SCE 594: Special Topics in Intelligent Automation & Robotics

Lecture 23: Stabilization Control on SO(3) III and SE(3)

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Outline

- Recap last lectures
- Lyapunov analysis of closed loop system
- From SO(3) to SE(3)
- From SE(3) to MAV control

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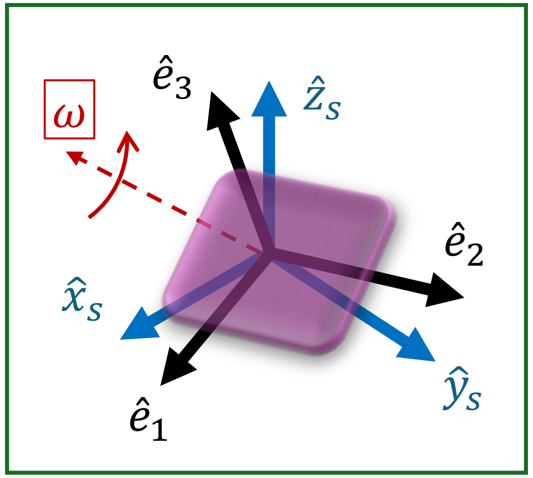
Recap: Rigid Body Rotation Dynamics

- The equations of a rotating rigid body with control torques τ are:
  - $x = (R, p) \in SO(3) \times \mathbb{R}^3$
  - $\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega \end{pmatrix} + \begin{pmatrix} 0 \\ I_3 \end{pmatrix} \tau$

  with $\omega = J^{-1} p$

  and $\beta_R : \mathbb{R}^3 \to T_R SO(3), \qquad \omega \mapsto \beta_R(\omega) := R\,\tilde{\omega}$



- Total energy of the system
  - $H(x) = H(p) = \frac{1}{2} p^\top J^{-1} p$

Recap: Stabilization Control on SO(3)

- We aim to design a Geometric PD controller

$$\tau = \tau_p + \tau_d$$

such that $x_d = (R_d, 0)$ is an asymptotically stable equilibrium point of the closed loop system

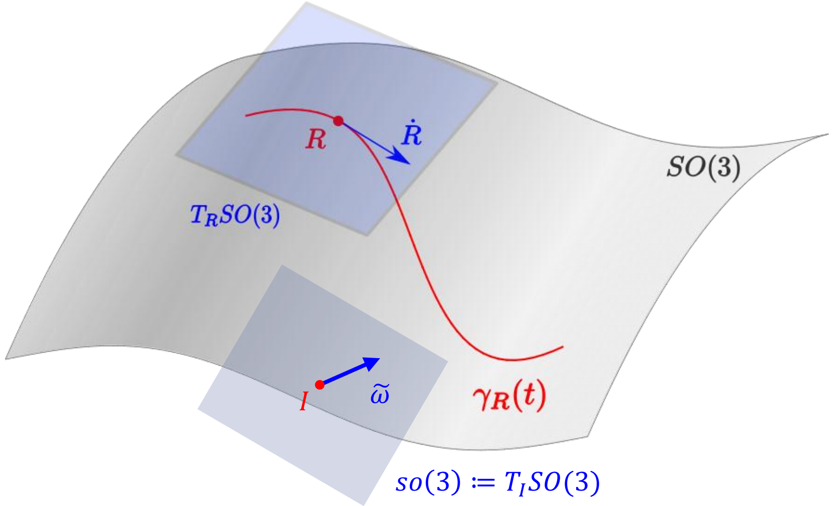
$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega + \tau_p + \tau_d \end{pmatrix}.$$

gradient of some pos.def. function $\Psi(R)$ on $SO(3)$ with a minimum at $R = R_d$.

Inject damping such that time derivative of $V(x) = H(p) + \Psi(R)$ is neg. def.

Recap: Geometric structure of SO(3)

- The geometric nature of SO(3) will be reflected in
  1. How to compute the error between $R, R_d \in SO(3)$?
  2. How to design $\Psi(R)$ to be positive definite ?
  3. How to compute $d\Psi(R) \in T_R^* SO(3)$?
  4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
  5. How to design the derivative torque $\tau_d \in \mathbb{R}^3$?



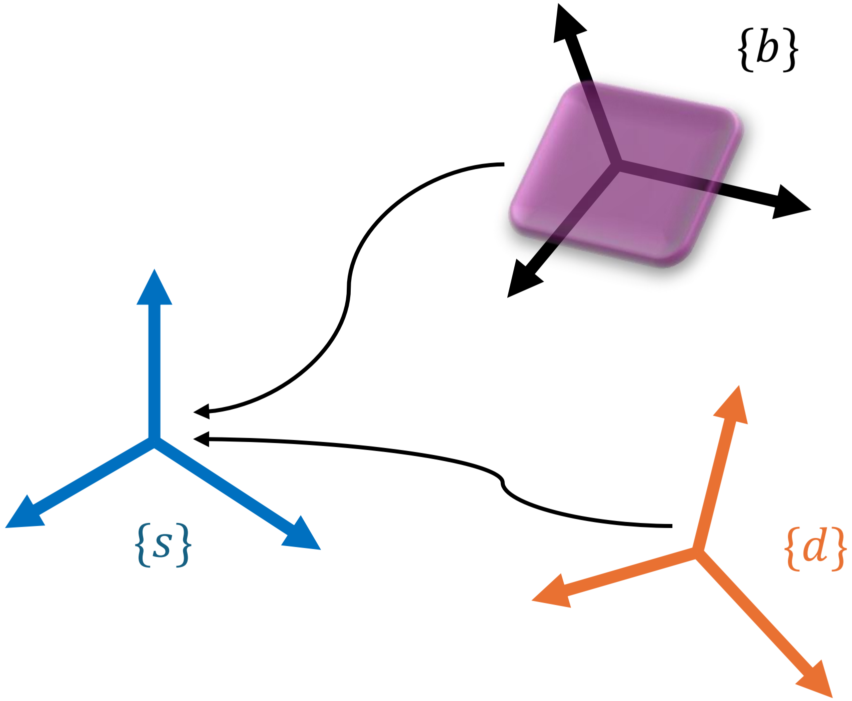
Recap: Stabilization Control on SO(3)

