SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 21: Stabilization Control on SO(3) I

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Outline

- Recap last lectures
- Energy balancing formulation
- Stabilization Control on $SO(3)$

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Recap: Lyapunov's Direct Method

$\dot{x} = f(x)$

x^* is an Eq. point of the system

Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) \leq 0$

$\dot{V}(x)$ is negative semi-definite

$V(x)$ is positive definite

Asymptotically Stable

Find a Lyapunov function $V(x)$ such that:

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$

$\dot{V}(x)$ is negative definite

Globally Asymptotically Stable

Find a Lyapunov function $V(x)$ such that :

- $V(x^*) = 0$
- $V(x) > 0$
- $\dot{V}(x) < 0$
- $V(x) \to \infty$ as $\|x - x^*\| \to \infty$

$V(x)$ is radially unbounded

Recap: La Salle's Invariance Principle

- Let $x_* \in \mathcal{X}$ be an equilibrium point of the dynamical system

$$\dot{x}(t) = f\big(x(t)\big).$$

Assume there exists a smooth Lyapunov function $V: \mathcal{X} \to \mathbb{R}$ such that in some neighborhood $\Omega \subset \mathcal{X}$ of x_*, we have that

  - V is positive definite
  - $\dot{V}$ is negative semi-definite
  - Let $R \coloneqq \{x \in \Omega \mid \dot{V}(x) = 0\} \subset \Omega$ and let $M \subset R$ be the largest invariant set in it.

- If M contains only the equilibrium point (i.e., $M = \{x_*\}$), then x_* is locally asymptotically stable with its domain of attraction defined by

$$\Omega_l \coloneqq \{x \in \Omega \mid V(x) < l\} \subset \Omega.$$

Recap: PD Control of Point Mass

- Point mass dynamics
  - $\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_3 \end{pmatrix} u,$ with $v = \frac{p}{m}$
- Control objective:
  - Stabilization at $(\xi_d, 0)$
- PD Control law:
  - $u = -K_p\,(\xi - \xi_d) - K_d v,$ with $K_p, K_d \succ 0$
- Closed loop dynamics
  - $\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -K_p\,(\xi - \xi_d) - K_d v \end{pmatrix}$

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Reformulating PD Control on $\mathbb{R}^3$

- Our starting point to develop a geometric PD controller is to express the proportional term as the gradient of some potential function.
- For the PD controller

$$u = u_p + u_d = -K_p\, e_\xi \, - K_d \dot{e}_\xi$$

we can view the proportional part as

$$u_p = -\nabla\Psi(\xi) = -K_p\, e_\xi$$

if we pick

$$\Psi(\xi) \coloneqq \frac{1}{2}(\xi \, - \xi_d)^\top K_p (\xi \, - \xi_d)$$

which is a (global) positive definite function of $\xi \in \mathbb{R}^3$.

Recall:
$\nabla\Psi^\top(e_\xi) = d\Psi(e_\xi)$

Reformulating PD Control on $\mathbb{R}^3$

- Such interpretation allows us to perform Lyapunov analysis of the closed loop system easily.
- We can choose

$$V(x) = \Psi(\xi) + H(p)$$
$$= \frac{1}{2}(\xi - \xi_d)^\top K_p(\xi - \xi_d) + \frac{1}{2m}p^\top p$$

Closed loop system

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla\Psi(\xi) + u_d \end{pmatrix}$$

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- We have that $V(x_d) = V(\xi_d, 0) = 0$, and its Hessian given by

$$H_V(x_d) = \begin{pmatrix} K_p & 0_{3\times3} \\ 0_{3\times3} & \frac{1}{m} I_3 \end{pmatrix} \succ 0\,,$$

- Therefore, $V(x)$ is globally positive definite.

