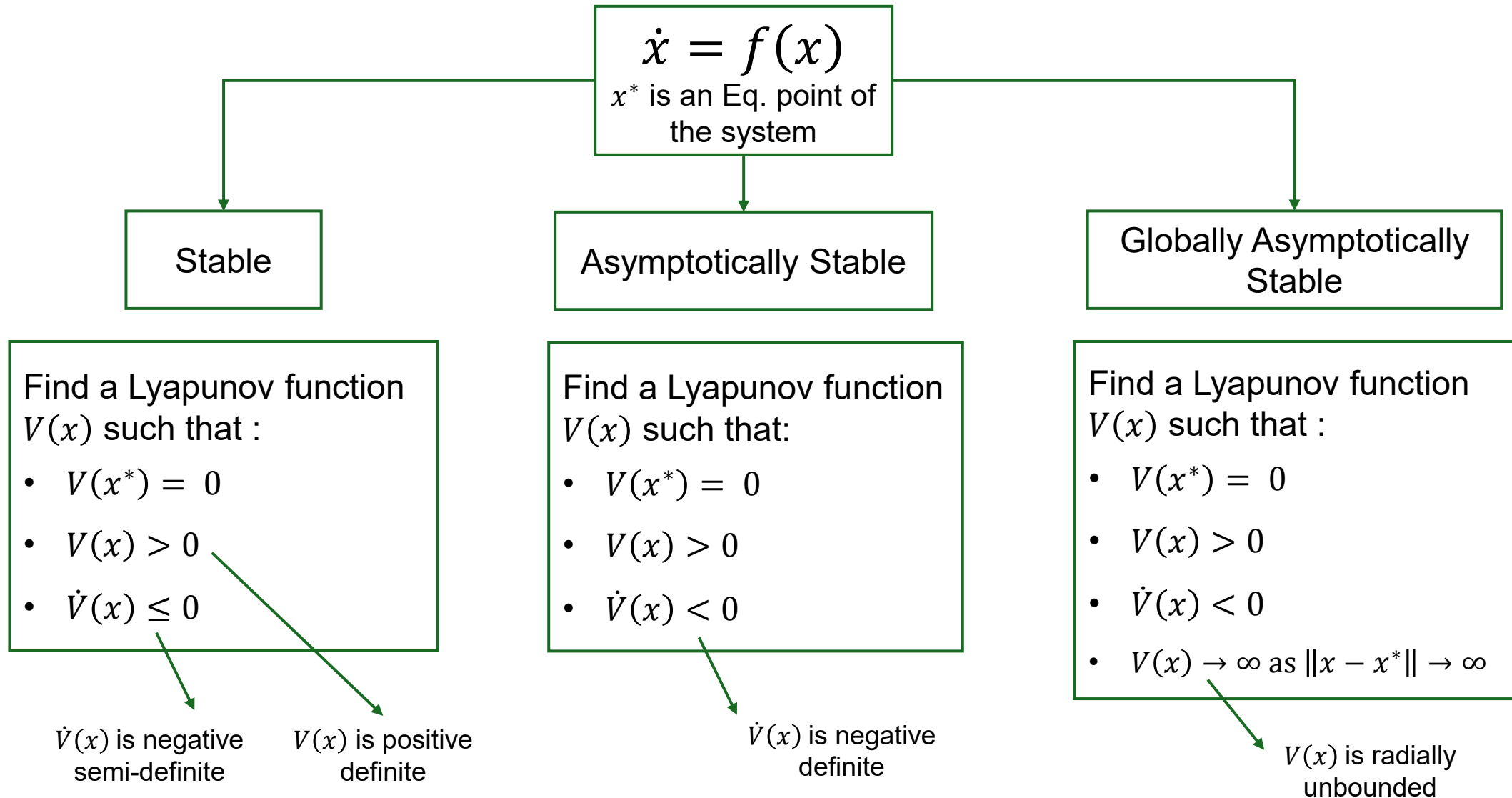
which is called the **state space model** of the dynamic system.

- We are focusing on analyzing the stability of the **equilibrium points** x_* systems of the form

$$\dot{x} = f(x)$$



Recap: Lyapunov's Direct Method



Recap: Local Definiteness

$$V(x) \approx \frac{1}{2} (x - x_*)^\top H_V(x_*) (x - x_*)$$

- Therefore, we have that $V(x)$ is a locally positive definite function if and only if the Hessian $H_V(x_*)$ is a positive definite matrix, which can be assessed from its eigenvalues.
- This result is a basic result from what is known as **Morse theory**.