1. How to compute the error between $R, R_d \in SO(3)$?
    - Error between actual and desired

$$R_e := R_d^\top R \in SO(3)$$

    - We have that

$$R_e \to I \text{ as } R \to R_d$$



Recap: Stabilization Control on $SO(3)$

2. How to design $\Psi(R)$ to be positive definite ?
    - One choice for $\Psi\colon \mathrm{SO}(3) \to \mathbb{R}$ is

$$\Psi(R) \coloneqq \frac{1}{2}\,\mathrm{tr}(I - R_d^{\top} R)$$

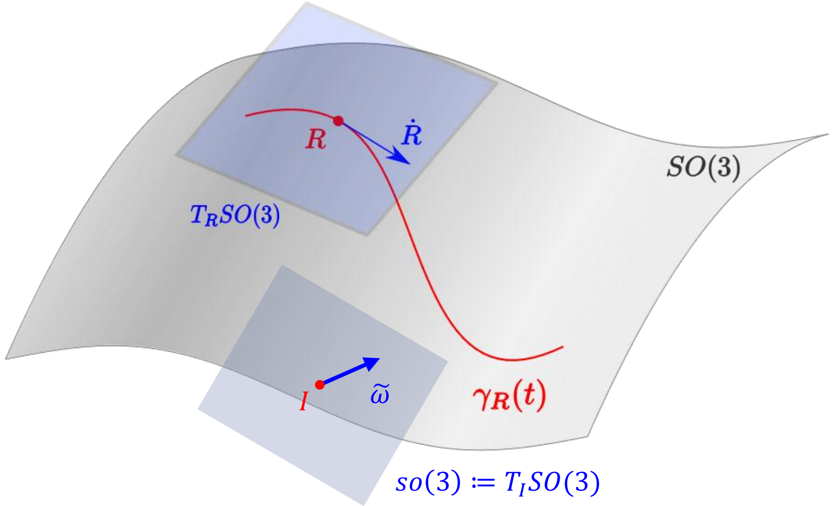
- $\Psi(R_d) = 0$
- We have shown before that it can be written as

$$\Psi(R) = 1 - \cos\theta$$

for some $\theta \in (-\pi, \pi]$.

- Therefore,

$$0 \leq \Psi(R) \leq 2$$



Recap: Stabilization Control on $SO(3)$

3. How to compute $d\Psi(R) \in T_R^*SO(3)$?
   - The differential of $\Psi : \mathrm{SO}(3) \to \mathbb{R}$ is defined as the unique covector $d\Psi(R) \in T_R^*SO(3)$ that satisfies

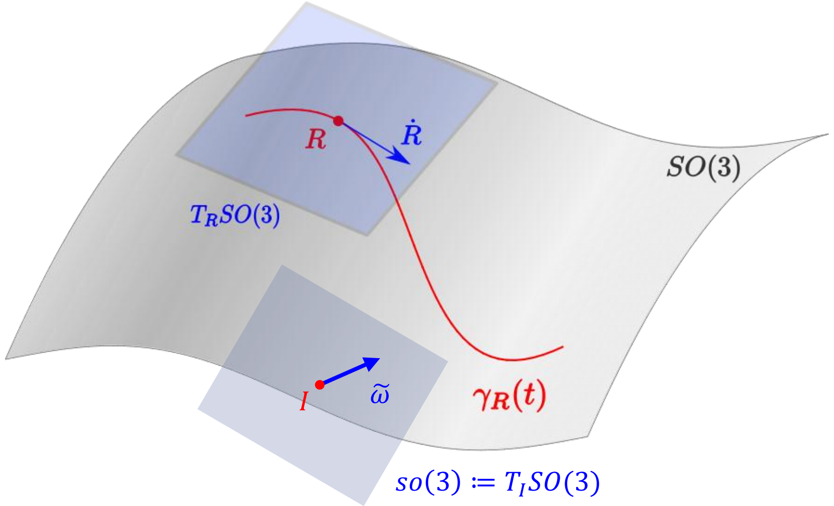
$$\langle d\Psi(R) |\, \delta R \rangle_{T_R SO(3)} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Psi(\mathrm{R}_\epsilon), \qquad \forall \delta R \in T_R SO(3)$$

where R_ϵ is a curve on $SO(3)$ that satisfies

$$\mathrm{R}_\epsilon|_{\epsilon=0} = R, \qquad \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathrm{R}_\epsilon = \delta R$$

The pairing between any covector in $T_R^*SO(3)$ and vector $T_R SO(3)$ is:

$$\langle \Gamma |\, \delta R \rangle_{T_R SO(3)} := \mathrm{tr}\,(\Gamma \delta R)$$

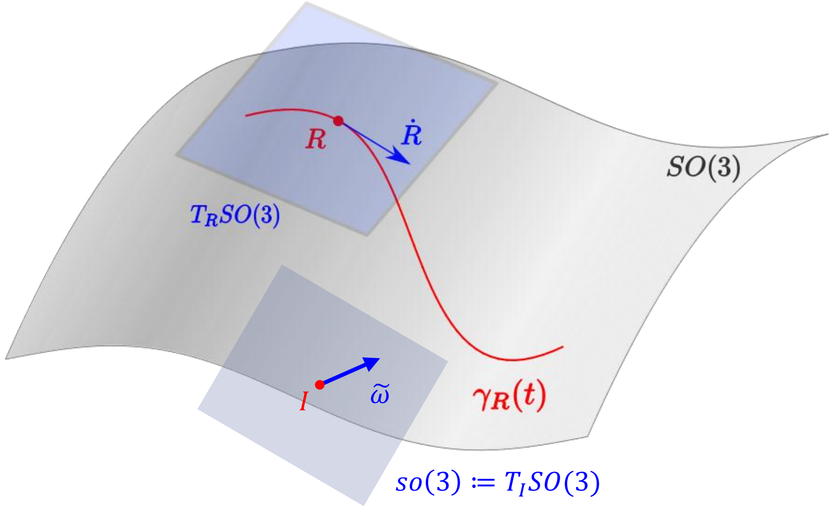


Recap: Stabilization Control on $SO(3)$

3. How to compute $d\Psi(R) \in T_R^* SO(3)$?
    - $\Psi(R_\epsilon) = \frac{1}{2}\,\mathrm{tr}\,(I - R_d^\top R_\epsilon)$
    - $\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Psi(R_\epsilon) = -\frac{1}{2}\ \mathrm{tr}\left(R_d^\top \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} R_\epsilon\right) = -\frac{1}{2}\ \mathrm{tr}\,(R_d^\top \delta R)$
    - $\langle d\Psi(R) |\, \delta R \rangle_{T_R SO(3)} = \mathrm{tr}\,(d\Psi(R)\delta R)$