Closed loop system

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla\Psi(\xi) + u_d \end{pmatrix}$$

Reformulating PD Control on $\mathbb{R}^3$

- Lyapunov function

$$V(x) = \Psi(\xi) + H(p)$$

- The time derivative $\dot{V}(x)$ along trajectories of the closed loop system can be written as

$$\begin{aligned}\dot{V}(x) &= \langle dV(x) \mid \dot{x}\rangle_{\mathbb{R}^6} \\ &= \left\langle d\Psi(\xi) \middle| \dot{\xi}\right\rangle_{\mathbb{R}^3} + \langle d\mathrm{H}(p) | \dot{p}\rangle_{\mathbb{R}^3} \\ &= \dot{\xi}^\top \nabla\Psi(\xi) + \nabla\mathrm{H}^\top(p)\,\dot{p} \\ &= v^\top\,\nabla\Psi(\xi) + v^\top\,[-\nabla\Psi(\xi) + u_d\,] \\ &= v^\top u_d\end{aligned}$$

Closed loop system

$$\begin{pmatrix}\dot{\xi}\\ \dot{p}\end{pmatrix} = \begin{pmatrix} v \\ -\nabla\Psi(\xi) + u_d\end{pmatrix}$$

Reformulating PD Control on $\mathbb{R}^3$

- Lyapunov function

$$V(x) = \Psi(\xi) + H(p)$$

- If we choose $u_d = -K_d v$, we have that

$$\dot{V}(x) = v^\top u_d = -v^\top K_d v \leq 0$$

- Using La Salle's invariance principle, it follows that $x_d = (\xi_d, 0)$ is a globally asymptotically stable equilibrium point of the closed loop system.

Closed loop system

$$\begin{pmatrix} \dot{\xi} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} v \\ -\nabla\Psi(\xi) + u_d \end{pmatrix}$$

Energy-balancing interpretation of PD Control on $\mathbb{R}^3$

- The PD controller

$$u = u_p + u_d$$

can be interpreted as a sum of an energy-shaping term u_p and a damping injection term u_d.

- For a chosen locally positive definite function $\Psi(\xi)$ designed such that ξ_d is a minimum, one has that $u_p = -\nabla\Psi(\xi)$ which yields $\dot{V}(x) = v^\top u_d$.

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- Choosing $u_d = \gamma(v)$ to inject damping such that $\dot{V}(x) \leq 0$, one has with La Salle's invariance principle that $x_d = (\xi_d, 0)$ is locally asymptotically stable.

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- For a chosen locally positive definite function $\Psi(\xi)$ designed such that ξ_d is a minimum, one has that $u_p = -\nabla\Psi(\xi)$ which yields $\dot{V}(x) = v^\top u_d$.
- Choosing $u_d = \gamma(v)$ to inject damping such that $\dot{V}(x) \leq 0$, one has with La Salle's invariance principle that $x_d = (\xi_d, 0)$ is locally asymptotically stable.
- If $\Psi(\xi)$ has ξ_d to be a global minimum, then $x_d = (\xi_d, 0)$ is globally asymptotically stable.

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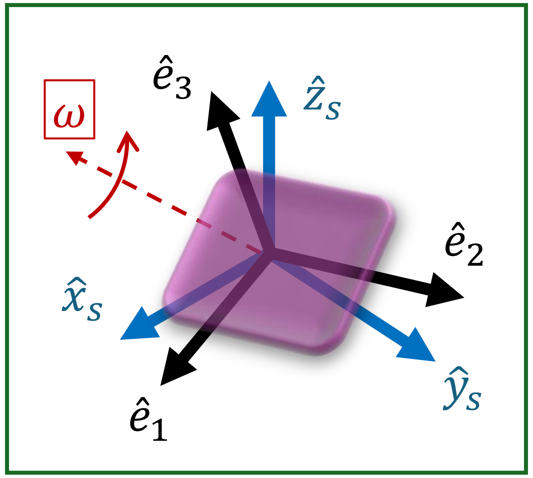
Rigid Body Rotation Dynamics

- The equations of a rotating rigid body* with control torques τ are:
  - $\dot{R} = R\,\widetilde{\omega}$
  - $J\dot{\omega} = -\omega \wedge J\omega + \tau$

    $= J\omega \wedge \omega + \tau$

- Which can be reformulated using angular momentum $p := J\omega$ as
  - $\dot{R} = R\,\widetilde{\omega}$
  - $\dot{p} = p \wedge \omega + \tau$



