# CIE 482: Path Planning & Navigation for Mobile Robots

Lecture 16: Kalman Filtering I

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# Outline

- Recap last lecture
- Introduction to Kalman Filtering
- State Transition and Measurement Probabilistic Models
- Examples

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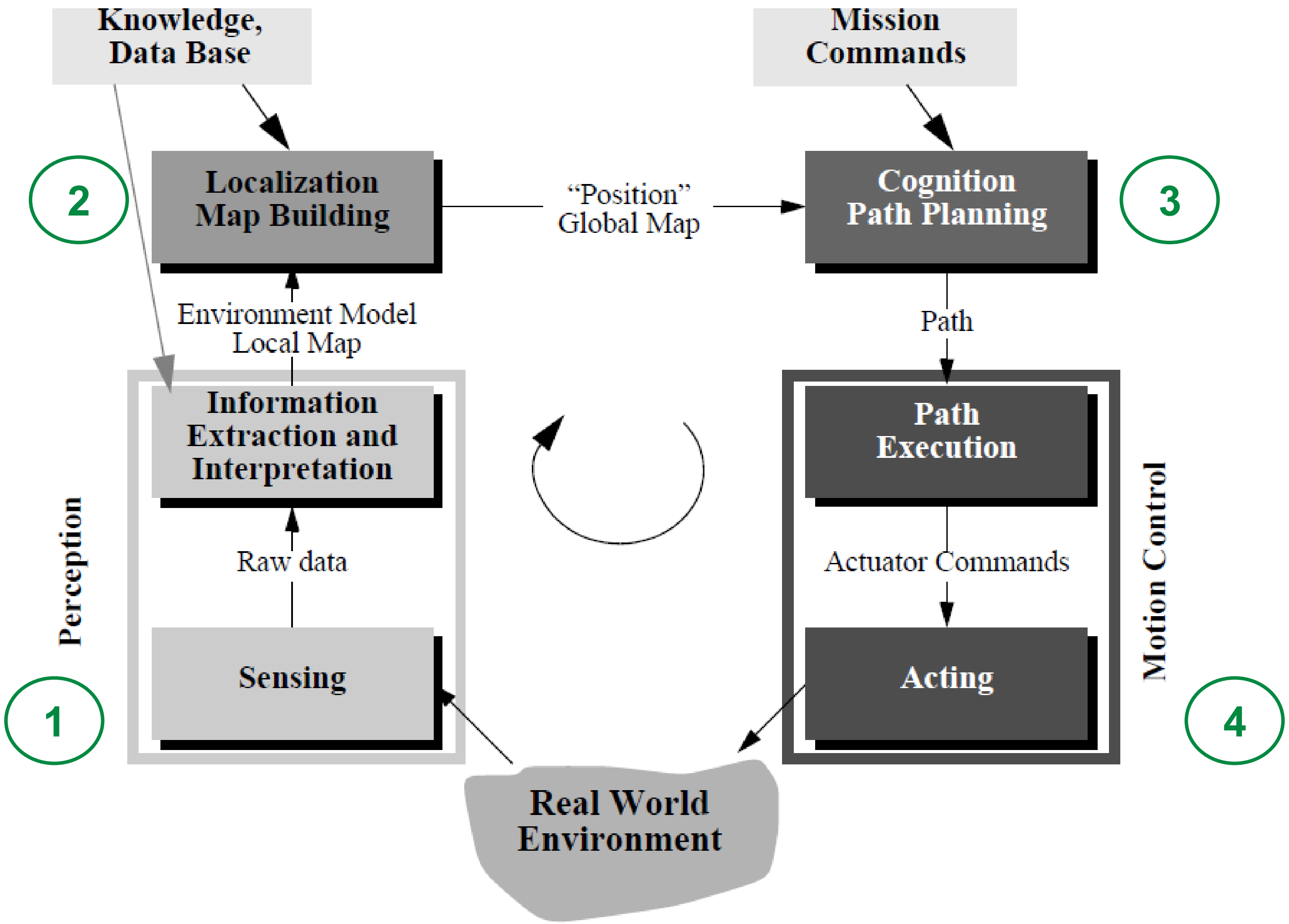
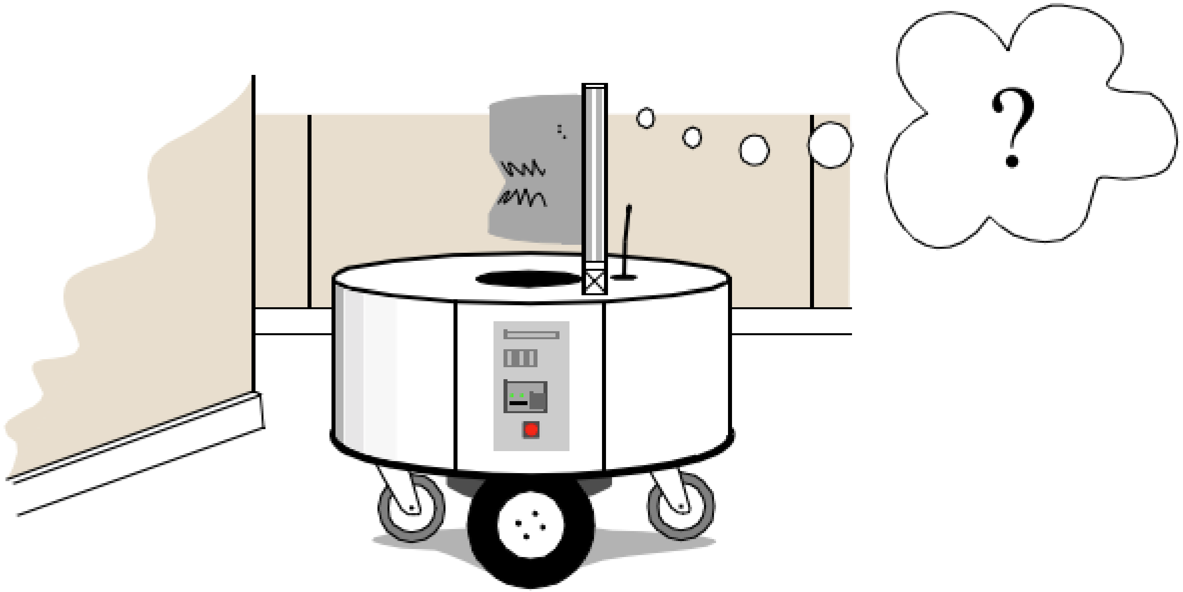
# Recap: Robot Navigation Architecture