- By comparing both sides, we have that

$$d\Psi(R) = -\frac{1}{2} R_d^\top$$

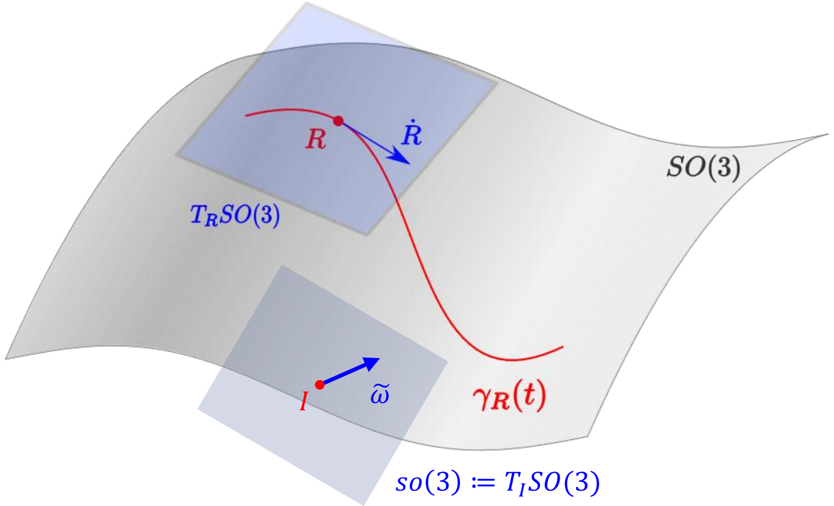


Recap: Stabilization Control on $SO(3)$

4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
   - We use the duality of linear maps !

- Let $A: U \to V$ be a linear map between the vector spaces U and V. Then its dual map $A^*: V^* \to U^*$ is defined (implicitly) by:

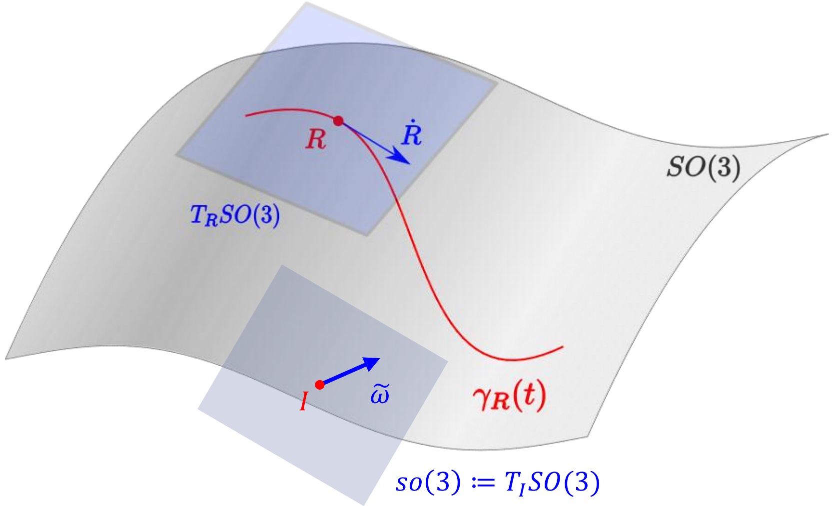
$$\langle A^*(\alpha) | u \rangle_U = \langle \alpha | A(u) \rangle_V, \qquad \forall u \in U, \alpha \in V^*$$



Recap: Stabilization Control on $SO(3)$

4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?

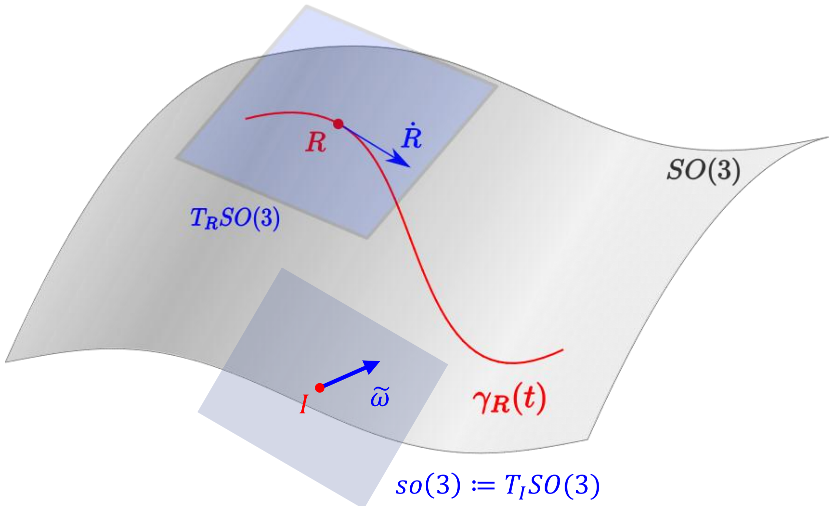
$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "S"] \arrow[rr, bend left, "\beta_R"] & so(3) \arrow[r, "L_{R,*}"] & T_R SO(3) \\
(\mathbb{R}^3)^* & so^*(3) \arrow[l, "S"'] & T_R^* SO(3) \arrow[l, "L_R^*"'] \arrow[ll, bend left, "\beta_R^*"]
\end{tikzcd}$$



Recap: Stabilization Control on $SO(3)$

4. How to convert $d\Psi(R)$ to the proportional torque $\tau_p \in \mathbb{R}^3$?
   - $d\Psi(R) = -\frac{1}{2} R_d^\top \in T_R^* SO(3)$
   - $\tilde{e}_R := L_R^*\big(d\Psi(R)\big) = \mathrm{sk}\big(R_d^\top R\big) = \frac{1}{2}(R_d^\top R - R^\top R_d) \in so^*(3)$
   - $e_R := \beta_R^*\big(d\Psi(R)\big) = S^{-1}(\tilde{e}_R) = \frac{1}{2} S^{-1}\big(R_d^\top R - R^\top R_d\big) \in (\mathbb{R}^3)^*$
   - The proportional torque is then designed as

$$\tau_p := -k_p e_R$$



Proportional gain: $k_p \in \mathbb{R}_+$

Recap: Stabilization Control on $SO(3)$

5. How to design the derivative torque $\tau_d \in \mathbb{R}^3$?
   - Let's conduct a Lyapunov analysis for the closed loop system

$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega + \tau_p + \tau_d \end{pmatrix} = \begin{pmatrix} \beta_R(\omega) \\ \tilde{p}\,\omega - k_p\,\beta_R^*\big(d\Psi(R)\big) + \tau_d \end{pmatrix}$$

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Stabilization Control on $SO(3)$

