SCE 594: Special Topics in Intelligent Automation & Robotics

Topic 5: Control of Fixed-Base Manipulators

Lecture 24: Motion Control

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Outline

- Open loop stability analysis
- Stabilization Control
- Feedback Linearization Control

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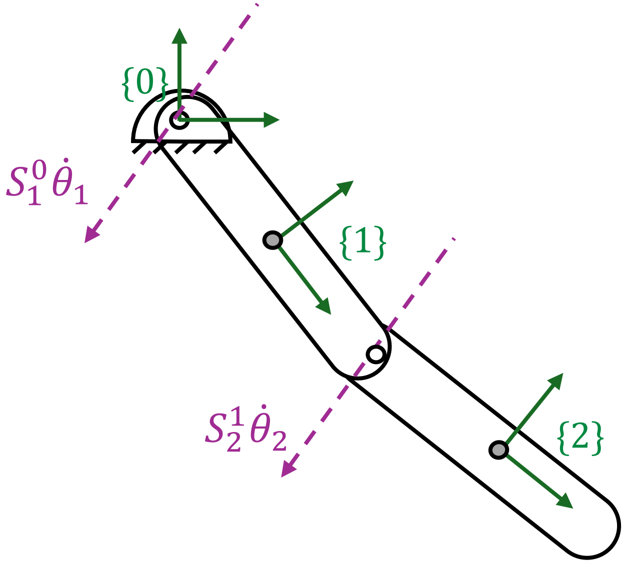
State Space Dynamics

- The governing equations of an n-link manipulator* with control torques τ are:

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + B(\dot{\theta})\dot{\theta} + g(\theta) = \tau$$

where $\theta \in Q = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \cong \mathbb{T}^n$.



*We assume for simplicity all joints are revolute.

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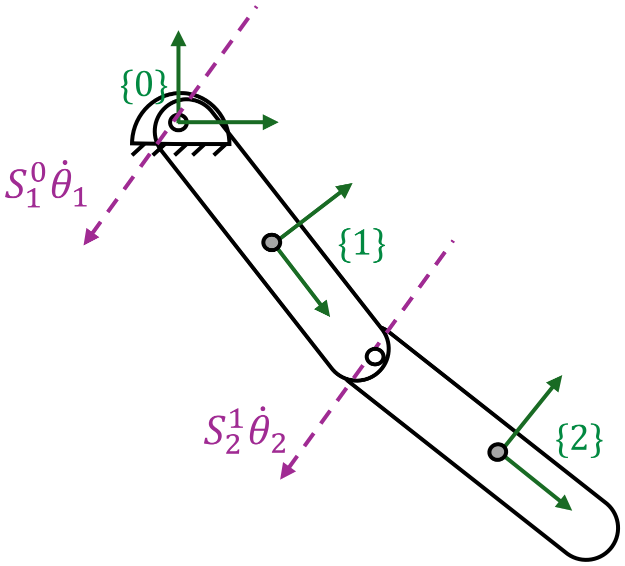
where $\theta \in Q = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \cong \mathbb{T}^n$.

- We can cast it into state space form
  - $x = (x_1, x_2) = \left(\theta, \dot{\theta}\right) \in TQ$
  - $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)\left[c(x) + b(x_2) + g(x_1)\right] \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} \tau$



$c(x) \coloneqq C(x_1, x_2)x_2 \in \mathbb{R}^n, \qquad b(x_2) \coloneqq B(x_2)x_2 \in \mathbb{R}^n, \qquad g(x_1) \in \mathbb{R}^n$

Equilibrium points of open loop system

- Proposition:
  - For $\tau = 0$, the equilibrium points of any n-link manipulator are states $x_* \coloneqq (x_{1,*}, x_{2,*})$ that satisfy

$$g(x_{1,*}) = 0_n, \qquad\qquad x_{2,*} = 0_n.$$

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- Proof:
  - $\begin{pmatrix} 0_n \\ 0_n \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)\,[c(x) + b(x_2) + g(x_1)] \end{pmatrix} \Longrightarrow x_2 = 0_n \Longrightarrow c(x) = b(x_2) = 0_n.$
  - Thus, the equilibrium points should satisfy

$$0_n = M^{-1}(x_1)g(x_1) \quad \text{or equivalently} \quad 0 = g^\top(x_1)M^{-1}(x_1)g(x_1)$$

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  - Since $M(x_1) \succ 0$, we have that $M^{-1}(x_1) \succ 0$
  - Consequently, $g^\top(x_1)M^{-1}(x_1)g(x_1) = 0$ if an only if $g(x_1) = 0_n$.