# Recap: Robot Localization

- Out of these four components of the robot navigation architecture, localization has received the greatest research attention due to its complexity.
- Robot localization focuses on the question:
  - “Where am I ?”



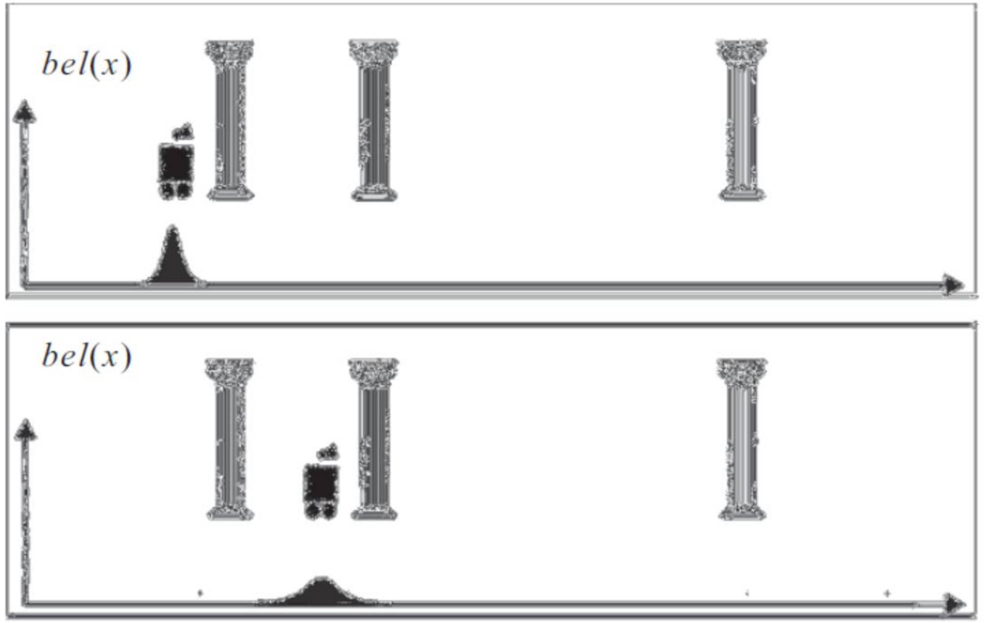
# Recap: Belief Representation

- We represent the robot not just using a single possible position but also a possibly infinite set of positions.
- In this approach, the belief is modeled by a probability distribution function $bel(x)$.
- In this approach, beliefs are