- Consider the Lyapunov function:

$$V(R,p) = H(p) + k_p \Psi(R)$$

where

$$H(p) \coloneqq \frac{1}{2}\, p^\top J^{-1} p, \qquad \text{and} \qquad \Psi(R) \coloneqq \frac{1}{2}\, \mathrm{tr}(I - R_d^\top R)$$

with

$$dH(p) = (J^{-1}p)^\top = \omega^\top, \qquad \text{and} \qquad d\Psi(R) = -\frac{1}{2} R_d^\top$$

Stability Analysis

Compute the time derivative of V

- $V(R,p) = H(p) + k_p\Psi(R)$
- $\dot{V}(R,p) = \dot{H}(p) + k_p\dot{\Psi}(R) = \langle dH(p)|\dot{p}\rangle_{\mathbb{R}^3} + k_p\left\langle d\Psi(R)|\,\dot{R}\right\rangle_{T_RSO(3)}$

Stability Analysis

Compute the time derivative of V

- $V(R,p) = H(p) + k_p\Psi(R)$
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$$= \langle dH(p)|\dot{p}\rangle_{\mathbb{R}^3} + k_p\langle d\Psi(R)|\beta_R(\omega)\rangle_{T_RSO(3)}$$

Stability Analysis

Compute the time derivative of V

- $V(R,p) = H(p) + k_p \Psi(R)$
- $\dot{V}(R,p) = \dot{H}(p) + k_p \dot{\Psi}(R) = \langle dH(p) | \dot{p} \rangle_{\mathbb{R}^3} + k_p \left\langle d\Psi(R) \middle| \dot{R} \right\rangle_{T_R SO(3)}$

$$= \langle dH(p) | \dot{p} \rangle_{\mathbb{R}^3} + k_p \langle d\Psi(R) | \beta_R(\omega) \rangle_{T_R SO(3)}$$

$$= \langle dH(p) | \dot{p} \rangle_{\mathbb{R}^3} + k_p \left\langle \beta_R^*\big(d\Psi(R)\big) \middle| \omega \right\rangle_{\mathbb{R}^3}$$

$$= \langle dH(p) | \dot{p} \rangle_{\mathbb{R}^3} + \left\langle k_p e_R \middle| \omega \right\rangle_{\mathbb{R}^3}$$

$e_R \coloneqq \beta_R^*\big(d\Psi(R)\big)$

Dual map: $\langle A^*(\alpha) | u \rangle_U = \langle \alpha | A(u) \rangle_V$

Stability Analysis

Compute the time derivative of V

- $V(R, p) = H(p) + k_p \Psi(R)$
- $\dot{V}(R, p) = \dot{H}(p) + k_p \dot{\Psi}(R) = \langle dH(p) | \dot{p} \rangle_{\mathbb{R}^3} + \langle k_p e_R | \omega \rangle_{\mathbb{R}^3}$

Stability Analysis

Compute the time derivative of V

- $V(R,p) = H(p) + k_p\Psi(R)$
- $\dot{V}(R,p) = \dot{H}(p) + k_p\dot{\Psi}(R) = \langle dH(p)|\dot{p}\rangle_{\mathbb{R}^3} + \left\langle k_p e_R|\omega\right\rangle_{\mathbb{R}^3}$

$$= \omega^\top\left(\tilde{p}\,\omega - k_p e_R + \tau_d\right) + \omega^\top k_p e_R$$

$$= \omega^\top\tilde{p}\,\omega\, - \omega^\top\left(k_p - k_p\right)e_R + \omega^\top\tau_d$$

$e_R \coloneqq \beta_R^*\left(d\Psi(R)\right)$

$\tau_p = -K_p\, e_R$

Stability Analysis

Compute the time derivative of V

- $V(R,p) = H(p) + k_p \Psi(R)$
- $\dot{V}(R,p) = \dot{H}(p) + k_p \dot{\Psi}(R) = \langle dH(p) | \dot{p} \rangle_{\mathbb{R}^3} + \langle k_p e_R | \omega \rangle_{\mathbb{R}^3}$

$$= \omega^\top \left(\tilde{p}\, \omega - k_p e_R + \tau_d \right) + \omega^\top k_p e_R$$

$$= \omega^\top \tilde{p}\, \omega \; - \omega^\top \left(k_p - k_p \right) e_R + \omega^\top \tau_d$$

$$= \underbrace{\omega^\top \tilde{p}\, \omega}_{=0} - \underbrace{\omega^\top \left(k_p - k_p \right) e_R}_{=0} + \omega^\top \tau_d$$

$$= \omega^\top \tau_d$$

$e_R \coloneqq \beta_R^* \left(d\Psi(R) \right)$

$\tau_p = -K_p\, e_R$

Stability Analysis

Compute the time derivative of V

- $V(R,p) = H(p) + k_p \Psi(R)$
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$$= \omega^\top \left(\tilde{p}\, \omega - k_p e_R + \tau_d \right) + \omega^\top k_p e_R$$

$$= \omega^\top \tilde{p}\, \omega \; - \omega^\top \left(k_p - k_p \right) e_R + \omega^\top \tau_d$$

$$= \underbrace{\omega^\top \tilde{p}\, \omega}_{=0} - \underbrace{\omega^\top \left(k_p - k_p \right) e_R}_{=0} + \omega^\top \tau_d$$

$$= \omega^\top \tau_d = -\omega^\top K_d \omega \leq 0$$

neg. semi-def.

Choose $\tau_d = -K_d \omega$ with $K_d \succ 0$

La Salle’s Invariance Principle

- Let $\Omega = \mathcal{X} = SO(3) \times \mathbb{R}^3$
- $\mathcal{R} \coloneqq \{(R, p) \in \Omega \,|\dot{V}(R, p)\}$
  - $\dot{V}(R, p) = \omega^\top K_d \omega$ $\Longrightarrow \omega = p = 0$

La Salle’s Invariance Principle

- Let $\Omega = \mathcal{X} = SO(3) \times \mathbb{R}^3$
- $\mathcal{R} := \{(R, p) \in \Omega \,|\dot{V}(R, p)\}$
  - $\dot{V}(R, p) = \omega^\top K_d \omega$ $\Longrightarrow \omega = p = 0$
- $\mathcal{M} := \{(R, p) \in \mathcal{R} \,|\dot{p} = 0\}$
  - $\dot{p} = \tilde{p}\,\omega - k_p e_R - K_d \omega$