Recap: Computing $\dot{V}(x)$

- We have in general for $x \in \mathcal{X}$ that $\dot{x} \in T_x \mathcal{X}$, with $\dot{x} = f(x)$
- Therefore, we have

$$\dot{V}(x) = \mathcal{L}_{\sigma_f} V(x) = \langle dV(x) | f(x) \rangle = \langle dV(x) | \dot{x} \rangle$$

where $dV(x) \in T_x^* \mathcal{X}$ and

$$\langle \cdot | \cdot \rangle : T_x^* \mathcal{X} \times T_x \mathcal{X} \rightarrow \mathbb{R}$$

is called the duality product on tangent spaces of \mathcal{X} .

dV is called the differential of the scalar function V



Recap: Computing $\dot{V}(x)$

- For the case $\mathcal{X} = \mathbb{R}^n$ we have that

$$\dot{V}(x) = \langle dV(x) | \dot{x} \rangle = dV(x)\dot{x}$$

where $dV(x) \in (\mathbb{R}^n)^*$ is given by

$$dV(x) = \left(\frac{\partial V}{\partial x_1}(x), \dots, \frac{\partial V}{\partial x_n}(x) \right) \in \mathbb{R}^{1 \times n}$$



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- It is more common to write the above using the **gradient vector**

$$\nabla V(x) = \begin{pmatrix} \frac{\partial V}{\partial x_1}(x) \\ \vdots \\ \frac{\partial V}{\partial x_n}(x) \end{pmatrix} \in \mathbb{R}^n$$

The differential and gradient are dual to each other
 $dV(x) = \nabla V^\top(x)$



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Introduction

- LaSalle's Invariance Principle is a fundamental result in the stability analysis of dynamical systems.
- It generalizes Lyapunov's direct method to handle the case where a Lyapunov-like function's time derivative $\dot{V}(x)$ is only negative semi-definite rather than strictly negative.



Invariant set

- Before discussing LaSalle's Invariance principle, we introduce the concept of an **invariant set**.
- **Definition:**
 - A set M is an invariant set for a dynamic system if every system trajectory which starts from a point in M remains in M for all future time.
- Examples of invariant sets:
 - Equilibrium points
 - Domain of attraction of an equilibrium point
 - The whole state space



La Salle's Invariance Principle

- Let $x_* \in \mathcal{X}$ be an equilibrium point of the dynamical system

$$\dot{x}(t) = f(x(t)).$$

Assume there exists a smooth Lyapunov function $V: \mathcal{X} \rightarrow \mathbb{R}$ such that in some neighborhood $\Omega \subset \mathcal{X}$ of x_* , we have that

- V is positive definite
- \dot{V} is negative semi-definite
- Let $R := \{x \in \Omega \mid \dot{V}(x) = 0\} \subset \Omega$ and let $M \subset R$ be the largest invariant set in it.



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- Let $R := \{x \in \Omega \mid \dot{V}(x) = 0\} \subset \Omega$ and let $M \subset R$ be the largest invariant set in it.
- If M contains only the equilibrium point (i.e., $M = \{x_*\}$), then x_* is locally asymptotically stable with its domain of attraction defined by

$$\Omega_l := \{x \in \Omega \mid V(x) < l\} \subset \Omega.$$



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$$\Omega_l := \{x \in \Omega \mid V(x) < l\} \subset \Omega.$$

This result is properly called **Barbashin-Krasovskii-LaSalle principle** !



Example: Pendulum

- State space model:

- $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -c_2 x_2 - c_1 \sin x_1 \end{pmatrix}, \quad c_1, c_2 > 0$

- Lyapunov's direct method:

- $V(x) = \frac{1}{2}x_2^2 + c_1(1 - \cos x_1)$

- $\dot{V}(x) = -c_2 x_2^2 \leq 0$

$$V(x_{*,1}) = 0, \quad V(x) > 0 \quad \forall x \in \mathcal{X} / \{x_{*,1}\}$$

$\dot{V}(x)$ is negative semi-definite $\therefore x_{*,1} := (0,0)$ is stable according to Lyapunov's direct method.