represented by conditional

probability distributions:

$$bel(x) = p(x|y)$$



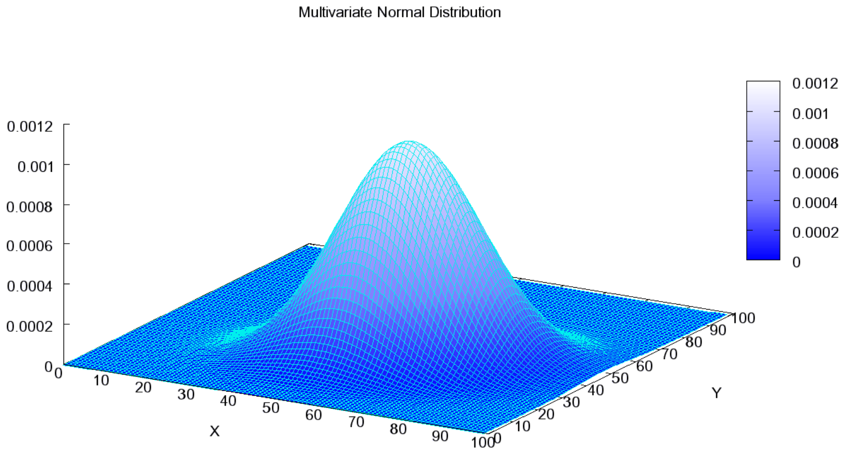

The belief function $bel(x)$ is our best guess of the robot's state $x$.

# Recap: Multivariate Gaussian Distribution

- In case the random variable $x \in \mathbb{R}^k$ is a $k$-dimensional vector, we will have a multivariate normal distribution characterized by the density function:

$$p(x) = \frac{1}{(2\pi)^{k/2}\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right)$$

where $\mu \in \mathbb{R}^k$ & $\Sigma \in \mathbb{R}^{k\times k}$ denote the mean & covariance matrix.



# Recap: Probability basics

**Joint** probability
of $X = x$ and $Y = y$

**Conditional** probability
of $X = x$ given $Y = y$

$$\boxed{p(x|y) = \frac{p(x,y)}{p(y)}}$$

**Marginal** probability
of $Y = y$

**Functions of one variable:**

- Conditional probability
- Marginal probability

**Functions of multiple variable:**

- Joint probability

**Recall that**

$p(y) = \int p(x,y)dx$

**or** $p(y) = \sum_x p(x,y)$

# Recap: Bayes rule

$$p(x|z) = \eta\, p(z|x)p(x)$$

- $p(x)$ is our prior probability of $X = x$ (before incorporating any external data).
- $p(x|z)$ is our posterior probability of $X = x$ after considering the data $Z = z$.
- $p(z|x)$ is the likelihood that we see the data $Z = z$ if $X = x$ is true.
- $\eta$ is a normalization factor

# Recap: Robot Localization Problem

- The Localization problem is equivalent to finding

$$\mathrm{bel}(\mathrm{x}_k) \coloneqq p(x_k | Z_k, U_k)$$

where

- The robot's path / history of states is given by

$$X_k \coloneqq \{x_0, x_1, \cdots, x_{k-1}, x_k\} = \{X_{k-1}, x_k\}$$

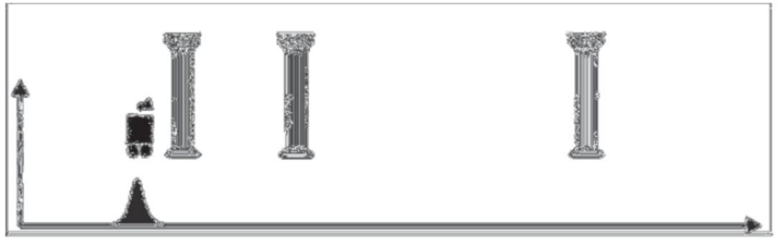
- The history of control inputs (or proprioceptive sensor readings)