La Salle’s Invariance Principle

- Let $\Omega = \mathcal{X} = SO(3) \times \mathbb{R}^3$
- $\mathcal{R} \coloneqq \{(R, p) \in \Omega \mid \dot{V}(R, p)\}$
  - $\dot{V}(R, p) = \omega^\top K_d \omega$ $\quad \Longrightarrow \omega = p = 0$
- $\mathcal{M} \coloneqq \{(R, p) \in \mathcal{R} \mid \dot{p} = 0\}$
  - $\dot{p} = \tilde{p}\,\omega - k_p e_R - K_d \omega$ $\quad \Longrightarrow e_R = 0$

$\Longrightarrow \tilde{e}_R = \operatorname{sk}\left(R_d^\top R\right) = 0$

$\Longrightarrow R_e - R_e^\top = 0$

$\Longrightarrow R_e = R_e^\top$

La Salle’s Invariance Principle

- Let $\Omega = \mathcal{X} = SO(3) \times \mathbb{R}^3$
- $\mathcal{R} \coloneqq \{(R,p) \in \Omega \mid \dot{V}(R,p)\}$
  - $\dot{V}(R,p) = \omega^\top K_d \omega$ $\quad \Longrightarrow \omega = p = 0$
- $\mathcal{M} \coloneqq \{(R,p) \in \mathcal{R} \mid \dot{p} = 0\}$
  - $\dot{p} = \tilde{p}\,\omega - k_p e_R - K_d \omega$ $\quad \Longrightarrow e_R = 0$

$$\Longrightarrow \tilde{e}_R = \operatorname{sk}\left(R_d^\top R\right) = 0$$

$$\Longrightarrow R_e - R_e^\top = 0$$

$$\Longrightarrow R_e = R_e^\top$$

Therefore, largest invariant set in $\mathcal{X}$ is:

$$\boxed{\mathcal{M} \coloneqq \{(R,p) \in \mathcal{R} \mid p = 0, R_d^\top R = R R_d^\top\}}$$

→ How many $R \in SO(3)$ satisfy this ?

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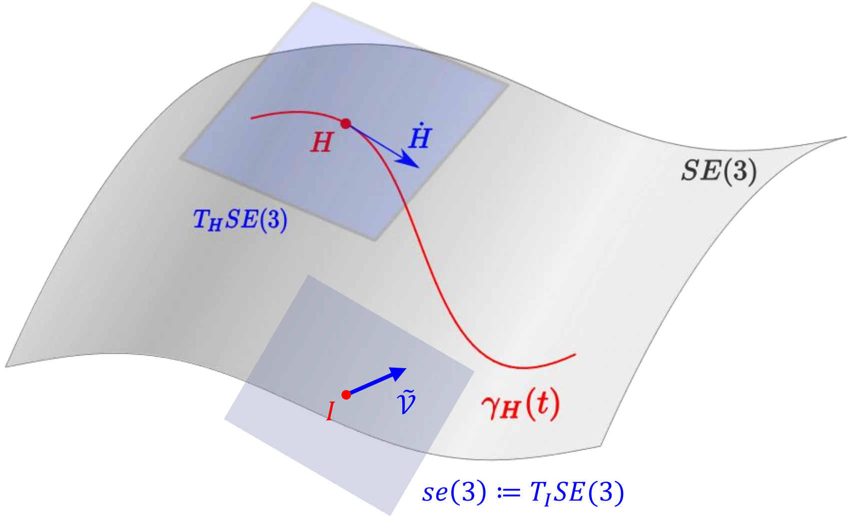
Geometric structure of SE(3)

- The geometric nature of SE(3) will be reflected in
  1. How to compute the error between $H, H_d \in SE(3)$?
  2. How to design $\Psi(H)$ to be positive definite ?
  3. How to compute $d\Psi(H) \in T_H^* SE(3)$?
  4. How to convert $d\Psi(H)$ to the proportional wrench $\mathcal{W}_p \in (\mathbb{R}^6)^*$?
  5. How to design the derivative wrench $\mathcal{W}_d \in (\mathbb{R}^6)^*$?



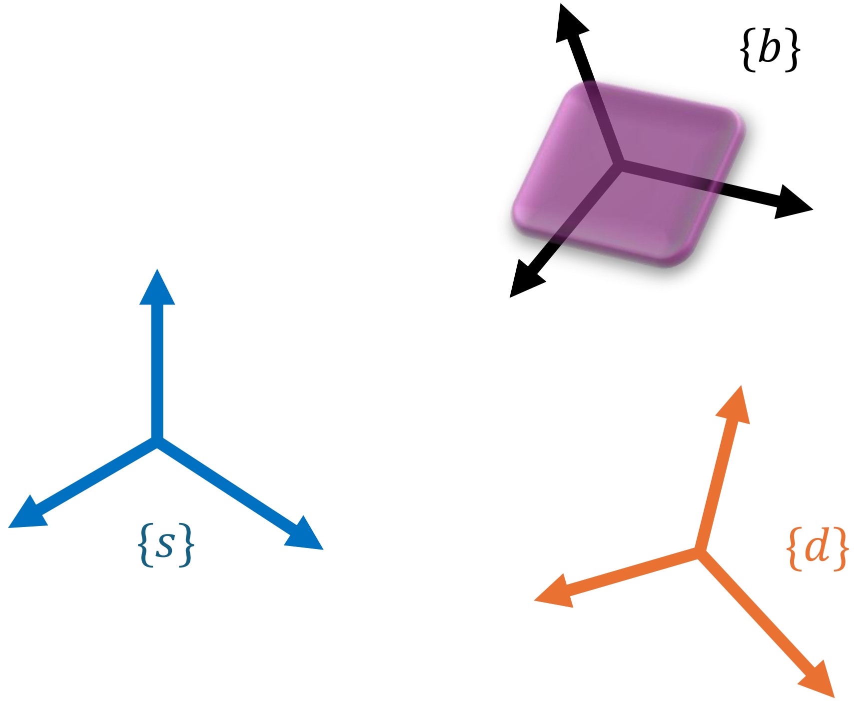
Stabilization Control on SE(3)

1. How to compute the error between $H, H_d \in SE(3)$?
   - Error between actual and desired

$$H_e := H_d^{-1} H \in SE(3)$$

   - We have that

$$H_e \to I \text{ as } H \to H_d$$



Stabilization Control on $SE(3)$

2. How to design $\Psi(H)$ to be positive definite ?
    - One choice for $\Psi\colon \mathrm{SE}(3) \to \mathbb{R}$ is to split rotation and translation

$$\Psi(H) = \Psi_{\mathrm{t}}(\xi) + \Psi_{\mathrm{o}}(R) = \frac{1}{2}(\xi - \xi_d)^\top K_t(\xi - \xi_d) + \frac{1}{2}\,\mathrm{tr}\left(G_o\left(I - R_d^\top R\right)\right)$$

