SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 22: Stabilization Control on SO(3) II

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Outline

- Recap last lectures
- Stabilization Control on $SO(3)$ …

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Recap: Energy-balancing interpretation of PD Control

- The PD controller

$$u = u_p + u_d$$

can be interpreted as a sum of an energy-shaping term u_p and a damping injection term u_d.

- For a chosen locally positive definite function $\Psi(\xi)$ designed such that ξ_d is a minimum, one has that $u_p = -\nabla\Psi(\xi)$ which yields $\dot{V}(x) = v^\top u_d$.

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- For a chosen locally positive definite function $\Psi(\xi)$ designed such that ξ_d is a minimum, one has that $u_p = -\nabla\Psi(\xi)$ which yields $\dot{V}(x) = v^\top u_d$.
- Choosing $u_d = \gamma(v)$ to inject damping such that $\dot{V}(x) \leq 0$, one has with La Salle's invariance principle that $x_d = (\xi_d, 0)$ is locally asymptotically stable.
- If $\Psi(\xi)$ has ξ_d to be a global minimum, then $x_d = (\xi_d, 0)$ is globally asymptotically stable.

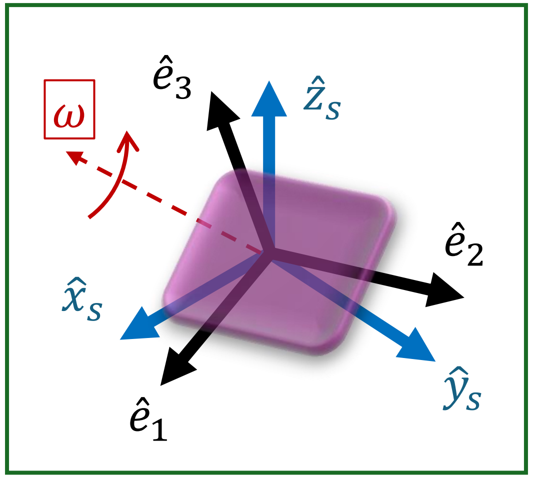
Recap: Rigid Body Rotation Dynamics

- The equations of a rotating rigid body with control torques τ are:
  - $x = (R, p) \in SO(3) \times \mathbb{R}^3$
  - $\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega \end{pmatrix} + \begin{pmatrix} 0 \\ I_3 \end{pmatrix} \tau$

  with $\omega = J^{-1} p$

  and $\beta_R \colon \mathbb{R}^3 \to T_R SO(3), \qquad \omega \mapsto \beta_R(\omega) := R\,\widetilde{\omega}$

- Total energy of the system
  - $H(x) = H(p) = \frac{1}{2} p^\top J^{-1} p$



Recap: Stabilization Control on SO(3)

- We aim to design a Geometric PD controller

$$\tau = \tau_p + \tau_d$$

such that $x_d = (R_d, 0)$ is an asymptotically stable equilibrium point of the closed loop system

$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega + \tau_p + \tau_d \end{pmatrix}.$$

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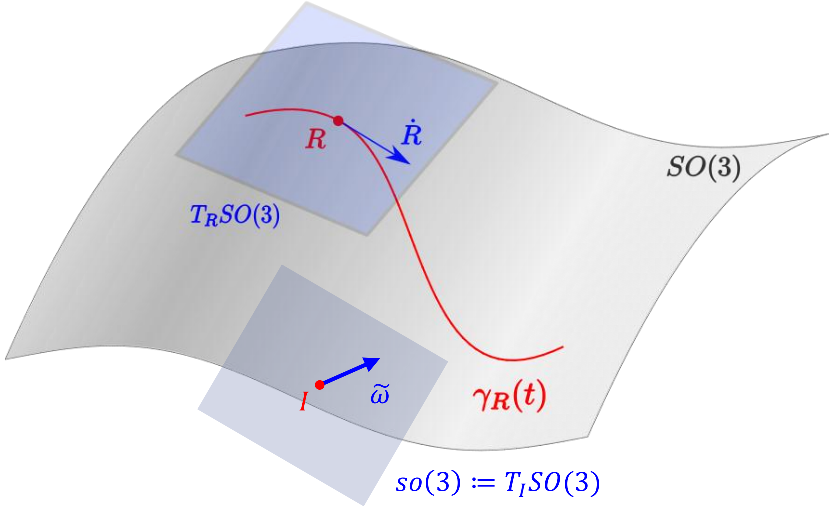
gradient of some pos.def. function $\Psi(R)$ on $SO(3)$ with a minimum at $R = R_d$.