$$U_k \coloneqq \{u_1, \cdots, u_{k-1}, u_k\} = \{U_{k-1}, u_k\}$$

- The history of observations (or exteroceptive sensor readings)

$$Z_k \coloneqq \{z_1, \cdots, z_{k-1}, z_k\} = \{Z_{k-1}, z_k\}$$

$\mathrm{bel}(\mathrm{x}_{k-1})$



$\mathrm{bel}(\mathrm{x}_k)$

# Recap: Summary of Bayesian Filtering

Propreoceptive sensor measurement $u_k$

**State Transition Model**
$p(x_k | x_{k-1}, Z_{k-1}, U_k)$

1 **Prediction update**

$\overline{\boldsymbol{bel}}\,(\boldsymbol{x_k}) = p(x_k | Z_{k-1}, U_k)$

**Correction update** 2

**Sensor Model**
$p(z_k | x_k)$

Exteroceptive sensor measurement $z_k$

$\boldsymbol{bel}\,(\boldsymbol{x_{k-1}}) = p(x_{k-1} | Z_{k-1}, U_{k-1})$

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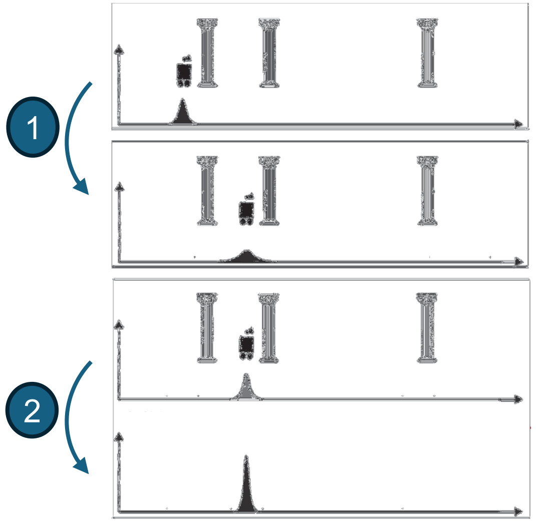
$\boldsymbol{bel}\,(\boldsymbol{x_k}) = p(x_k | Z_k, U_k)$