Stabilization Control on $SE(3)$

2. How to design $\Psi(H)$ to be positive definite ?
   - One choice for $\Psi: \mathrm{SE}(3) \to \mathbb{R}$ is to split rotation and translation

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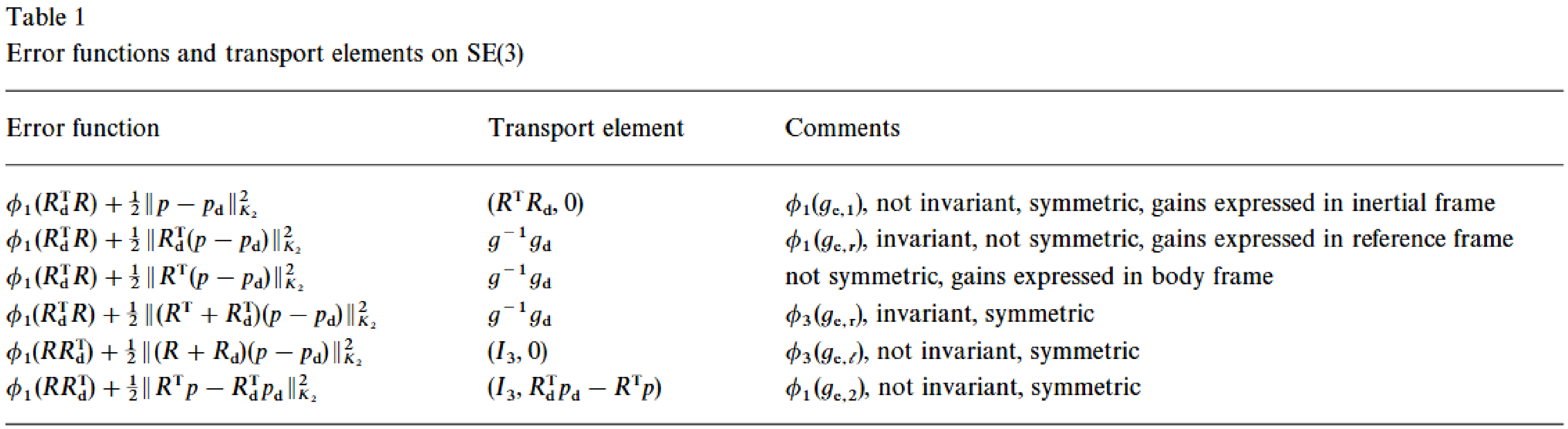


Table 1
Error functions and transport elements on SE(3)

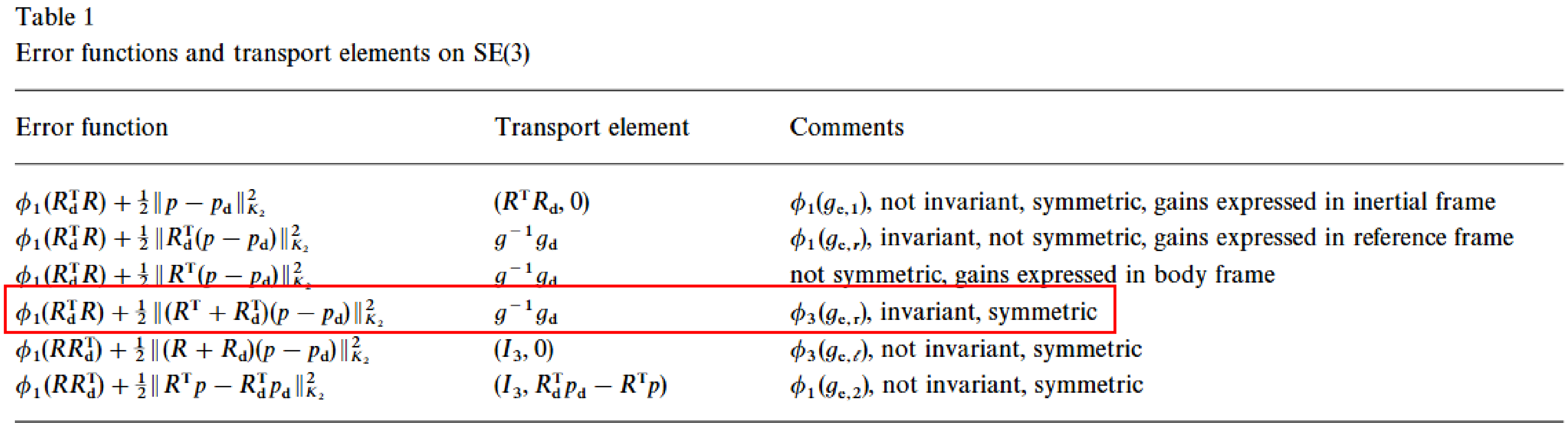
Error function	Transport element	Comments
$\phi_1(R_\mathrm{d}^\mathrm{T} R) + \frac{1}{2}\|p - p_\mathrm{d}\|_{K_2}^2$	$(R^\mathrm{T} R_\mathrm{d}, 0)$	$\phi_1(g_{\mathrm{e},1})$, not invariant, symmetric, gains expressed in inertial frame
$\phi_1(R_\mathrm{d}^\mathrm{T} R) + \frac{1}{2}\|R_\mathrm{d}^\mathrm{T}(p - p_\mathrm{d})\|_{K_2}^2$	$g^{-1} g_\mathrm{d}$	$\phi_1(g_{\mathrm{e},r})$, invariant, not symmetric, gains expressed in reference frame
$\phi_1(R_\mathrm{d}^\mathrm{T} R) + \frac{1}{2}\|R^\mathrm{T}(p - p_\mathrm{d})\|_{K_2}^2$	$g^{-1} g_\mathrm{d}$	not symmetric, gains expressed in body frame
$\phi_1(R_\mathrm{d}^\mathrm{T} R) + \frac{1}{2}\|(R^\mathrm{T} + R_\mathrm{d}^\mathrm{T})(p - p_\mathrm{d})\|_{K_2}^2$	$g^{-1} g_\mathrm{d}$	$\phi_3(g_{\mathrm{e},\mathrm{r}})$, invariant, symmetric
$\phi_1(R R_\mathrm{d}^\mathrm{T}) + \frac{1}{2}\|(R + R_\mathrm{d})(p - p_\mathrm{d})\|_{K_2}^2$	$(I_3, 0)$	$\phi_3(g_{\mathrm{e},\ell})$, not invariant, symmetric
$\phi_1(R R_\mathrm{d}^\mathrm{T}) + \frac{1}{2}\|R^\mathrm{T} p - R_\mathrm{d}^\mathrm{T} p_\mathrm{d}\|_{K_2}^2$	$(I_3, R_\mathrm{d}^\mathrm{T} p_\mathrm{d} - R^\mathrm{T} p)$	$\phi_1(g_{\mathrm{e},2})$, not invariant, symmetric

Bullo, F., & Murray, R. M. (1999). Tracking for fully actuated mechanical systems:
a geometric framework. *Automatica*, *35*(1), 17-34.

