3 Compute $d\Psi(R) \in T_R^* SO(3)$:

a) Differential of a function Ψ on a vector space W:

- Let $\Psi : W \longrightarrow \mathbb{R}$, $x \longmapsto \Psi(x)$
- The differential of Ψ is the unique covector $d\Psi(x) \in W^*$ that satisfies:

$$\boxed{\left\langle d\Psi(x) \,\middle|\, \delta x \right\rangle_W = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Psi(x + \varepsilon\,\delta x)} \quad \forall\, \delta x \in W, \ \varepsilon \in \mathbb{R}$$

b) Example $\mathbb{R}^n$: $x \in \mathbb{R}^n$

$$\nabla \Psi(x) = K x$$
$$d\Psi(x) = (Kx)^T$$

$\Rightarrow$ let $\Psi(x) = \frac{1}{2} x^T K x$, $K \succ 0 \Rightarrow K = K^T$

$$\Rightarrow \Psi(x + \varepsilon \delta x) = \frac{1}{2}(x + \varepsilon \delta x)^T K (x + \varepsilon \delta x)$$
$$= \frac{1}{2} x^T K x + \underbrace{\frac{1}{2} \varepsilon (\delta x)^T K x + \frac{1}{2} \varepsilon x^T K (\delta x)}$$
$$= \quad + \quad \frac{1}{2}\varepsilon \left[(\delta x)^T K x + (\delta x)^T K^T x \right]$$
$$\cancel{\frac{2}{2}} \varepsilon \left[(\delta x)^T K x \right]$$

$(a+b)(a+b)$
$a^2 + 2ab + b^2$

$$\Rightarrow \psi(x+\varepsilon\delta x) = \frac{1}{2}(x+\varepsilon\delta x)^T K (x+\varepsilon\delta x)$$

$$= \underbrace{\frac{1}{2} x^T K x}_{0} + \varepsilon(\delta x)^T K x + \underbrace{\frac{1}{2}\varepsilon^2(\delta x)^T K(\delta x)}_{0}$$

$$\Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \psi(\ ) = 0 + (\delta x)^T K x + 0$$

$$= (\delta x)^T K x$$

$$= (Kx)^T \delta x$$

$$\boxed{d\psi(x) = (Kx)^T}$$

* let $y \in W^* = (\mathbb{R}^n)^*$, $x \in W = \mathbb{R}^n$

$$\langle y | x \rangle_{\mathbb{R}^n} = y\,x$$

$y = [y_1, \ldots, y_n]$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

* on $\mathbb{R}^n$

$$\langle d\psi(x) | \delta x \rangle = (\quad) \, \delta x$$

c) Differential of a function on a manifold M:

- let $\Psi : M \to \mathbb{R}$, $x \mapsto \Psi(x)$
- The differential of Ψ is the unique covector $d\Psi(x) \in T_x^* M$ that satisfies for all $\delta x \in T_x M$:

$$\boxed{\langle d\Psi(x) \mid \delta x \rangle_{T_x M} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Psi(x_\varepsilon)}$$

where x_ε is a curve on M defined s.t. $x_\varepsilon(0) = x$, $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} x_\varepsilon = \delta x \in T_x M$