Equilibrium Points:

$$x_{*,1} := (0,0), \quad x_{*,2} := (\pi, 0)$$



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- **Further analysis:**

1. Start with $\Omega = \mathcal{X}$

2. Define the set $R := \{x \in \Omega \mid \dot{V}(x) = 0\}$

- $R = \{(x_1, x_2) \in \mathcal{X} \mid x_2 = 0\}$

Equilibrium Points:

$$x_{*,1} := (0,0), \quad x_{*,2} := (\pi, 0)$$

$$V(x) = \frac{1}{2}x_2^2 + c_1(1 - \cos x_1)$$

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3. Define the points that form the invariant set in R

- i.e., points that have x_2 equal to zero

- $\dot{x}_2 = 0 = c_2 x_2 + c_1 \sin x_1 \implies \sin x_1 = 0$

- Thus, the invariant set M is given by $M = \{x_{*,1}, x_{*,2}\}$

Equilibrium Points:

$$x_{*,1} := (0,0), \quad x_{*,2} := (\pi, 0)$$

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4. Compute level sets of $V(x)$

- $V(x_{*,1}) = \frac{1}{2} (0)^2 + c_1(1 - \cos 0) = 0$
- $V(x_{*,2}) = \frac{1}{2} (0)^2 + c_1(1 - \cos \pi) = 2 c_1$

Equilibrium Points:

$$x_{*,1} := (0,0), \quad x_{*,2} := (\pi, 0)$$

$$V(x) = \frac{1}{2} x_2^2 + c_1(1 - \cos x_1)$$

$$\dot{V}(x) = -c_2 x_2^2 \leq 0$$



Example: Pendulum

- Further analysis:

5. If we consider only the region

- $\Omega = \Omega_{2c_1} = \{x \in \mathcal{X} \mid V(x) < 2c_1\}$
- Refine $R = \{(x_1, x_2) \in \Omega_{2c_1} \mid x_2 = 0\}$
- The largest invariant set in $R \subset \Omega_{2c_1}$ becomes $M = \{x_{*,1}\}$

- Therefore, using La Salle's Invariance Principle, every solution originating in the region Ω_{2c_1} will tend to $\{x_{*,1}\}$ as $t \rightarrow \infty$.
- Therefore, $x_{*,1} := (0,0)$ is locally asymptotically stable.

Equilibrium Points:

$$x_{*,1} := (0,0), \quad x_{*,2} := (\pi, 0)$$

$$V(x) = \frac{1}{2}x_2^2 + c_1(1 - \cos x_1)$$

$$\dot{V}(x) = -c_2x_2^2 \leq 0$$



Summary

- Lyapunov's direct method:
 - If you have $\dot{V}(x) < 0$ (negative definite), you can conclude asymptotic stability of that equilibrium.
- LaSalle's Invariance Principle:
 - Works with $\dot{V}(x) \leq 0$ (negative semi-definite).
 - This weaker condition only implies that the trajectory stays in or approaches the set R where $\dot{V}(x) = 0$.
 - You then must check what the motion does within R to conclude asymptotic convergence.



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- **PD Control of a Point Mass on \mathbb{R}^3**
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PD Control of Point Mass

- To provide some intuition, let's start in a simple Euclidean space \mathbb{R}^n .
- The governing equations of a point mass (with no gravity) are:

- $\dot{\xi} = v, \quad \dot{v} = \frac{1}{m}u$

where $\xi, v, u \in \mathbb{R}^3$ denote the position, velocity and control forces.



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where $\xi, v, u \in \mathbb{R}^3$ denote the position, velocity and control forces.

- We can rewrite it as the state space model

- $x = (\xi, p) = (\xi, mv) \in \mathbb{R}^3 \times \mathbb{R}^3$

- $\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_3 \end{pmatrix} u \quad \Rightarrow \dot{x} = f(x) + g u$

with $v = \frac{p}{m}$.



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with $v = \frac{p}{m}$.

Note that the total energy of the system is given by

$$H(p) = \frac{1}{2m} p^\top p = \frac{1}{2} m v^\top v$$



PD Control of Point Mass

- Our control objective is to stabilize the system at the desired state

$$x_d = (\xi_d, 0)$$

- A classical proportional-derivative (PD) controller can often be written as

- $e_\xi := \xi - \xi_d \in \mathbb{R}^3$

- $u = -K_p e_\xi - K_d \dot{e}_\xi$

$$= -K_p (\xi - \xi_d) - K_d v \quad \Rightarrow u = \gamma(x)$$

where K_p and K_d are positive-definite gain matrices



PD Control of Point Mass

- Closed loop system can be written as

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -K_p (\xi - \xi_d) - K_d v \end{pmatrix}$$

$$\Rightarrow \dot{x} = f(x) + g \gamma(x)$$

We will show later the stability of this system using Lyapunov's direct method



Moving to a “Geometric” Setting

- It is not straightforward to extend such PD controller to a non-Euclidean space.
- For example, for the satellite problem we have that

$$x = (R, \omega) \in SO(3) \times \mathbb{R}^3$$

- If we wish to stabilize the system at the desired state $x_d = (R_d, 0)$, one **cannot** simply compute

$$u = -K_p e_R - K_d \dot{e}_R$$

with $e_R = R - R_d \notin SO(3)$ and $\dot{e}_R = \dot{R} \in T_R SO(3)$ to compute the control torques $u \in (\mathbb{R}^3)^*$.