Inject damping such that time derivative of $V(x) = H(p) + \Psi(R)$ is neg. def.

Recap: Geometric structure of SO(3)

- The geometric nature of SO(3) will be reflected in
  1. How to compute the error between $R, R_d \in SO(3)$?
  2. How to design $\Psi(R)$ to be positive definite ?
  3. How to compute $d\Psi(R) \in T_R^* SO(3)$?
  4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
  5. How to design the derivative torque $\tau_d \in \mathbb{R}^3$?



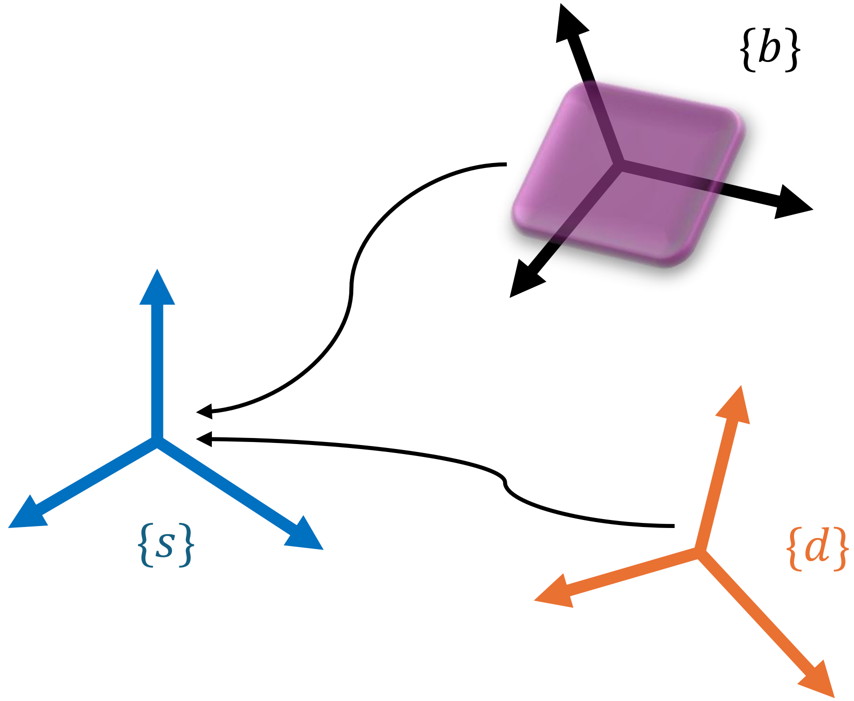
Recap: Stabilization Control on SO(3)