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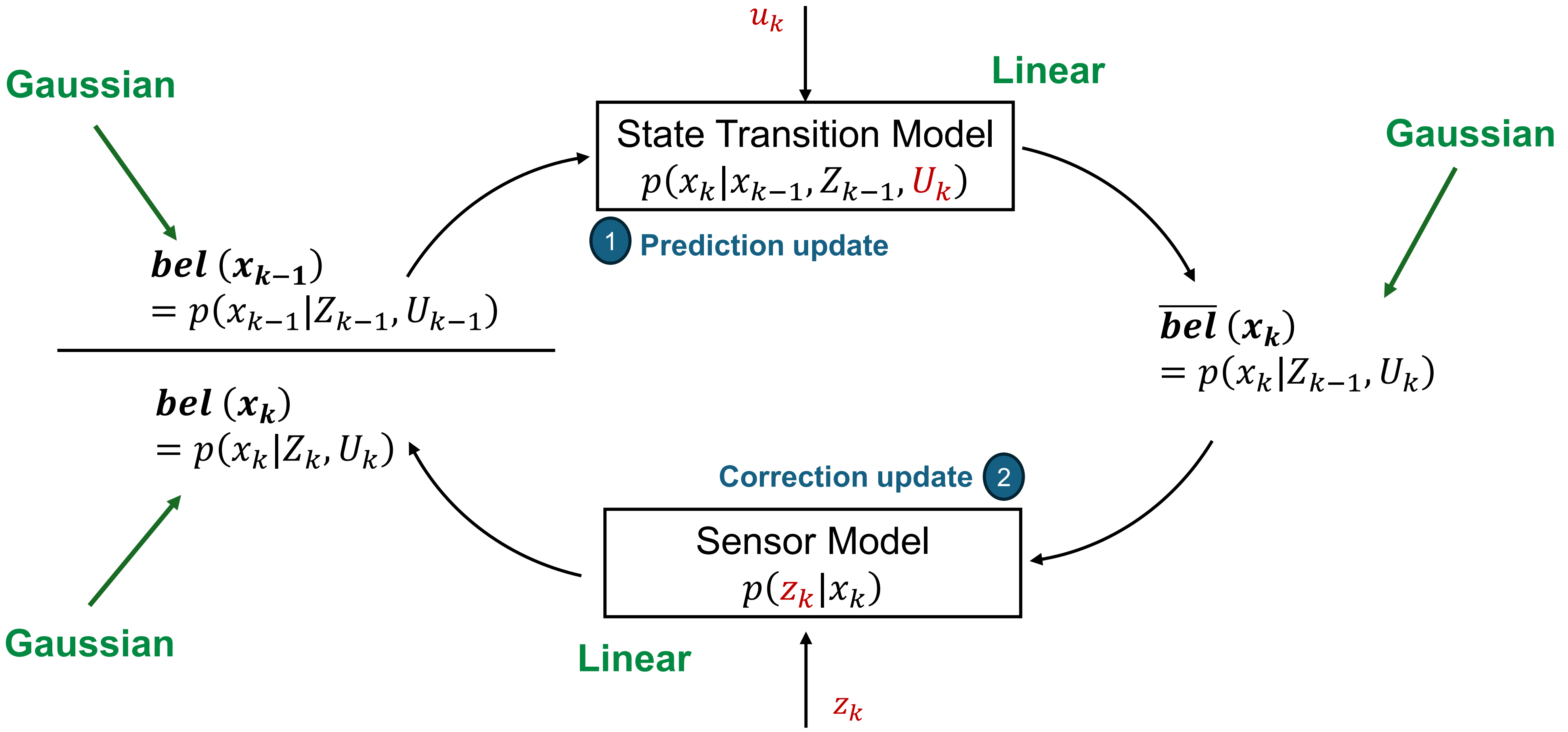
# Kalman Filter

- The Kalman Filter is a particular kind of recursive Bayesian filter specialized for linear systems with Gaussian noise.
- It exploits the fact that Gaussians remain Gaussians under linear transformations, enabling closed-form solutions at every step.
- It assumes that both the State Transition Model and Sensor Model are both linear.

# Kalman Filter



# Linear State Transition Models

- The state transition model describes how the state at the next time step $x_k$ evolves.
- A Kalman Filter assumes the state transition model to satisfy a linear equation of the form:

$$\boxed{x_k = F_k\, x_{k-1} + B_k\, u_k + v_k}$$

where:

- $x_k \in \mathbb{R}^n$ is the state vector of interest,
- $u_k \in \mathbb{R}^m$ is a known control input,
- $v_k \in \mathbb{R}^n$ is a random variable describing model uncertainty,
- $F_k \in \mathbb{R}^{n\times n}, B_k \in \mathbb{R}^{n\times m}$ are matrices describing the contribution of states, and controls to state transition.

# Linear Measurement Models

- The measurement model describes how the measurement $z_k$ is related to the state $x_k$.
- A Kalman Filter assumes the measurement model to satisfy a linear equation of the form:

$$\boxed{z_k = H_k\, x_k + w_k}$$

where:

- $x_k \in \mathbb{R}^n$ is the state vector of interest,
- $z_k \in \mathbb{R}^q$ is the observation vector,
- $w_k \in \mathbb{R}^q$ is a random variable describing observation uncertainty,
- $H_k \in \mathbb{R}^{q\times n}$ is a matrix describing the contribution of state to observation.