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Reformulating PD Control on \mathbb{R}^3

- Our starting point to develop a geometric PD controller is to express the proportional term as the **gradient of some potential function**.
- For the PD controller

$$u = u_p + u_d = -K_p e_\xi - K_d \dot{e}_\xi$$

we can view the proportional part as

$$u_p = -\nabla\Psi(\xi) = -K_p e_\xi$$

if we pick

$$\Psi(\xi) := \frac{1}{2} (\xi - \xi_d)^\top K_p (\xi - \xi_d)$$

which is a (global) positive definite function of $\xi \in \mathbb{R}^3$.

Recall:

$$\nabla\Psi^\top(e_\xi) = d\Psi(e_\xi)$$



Reformulating PD Control on \mathbb{R}^3

- Such interpretation allows us to perform **Lyapunov analysis** of the closed loop system easily.
- We can choose

$$\begin{aligned} V(x) &= \Psi(\xi) + H(p) \\ &= \frac{1}{2} (\xi - \xi_d)^\top K_p (\xi - \xi_d) + \frac{1}{2m} p^\top p \end{aligned}$$

Closed loop system

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla \Psi(\xi) + u_d \end{pmatrix}$$



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- We have that $V(x_d) = V(\xi_d, 0) = 0$, and its Hessian given by

$$H_V(x_d) = \begin{pmatrix} K_p & 0_{3 \times 3} \\ 0_{3 \times 3} & \frac{1}{m} I_3 \end{pmatrix} \succ 0,$$

- Therefore, $V(x)$ is globally positive definite.

Closed loop system

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla \Psi(\xi) + u_d \end{pmatrix}$$



Reformulating PD Control on \mathbb{R}^3

- Lyapunov function

$$V(x) = \Psi(\xi) + H(p)$$

- The time derivative $\dot{V}(x)$ along trajectories of the closed loop system can be written as

$$\begin{aligned}\dot{V}(x) &= \langle dV(x) | \dot{x} \rangle_{\mathbb{R}^6} \\ &= \langle d\Psi(\xi) | \dot{\xi} \rangle_{\mathbb{R}^3} + \langle dH(p) | \dot{p} \rangle_{\mathbb{R}^3} \\ &= \dot{\xi}^\top \nabla \Psi(\xi) + \nabla H^\top(p) \dot{p} \\ &= v^\top \nabla \Psi(\xi) + v^\top [-\nabla \Psi(\xi) + u_d] \\ &= v^\top u_d\end{aligned}$$

Closed loop system

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Reformulating PD Control on \mathbb{R}^3

- Lyapunov function

$$V(x) = \Psi(\xi) + H(p)$$

- If we choose $u_d = -K_d v$, we have hat

$$\dot{V}(x) = v^\top u_d = -v^\top K_d v \leq 0$$

- Using La Salle's invariance principle, it follows that $x_d = (\xi_d, 0)$ is a **globally asymptotically stable** equilibrium point of the closed loop system.

Closed loop system

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla\Psi(\xi) + u_d \end{pmatrix}$$



Energy-balancing interpretation of PD Control on \mathbb{R}^3

- The PD controller

$$u = u_p + u_d$$

can be interpreted as a sum of an energy-shaping term u_p and a damping injection term u_d .

- For a chosen locally positive definite function $\Psi(\xi)$ designed such that ξ_d is a minimum, one has that $u_p = -\nabla\Psi(\xi)$ which yields $\dot{V}(x) = v^\top u_d$.



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- Choosing $u_d = \gamma(v)$ to inject damping such that $\dot{V}(x) \leq 0$, one has with La Salle's invariance principle that $x_d = (\xi_d, 0)$ is locally asymptotically stable.



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- For a chosen locally positive definite function $\Psi(\xi)$ designed such that ξ_d is a minimum, one has that $u_p = -\nabla\Psi(\xi)$ which yields $\dot{V}(x) = v^\top u_d$.
- Choosing $u_d = \gamma(v)$ to inject damping such that $\dot{V}(x) \leq 0$, one has with La Salle's invariance principle that $x_d = (\xi_d, 0)$ is locally asymptotically stable.
- If $\Psi(\xi)$ has ξ_d to be a **global minimum**, then $x_d = (\xi_d, 0)$ is **globally asymptotically stable**.