Stabilization Control on $SE(3)$

2. How to design $\Psi(H)$ to be positive definite ?
   - One choice for $\Psi\colon \mathrm{SE}(3) \to \mathbb{R}$ is to split rotation and translation

$$\Psi(H) = \Psi_{\mathrm{t}}(\xi) + \Psi_{\mathrm{o}}(R) = \frac{1}{2}(\xi - \xi_d)^{\top} K_t (\xi - \xi_d) + \frac{1}{2}\,\mathrm{tr}\Big(G_o\big(I - R_d^{\top} R\big)\Big)$$



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$\phi_1(R_{\mathrm{d}}^{\mathrm{T}}R) + \frac{1}{2}\|R_{\mathrm{d}}^{\mathrm{T}}(p - p_{\mathrm{d}})\|_{K_2}^2$	$g^{-1}g_{\mathrm{d}}$	$\phi_1(g_{\mathrm{e},\mathrm{r}})$, invariant, not symmetric, gains expressed in reference frame
$\phi_1(R_{\mathrm{d}}^{\mathrm{T}}R) + \frac{1}{2}\|R^{\mathrm{T}}(p - p_{\mathrm{d}})\|_{K_2}^2$	$g^{-1}g_{\mathrm{d}}$	not symmetric, gains expressed in body frame
$\phi_1(R_{\mathrm{d}}^{\mathrm{T}}R) + \frac{1}{2}\|(R^{\mathrm{T}} + R_{\mathrm{d}}^{\mathrm{T}})(p - p_{\mathrm{d}})\|_{K_2}^2$	$g^{-1}g_{\mathrm{d}}$	$\phi_3(g_{\mathrm{e},\mathrm{r}})$, invariant, symmetric
$\phi_1(RR_{\mathrm{d}}^{\mathrm{T}}) + \frac{1}{2}\|(R + R_{\mathrm{d}})(p - p_{\mathrm{d}})\|_{K_2}^2$	$(I_3, 0)$	$\phi_3(g_{\mathrm{e},\ell})$, not invariant, symmetric
$\phi_1(RR_{\mathrm{d}}^{\mathrm{T}}) + \frac{1}{2}\|R^{\mathrm{T}}p - R_{\mathrm{d}}^{\mathrm{T}}p_{\mathrm{d}}\|_{K_2}^2$	$(I_3, R_{\mathrm{d}}^{\mathrm{T}}p_{\mathrm{d}} - R^{\mathrm{T}}p)$	$\phi_1(g_{\mathrm{e},2})$, not invariant, symmetric

Bullo, F., & Murray, R. M. (1999). Tracking for fully actuated mechanical systems: a geometric framework. *Automatica*, *35*(1), 17-34.

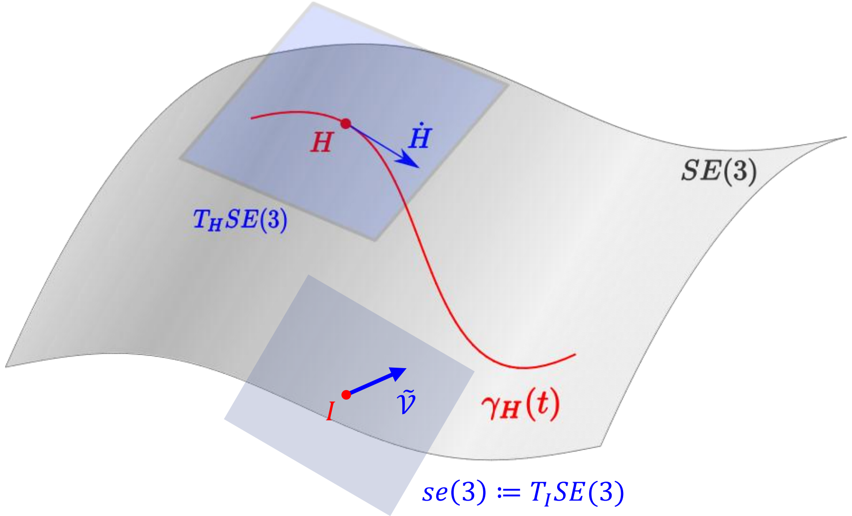
Stabilization Control on $SE(3)$

4. How to convert $d\Psi(H)$ to proportional wrench $\mathcal{W}_p \in (\mathbb{R}^6)^*$?

$$\begin{tikzcd}
\mathbb{R}^6 \arrow[r, "\tilde{S}"] \arrow[rr, bend left, "\beta_H"] & se(3) \arrow[r, "L_{H,*}"] & T_H SE(3) \\
(\mathbb{R}^6)^* & se^*(3) \arrow[l, "\tilde{S}"'] & T_H^* SE(3) \arrow[l, "L_H^*"'] \arrow[ll, bend left, "\beta_H^*"]
\end{tikzcd}$$

$$\boxed{\mathcal{W}_p := \beta_H^*\big(d\Psi(H)\big)}$$

$$\boxed{\beta_H(\delta\mathcal{V}) := H\,\widetilde{\delta\mathcal{V}}}$$



$$\langle \beta_H^*\big(d\Psi(H)\big) | \delta\mathcal{V} \rangle = \langle d\Psi(H) | \beta_H(\delta\mathcal{V}) \rangle$$

Outline

- Recap last lectures
- Lyapunov analysis of closed loop system
- From SO(3) to SE(3)
- From SE(3) to MAV control

Multi-rotor aerial vehicles

- Multi-rotor aerial vehicles (MAVs) are the most popular choice of aerial robotics platform.
- Usually, they have a simple mechanical structure and few moving parts.
- MAVs are usually modeled as a single rigid body floating in space.

Rashad, R., Goerres, J., Aarts, R., Engelen, J. B., & Stramigioli, S. (2020). Fully actuated multirotor UAVs: A literature review. *IEEE Robotics & Automation Magazine*, *27*(3), 97-107.