R: orientation of {black} w.r.t. {blue}

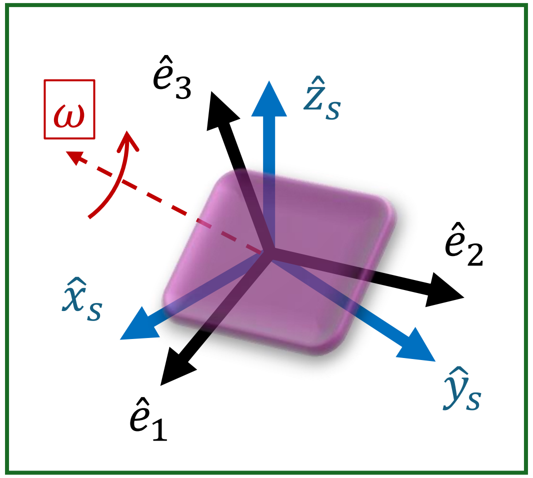
* Example applications: satellite/MAV attitude control

Rigid Body Rotation Dynamics

- The equations of a rotating rigid body* with control torques τ are:
  - $\dot{R} = R\,\widetilde{\omega}$
  - $\dot{p} = p \wedge \omega + \tau$
- We can cast it into state space form*
  - $x = (R, p) \in SO(3) \times \mathbb{R}^3$
  - $\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega \end{pmatrix} + \begin{pmatrix} 0 \\ I_3 \end{pmatrix} \tau$

  with $\omega = J^{-1} p$

  and $\beta_R : \mathbb{R}^3 \to T_R SO(3), \qquad \omega \mapsto \beta_R(\omega) := R\,\widetilde{\omega}$



*in the form: $\dot{x} = f(x) + g\,\tau$

Stabilization Control on SO(3)

- We aim to design a Geometric PD controller

$$\tau = \tau_p + \tau_d$$

such that $x_d = (R_d, 0)$ is an asymptotically stable equilibrium point of the closed loop system

$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega + \tau_p + \tau_d \end{pmatrix}.$$

Stabilization Control on SO(3)

- We aim to design a Geometric PD controller

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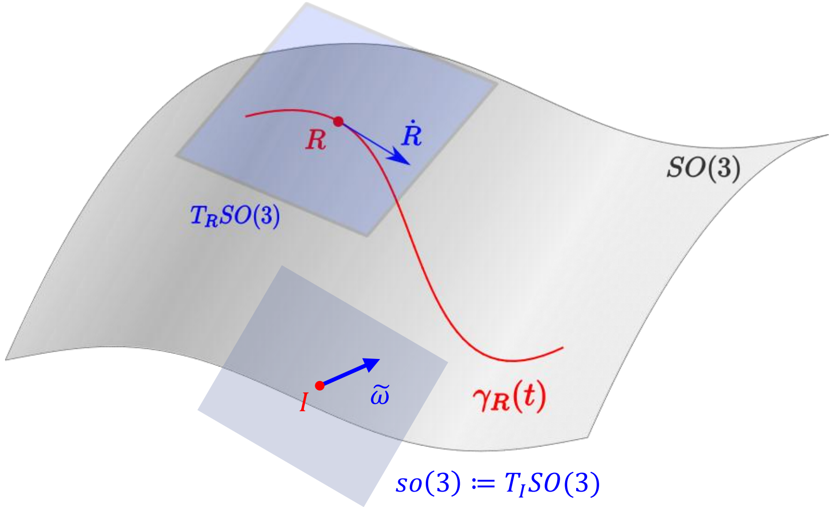
$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega + \tau_p + \tau_d \end{pmatrix}.$$

- The controller we seek is geometric in the sense that it respects the underlying non-Euclidean structure of $SO(3)$.
  - τ_p will be derived from the gradient of some positive definite potential function $\Psi(R)$ on $SO(3)$ with a minimum at $R = R_d$.
  - τ_d will be designed to inject damping.

Geometric structure of SO(3)

- The geometric nature of SO(3) will be reflected in
  1. How to compute the error between $R, R_d \in SO(3)$?
  2. How to design $\Psi(R)$ to be positive definite ?
  3. How to compute $d\Psi(R) \in T_R^* SO(3)$?
  4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
  5. How to design the derivative torque $\tau_d \in \mathbb{R}^3$?