Example: 2-link manipulator

- Recall that the gravity torques for a 2-link manipulator are given by:

$$g(\theta) = \begin{pmatrix} \gamma_1 \sin\theta_1 + \gamma_2 \sin(\theta_1 + \theta_2) \\ \gamma_2 \sin(\theta_1 + \theta_2) \end{pmatrix}$$

where $\theta \coloneqq (\theta_1, \theta_2) \in (-\pi, \pi] \times (-\pi, \pi]$ and $\gamma_1, \gamma_2 \in \mathbb{R}_+$ are constants.

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- The equilibrium points $x_* = (\theta_*, 0_2)$ of this manipulator should satisfy $g(\theta_*) = 0_n$:

$$\sin(\theta_1 + \theta_2) = 0 \qquad \text{and} \qquad \sin\theta_1 = 0$$

  - Thus, we have that

$$\theta_* = (0,0), \qquad \theta_* = (0,\pi), \qquad \theta_* = (\pi,0), \qquad \theta_* = (\pi,\pi),$$

  - Consequently:

$$x_* = (0,0,0,0), \qquad x_* = (0,\pi,0,0), \qquad x_* = (\pi,0,0,0), \qquad x_* = (\pi,\pi,0,0),$$

Stability properties of the origin

- Total energy of the system

$$E_{\text{tot}}\left(\theta,\dot{\theta}\right) = E_{\text{kin}}\left(\theta,\dot{\theta}\right) + E_{\text{pot}}(\theta) = \frac{1}{2}\,\dot{\theta}^{\top}M(\theta)\dot{\theta}\, + E_{\text{pot}}(\theta)$$

where $E_{\text{pot}}: Q \to \mathbb{R}$ is defined such that its gradient:

$$\nabla_{\theta}E_{\text{pot}}(\theta) = g(\theta) \in \mathbb{R}^{n}.$$

In general, we have that $E_{\text{kin}}: TQ \to \mathbb{R}$ and $E_{\text{pot}}: Q \to \mathbb{R}$

Stability properties of the origin

- Consider the Lyapunov function given by the total energy of the mechanical system

$$V_{OL}(x_1, x_2) = E_{\mathrm{tot}}\big(\theta, \dot{\theta}\big) = \frac{1}{2}\, x_2^\top M(x_1) x_2 \; + E_{\mathrm{pot}}(x_1)$$

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- Since it represents energy, $V_{OL}(x_1, x_2)$ is a positive definite Lyapunov function with a minimum at V_{OL} $V_{OL}(0_2, 0_2) = 0$.
- One can show* that

$$\dot{V}_{OL}(x_1, x_2) = -x_2^{\top} B(x_2) x_2 \leq 0$$

which along with LaSalle’s invariance principle shows that $x_* = (0_2, 0_2)$ is locally asymptotically stable.

*Homework

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- Open loop stability analysis
- Stabilization Control
- Feedback Linearization Control

Stabilization Control on $Q = \mathbb{T}^n \cong \mathbb{R}^n$

- We aim to design a energy-balancing controller

$$\tau = \tau_p + \tau_d$$

such that $x_d = (\theta_d, 0)$ is an asymptotically stable equilibrium point of the closed loop system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)\left[c(x) + b(x_2) + g(x_1) + \tau_p + \tau_d\right] \end{pmatrix}$$

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gradient of some pos.def. function $\Psi(\theta)$ on Q with a minimum at $\theta = \theta_d$.

Inject damping such that time derivative of $V(x) = V_{OL}(x) + \Psi(\theta)$ is neg. def.