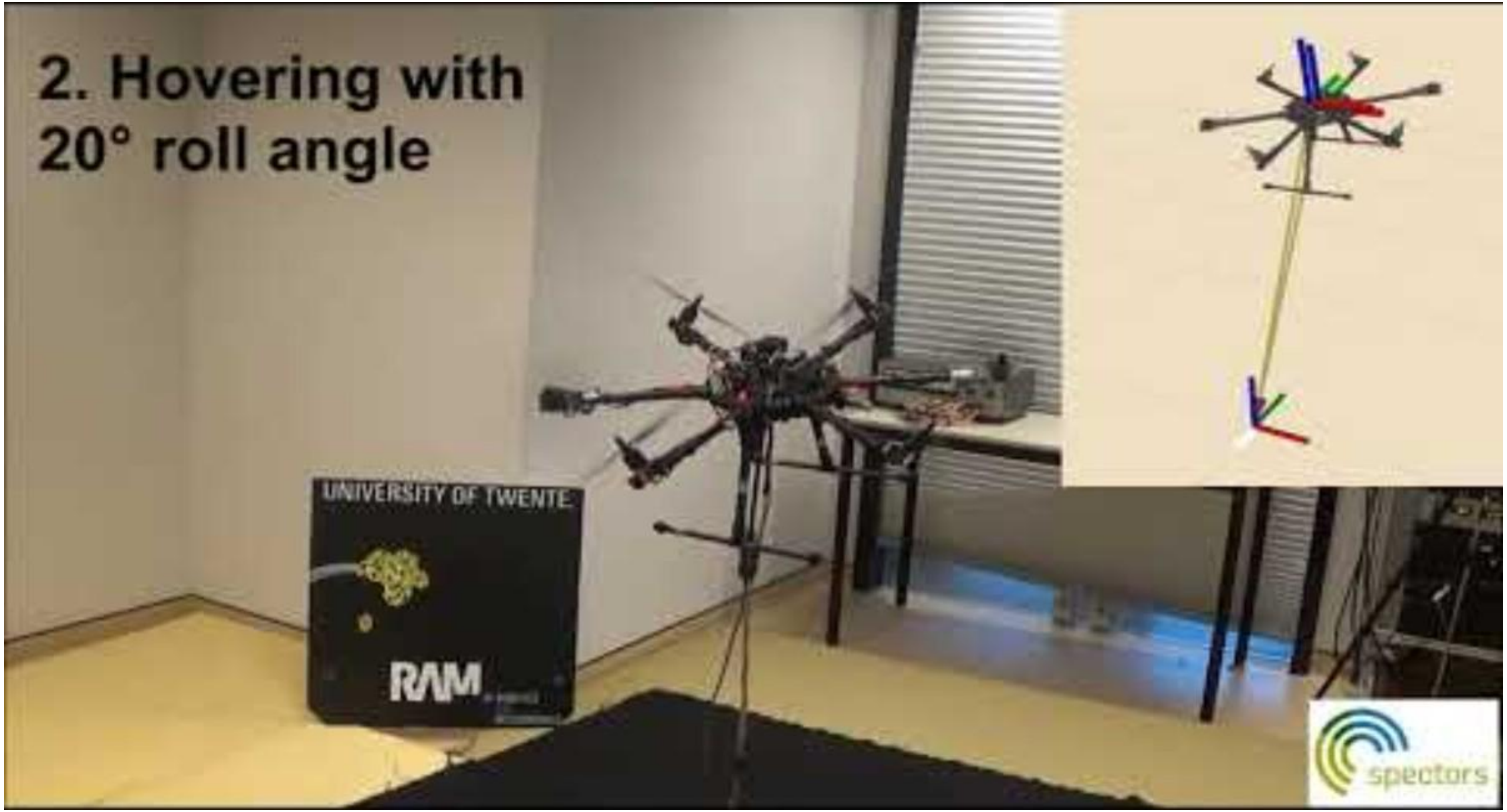
Multi-rotor aerial vehicles

- MAVs are classified based on properties of the map between individual rotor thrusts λ_i and the resultant wrench applied on the MAV’s body which is used for control.

$$\boxed{\mathcal{W}_{\text{con}}^{b,b} = \begin{pmatrix} \tau_{\text{con}}^{b,b} \\ f_{\text{con}}^{b,b} \end{pmatrix} = M(t)\lambda}$$

$\lambda = \left(\lambda_1, \cdots, \lambda_{N_p}\right) \in \mathbb{R}^{N_p}$ is the rotors thrust vector and N_p is the number of propellers on the MAV.

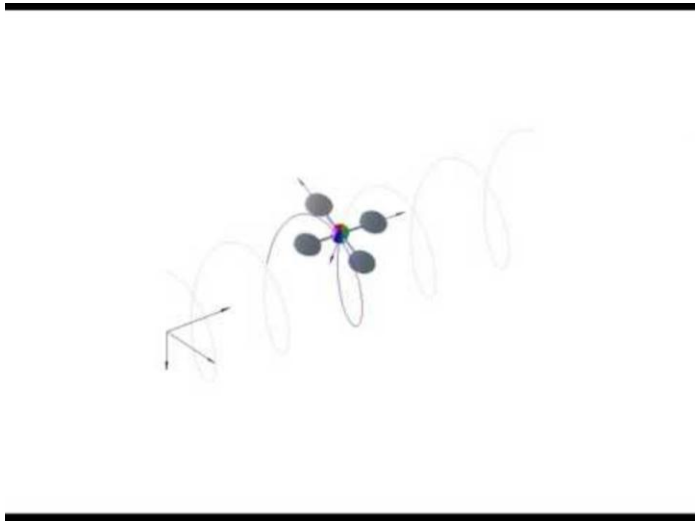
SE(3) Control of Fully-actuated MAV



Rashad, R., Kuipers, P., Engelen, J., & Stramigioli, S. (2017). Design, modeling, and geometric control on SE (3) of a fully-actuated hexarotor for aerial interaction. *arXiv preprint arXiv:1709.05398*.

SE(3) Control of Under-actuated MAV

Mellinger, D., & Kumar, V. (2011, May). Minimum snap trajectory generation and control for quadrotors. In *2011 IEEE international conference on robotics and automation* (pp. 2520-2525). IEEE.



Lee, T., Leok, M., & McClamroch, N. H. (2010, December). Geometric tracking control of a quadrotor UAV on SE (3). In *49th IEEE conference on decision and control (CDC)* (pp. 5420-5425). IEEE.