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# Probabilistic Models

- The state transition model and measurement models are not deterministic, but probabilistic with Gaussian distributions.

| | State transition model | Measurement model |
|---|---|---|
| Difference Equation | $x_k = F_k\, x_{k-1} + B_k\, u_k + v_k$ | $z_k = H_k\, x_k + w_k$ |
| Probability Distribution Function | $p(x_k \mid x_{k-1}, u_k)$ | $p(z_k \mid x_k)$ |
| Mean of Gaussian PDF | $\hat{x}_k$ | $\hat{z}_k$ |
| Covariance of Gaussian PDF | $P_k$ | $R_k$ |

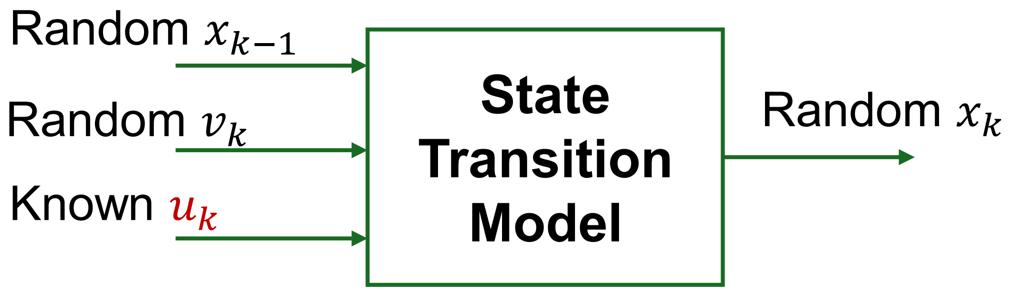
In summary, Kalman Filter assumes:

- $p(x_k \mid x_{k-1}, u_k) = N(\hat{x}_k, P_k)$
- $p(z_k \mid x_k) = N(\hat{z}_k, R_k)$

# State Transition Probabilistic Model

$$\boxed{x_k = F_k\, x_{k-1} + B_k\, u_k + v_k}$$

- If $x_{k-1}$ and $v_k$ are two random variables that are independent and have a Gaussian distribution, then $x_k$ will also have a Gaussian distribution.
- Question: If we know the mean and covariance of $x_{k-1}$ and $v_k$, how can we compute the mean and covariance of $x_k$?




**Note** that $u_k$ is not a random variable but a known input.

# State Transition Probabilistic Model

$$x_k = F_k \, x_{k-1} + B_k \, u_k + v_k$$

- If $x_{k-1}$ and $v_k$ are two random variables that are independent and have a Gaussian distribution, then $x_k$ will also have a Gaussian distribution.
- We will denote the mean and covariance of $x_{k-1}$, $v_k$, and $x_k$ by:
  - $x_{k-1} \sim N(\hat{x}_{k-1}, P_{k-1})$
  - $v_k \sim N(0, Q_k)$
  - $x_k \sim N(\hat{x}_k, P_k)$

# State Transition Probabilistic Model

$$x_k = F_k\, x_{k-1} + B_k\, u_k + v_k$$

- If $x_{k-1}$ and $v_k$ are two random variables that are independent and have a Gaussian distribution, then $x_k$ will also have a Gaussian distribution.
- If $x_{k-1} \sim N(\hat{x}_{k-1}, P_{k-1})$ and $v_k \sim N(0, Q_k)$ then $x_k \sim N(\hat{x}_k, P_k)$ with

**Mean**

$$\hat{x}_k = F_k \hat{x}_{k-1} + B_k\, u_k$$

**Covariance**

$$P_k = F_k P_{k-1} F_k^{\top} + Q_K$$

# Measurement Probabilistic Model

- Now let's consider the measurement model $z_k = H_k\, x_k + \ w_k$
- Note that in this equation the observation $z_k$ is written as an output.
- However, to use it in the Kalman filter, we need to use the observation as an input (similar to $u_k$ in the state transition model).

# Measurement Probabilistic Model

- Now let's consider the measurement model $z_k = H_k\, x_k + \ w_k$
- Note that in this equation the observation $z_k$ is written as an output.
- However, to use it in the Kalman filter, we need to use the observation as an input (similar to $u_k$ in the state transition model).
- The way we do this is by introducing the innovation variable $\beta_k \in \mathbb{R}^q$