1. How to compute the error between $R, R_d \in SO(3)$?
    - Error between actual and desired

$$R_e \coloneqq R_d^\top R \in SO(3)$$

    - We have that

$$R_e \to I \text{ as } R \to R_d$$



Recap: Stabilization Control on $SO(3)$

2. How to design $\Psi(R)$ to be positive definite ?
   - One choice for $\Psi : \mathrm{SO}(3) \to \mathbb{R}$ is

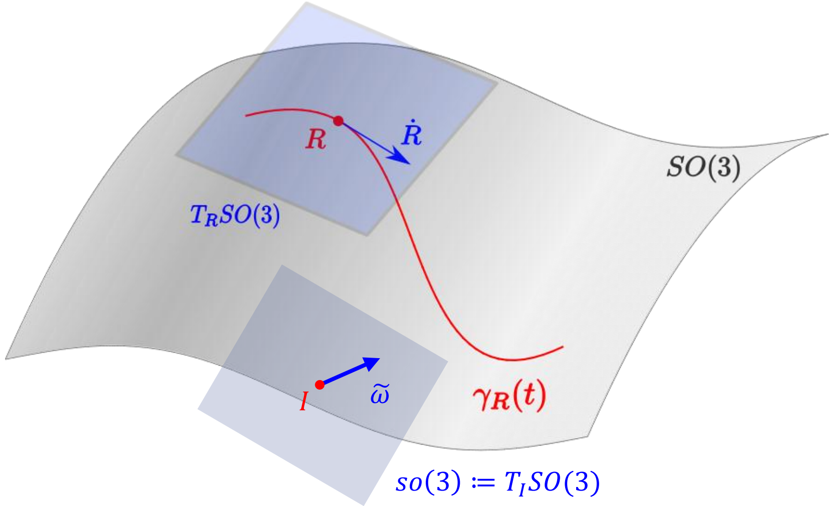
$$\Psi(R) := \frac{1}{2}\,\mathrm{tr}(I - R_d^\top R)$$

   - $\Psi(R_d) = 0$
   - We have shown before that it can be written as

$$\Psi(R) = 1 - \cos\theta$$

   for some $\theta \in (-\pi, \pi]$.
   - Therefore,

$$0 \le \Psi(R) \le 2$$



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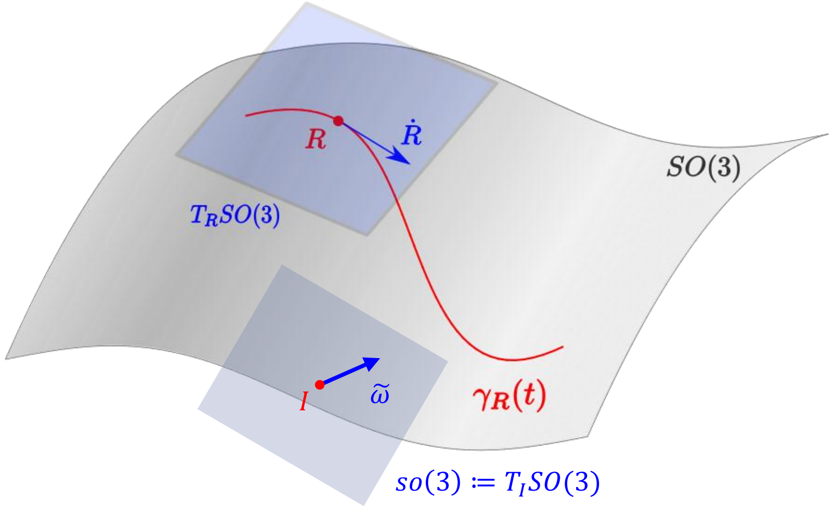
PD Control on $SO(3)$

3. How to compute $d\Psi(R) \in T_R^* SO(3)$?
   - The differential of $\Psi : \mathrm{SO}(3) \to \mathbb{R}$ is defined as the unique covector $d\Psi(R) \in T_R^* SO(3)$ that satisfies

$$\langle d\Psi(R) |\, \delta R \rangle_{T_R SO(3)} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Psi(\mathrm{R}_\epsilon), \qquad \forall \delta R \in T_R SO(3)$$

where R_ϵ is a curve on $SO(3)$ that satisfies

$$\mathrm{R}_\epsilon \big|_{\epsilon=0} = R, \qquad \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathrm{R}_\epsilon = \delta R$$



The pairing between any covector in $T_R^* SO(3)$ and vector $T_R SO(3)$ is:

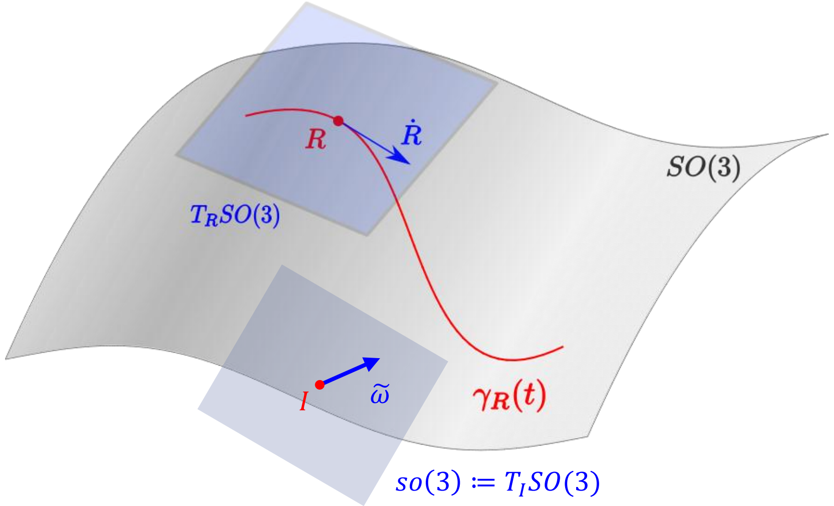
$$\langle \Gamma |\, \delta R \rangle_{T_R SO(3)} := \mathrm{tr}\,(\Gamma \delta R)$$

PD Control on $SO(3)$

3. How to compute $d\Psi(R) \in T_R^* SO(3)$?
   - $\Psi(\mathrm{R}_\epsilon) = \frac{1}{2}\,\mathrm{tr}\,(I - R_d^\top \mathrm{R}_\epsilon)$
   - $\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Psi(\mathrm{R}_\epsilon) = -\frac{1}{2}\ \mathrm{tr}\left(R_d^\top \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathrm{R}_\epsilon\right) = -\frac{1}{2}\ \mathrm{tr}\left(R_d^\top \delta R\right)$
   - $\langle d\Psi(R) |\, \delta R\rangle_{T_R SO(3)} = \mathrm{tr}\,(d\Psi(R)\delta R)$

- By comparing both sides, we have that

$$d\Psi(R) = -\frac{1}{2} R_d^\top$$

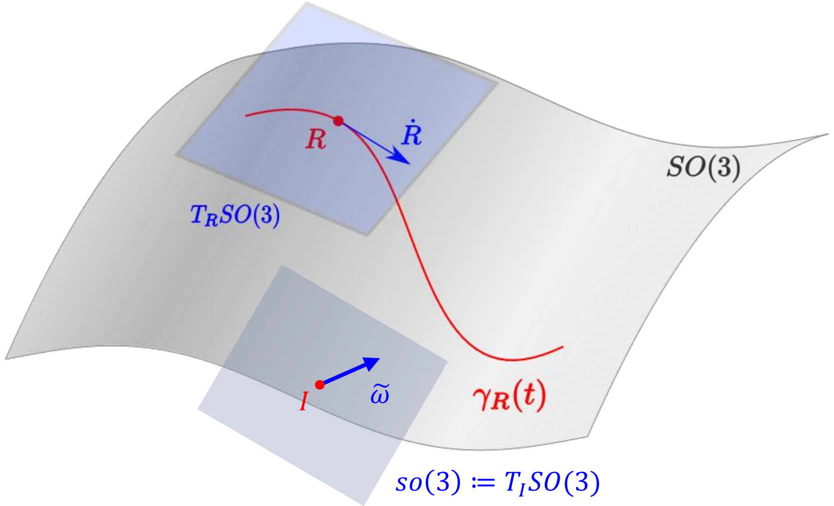


PD Control on $SO(3)$

4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
    - We use the duality of linear maps !

- Let $A\colon U \to V$ be a linear map between the vector spaces U and V. Then its dual map $A^*\colon V^* \to U^*$ is defined (implicitly) by:

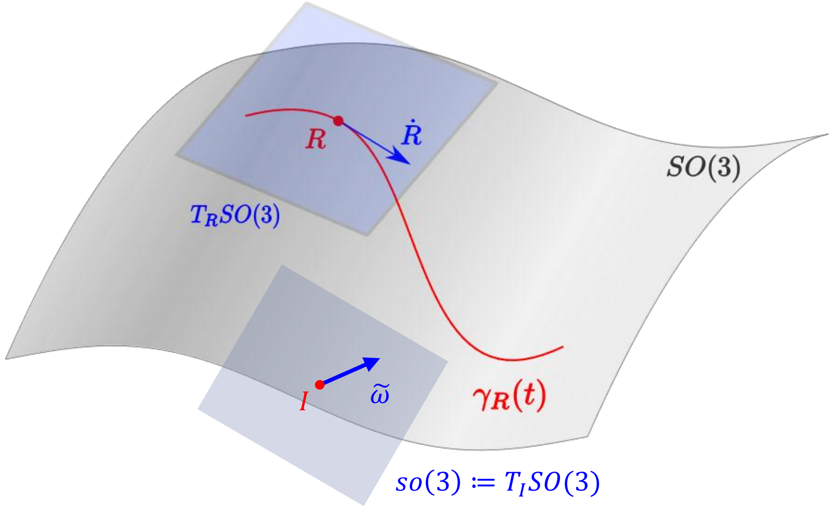
$$\langle A^*(\alpha) | u \rangle_U = \langle \alpha | A(u) \rangle_V, \qquad \forall u \in U, \alpha \in V^*$$



PD Control on $SO(3)$

4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?

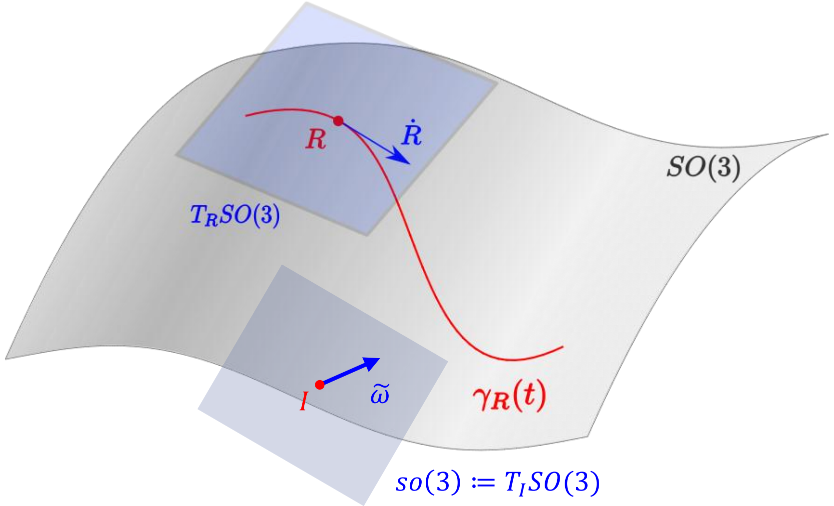
$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "S"] \arrow[rr, bend left, "\beta_R"] & so(3) \arrow[r, "L_{R,*}"] & T_R SO(3) \\
(\mathbb{R}^3)^* & so^*(3) \arrow[l, "S"'] & T_R^* SO(3) \arrow[l, "L_R^*"'] \arrow[ll, bend left, "\beta_R^*"]
\end{tikzcd}$$



PD Control on $SO(3)$

4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
   - $d\Psi(R) = -\frac{1}{2} R_d^\top \in T_R^* SO(3)$
   - $\tilde{e}_R := L_R^*\big(d\Psi(R)\big) = \mathrm{sk}\big(R_d^\top R\big) = \frac{1}{2}(R_d^\top R - R^\top R_d) \in so^*(3)$
   - $e_R := \beta_R^*\big(d\Psi(R)\big) = \mathrm{S}^{-1}(\tilde{e}_R) = \frac{1}{2} S^{-1}\big(R_d^\top R - R^\top R_d\big) \in (\mathbb{R}^3)^*$
   - The proportional torque is then designed as

$$\tau_p := -k_p e_R$$



Proportional gain: $k_p \in \mathbb{R}_+$

PD Control on $SO(3)$

5. How to design the derivative torque $\tau_d \in \mathbb{R}^3$?
    - Let's conduct a Lyapunov analysis for the closed loop system

$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega + \tau_p + \tau_d \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega - k_p\,\beta_R^*\big(d\Psi(R)\big) + \tau_d \end{pmatrix}$$