Summary

- The control law

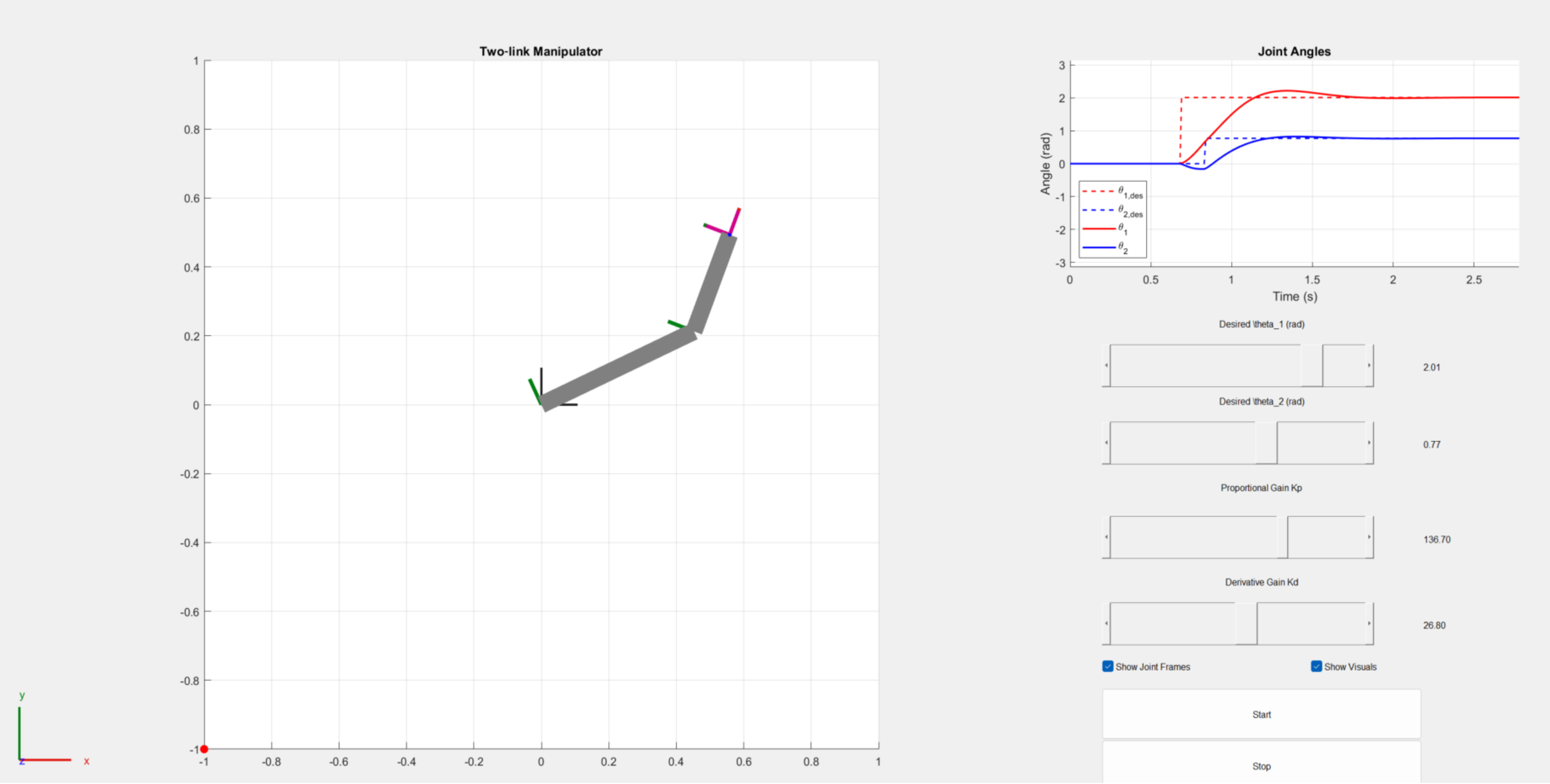
$$\tau = \tau_p + \tau_d = -K_p\,(x_1 - \theta_d) + g(x_1) - K_d x_2$$

makes the closed loop system

$$\begin{pmatrix} \dot{x}_1 \\ M(x_1)\dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -c(x) - b(x_2) - K_p\,(x_1 - \theta_d) - K_d x_2 \end{pmatrix}$$

have a globally asymptotically equilibrium point at $x_d = (\theta_d, 0_2)$.

Software Implementation



Desired \theta_1 (rad) 2.01

Desired \theta_2 (rad) 0.77

Proportional Gain Kp 136.70

Derivative Gain Kd 26.80

☑ Show Joint Frames

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Feedback Linearization

- Feedback linearization control aims to cancel out the known nonlinearities by applying a torque command τ that directly “inverts” the dynamics.
- The result (in an ideal, no-uncertainty scenario) is a closed-loop system that behaves like a simple linear system.

Feedback Linearization

- For the state space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)\,[c(x) + b(x_2) + g(x_1) - \tau] \end{pmatrix}$$

- The feedback linearization control law is given by

$$\tau = c(x) + b(x_2) + g(x_1) \; - M(x_1)\left[K_p\,(x_1 \; - \theta_d) + K_d x_2\right]$$

Known in the literature as the inverse dynamics or computed torque control method

Feedback Linearization

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$$\tau = c(x) + b(x_2) + g(x_1) - M(x_1)\left[K_p\,(x_1 - \theta_d) + K_d x_2\right]$$

yields the closed loop system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -K_p\,(x_1 - \theta_d) - K_d x_2 \end{pmatrix}$$

which is globally asymptotically stable at $x_d = (\theta_d, 0_2)$.

Known in the literature as the inverse dynamics or computed torque control method

Comparison

- **Computed Torque Method (CTM):**
  - Full dynamic compensation: Cancels out all nonlinearities (inertia coupling, Coriolis/centrifugal forces, and gravity) to transform the system into decoupled linear systems.
  - Model dependence: Highly sensitive to model errors; requires accurate knowledge of inertial, Coriolis, and gravity terms.
  - Performance: Can achieve very fast, accurate trajectory tracking if the model is good.
- **PD with Gravity Compensation:**
  - Partial dynamic compensation: Only cancels gravity effects.
  - Model dependence: Less sensitive to errors compared to CTM; only needs a reasonable estimate of the gravity vector.
  - Performance: Good for slow or moderate speed motions; tracking degrades at high speeds due to unmodeled dynamics