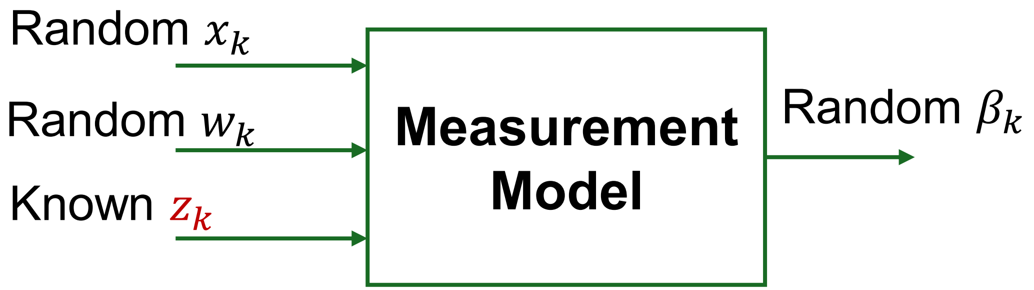
$$\beta_k \coloneqq z_k - H_k x_k + \ w_k$$

which is the difference between the observed measurements and what the measurement model predicts.

# Measurement Probabilistic Model

$$\boxed{\beta_k := z_k - H_k x_k + w_k}$$

- If $x_k$ and $w_k$ are two random variables that are independent and have a Gaussian distribution, then $\beta_k$ will also have a Gaussian distribution.
- Question: If we know the mean and covariance of $x_k$ and $w_k$, how can we compute the mean and covariance of $\beta_k$?




**Note** that $z_k$ is not a random variable but a known input.

# Measurement Probabilistic Model

$$\boxed{\beta_k \coloneqq z_k - H_k x_k + w_k}$$

- If $x_k$ and $w_k$ are two random variables that are independent and have a Gaussian distribution, then $\beta_k$ will also have a Gaussian distribution.
- If $x_k \sim N(\hat{x}_k, P_k)$ and $w_k \sim N(0, R_k)$ then $\beta_k \sim N\left(\hat{\beta}_k, S_k\right)$ with

**Mean**

$$\boxed{\hat{\beta}_k = z_k - H_k \hat{x}_k}$$

**Covariance**

$$\boxed{S_k = H_k P_k H_k^\top + R_K}$$

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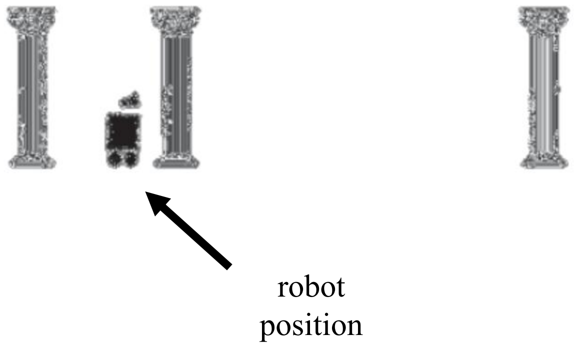
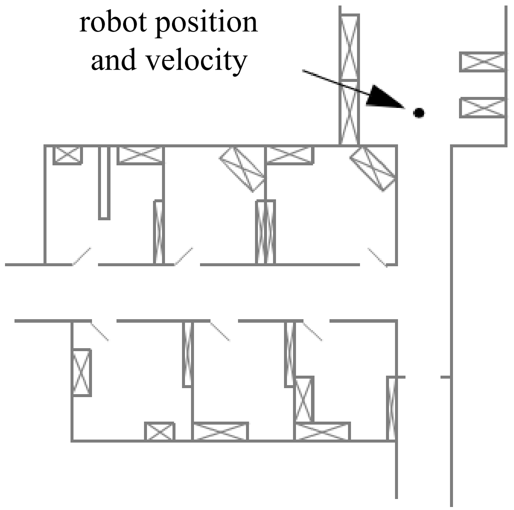
# Example Estimation Problems

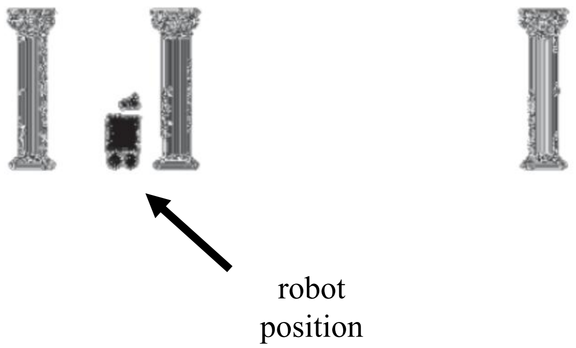
- 1D robot position:
  - State $x(t) = \xi(t) \in \mathbb{R}$
  - Velocity sensor $u(t) \in \mathbb{R}$
  - Position sensor $\mathrm{z}(t) \in \mathbb{R}$
- 2D robot position and velocity
  - State $x(t) = (\xi(t), v(t)) \in \mathbb{R}^4$
  - Acceleration sensor $u(t) \in \mathbb{R}^2$
  - Position sensor $\mathrm{z}(t) \in \mathbb{R}^2$