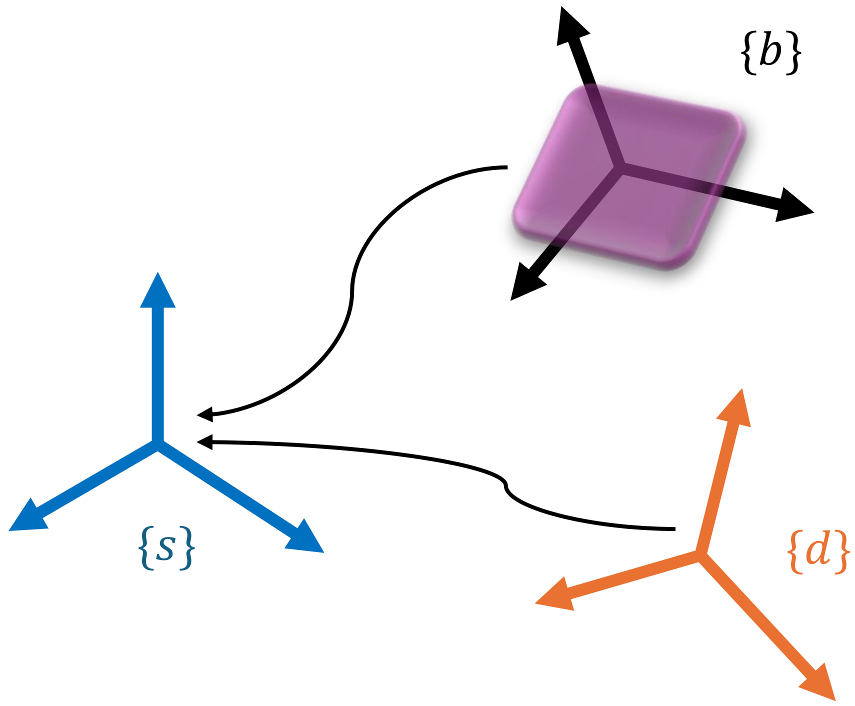
Stabilization Control on SO(3)

1. How to compute the error between $R, R_d \in SO(3)$?
   - Error between actual and desired

$$R_e := R_d^\top R \in SO(3)$$

   - We have that

$$R_e \to I \text{ as } R \to R_d$$

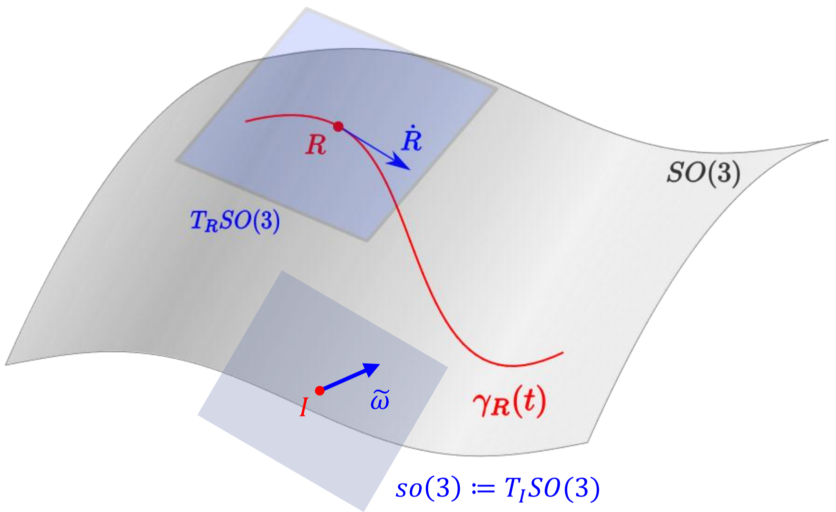


Identity on $SO(3)$ means zero error.

$R \equiv R_b^s, \quad R_d \equiv R_d^s, \quad R_e \equiv R_b^d$

PD Control on $SO(3)$

2. How to design $\Psi(R)$ to be positive definite ?
   - One choice for $\Psi: SO(3) \to \mathbb{R}$ is
$$\Psi(R) := \frac{1}{2}\,\mathrm{tr}(I - R_d^\top R)$$
   - $\Psi(R_d) = 0$



PD Control on $SO(3)$

2. How to design $\Psi(R)$ to be positive definite ?
   - One choice for $\Psi: SO(3) \to \mathbb{R}$ is

$$\Psi(R) := \frac{1}{2}\operatorname{tr}(I - R_d^\top R)$$

   - $\Psi(R_d) = 0$
   - We have shown before that it can be written as

$$\Psi(R) = 1 - \cos\theta$$

   for some $\theta \in (-\pi, \pi]$.
   - Therefore,

$$0 \le \Psi(R) \le 2$$