# Example 1:

- Robot State: 1D position
  - $x(t) = \xi(t) \in \mathbb{R}$
- Control input: PC velocity sensor at 1 Hz
  - $u(t) = v(t) + \eta(t), \qquad \eta(t) \sim N(0, \sigma_u^2)$
- Observation input: EC position sensor at 1/Ts Hz
  - $z(t) = \xi(t) + w(t), \qquad w(t) \sim N(0, \sigma_z^2)$
- Kinematic model:
  - $\dot{x}(t) = \dot{\xi}(t) = v(t)$




**PC**: proprioceptive
**EC**: exteroceptive

# Example 1:

- Kinematic model:
  - $\dot{x}(t) = \dot{\xi}(t) = v(t)$

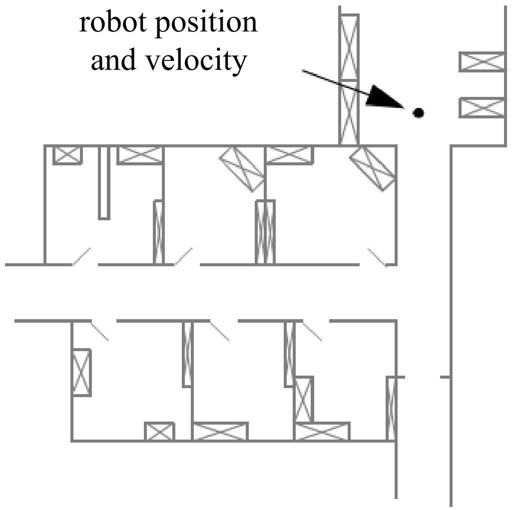
Discretization rule:
$\dot{x}_k \approx \frac{x_k - x_{k-1}}{\Delta t}$

- State Transition model:
  $$x_k = x_{k-1} + \Delta t\, u_k + \eta_k$$
- Measurement model:
  $$z_k = x_k + w_k$$

$$F_k = H_k = 1, \qquad B_k = \Delta t, \qquad Q_k = \sigma_u^2, \qquad R_k = \sigma_z^2$$

# Example 2:

- Robot State: 2D position and velocity
  - $x(t) = (\xi(t), v(t)) \in \mathbb{R}^4$
- Control input: PC acceleration sensor at 1 Hz
  - $u(t) = a(t) + \eta(t), \qquad \eta(t) \sim N(0, Q)$
- Observation input: EC position sensor at 1/Ts Hz
  - $z(t) = \xi(t) + w(t), \qquad w(t) \sim N(0, R)$
- Kinematic model:
  - $\dot{x}(t) = \begin{pmatrix} \dot{\xi}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ a(t) \end{pmatrix}$



# Example 2:

- Kinematic model:
  - $\dot{x}(t) = \begin{pmatrix} \dot{\xi}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ a(t) \end{pmatrix}$

Discretization:

$$\dot{x}_k \approx \frac{x_k - x_{k-1}}{\Delta t}$$

- State Transition model:

$$x_k = F_k x_{k-1} + \mathrm{B}_k\, u_k + \eta_k$$

- Measurement model:

$$z_k = H_k x_k + w_k$$

$$F_k = \begin{pmatrix} I_2 & \Delta t\, I_2 \\ 0_{2\times 2} & I_2 \end{pmatrix}, \quad H_k = \begin{pmatrix} I_2 & 0_{2\times 2} \end{pmatrix}, \quad B_k = \begin{pmatrix} (\Delta t)^2 I_2 \\ \Delta t\, I_2 \end{pmatrix},$$

$$Q_k = \mathrm{diag}\,(\sigma_{u1}^2, \sigma_{u2}^2), \quad R_k = \mathrm{diag}\,(\sigma_{z1}^2, \sigma_{z2}^2)$$